A Study of Multivariable Calculus Vector-Related Items in the WeBWorK Open Problem Library

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Research Article

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Abstract

WeBWorK is a free, online homework system that was developed in 1994. It has been supported by key organizations and is currently used by many in the postsecondary mathematics community. Its Open Problem Library (OPL) contains tens of thousands of items at the lower division undergraduate mathematics level. The goal of this study is to investigate multivariable calculus vector-related items in the WeBWorK OPL. We first consider the usage and nature of these items. Next, we consider the nature of the items by looking at each item's cognitive complexity, the vector ideas addressed in the item, and the vector representations used in the item by systematically adapting and employing previous coding schemes from documented mathematics education research. The findings in this study validate the critical features of vectors that were identified in earlier work. We conclude that the list of key vector ideas appears to be comprehensive since they addressed the topics in all multivariable calculus vector-related items studied. However, there are not many items of high or very high cognitive complexity. Based on our findings, there appears to be a need for specific types of vector-related items available in the WeBWorK OPL. We conclude there is a need for additional items that involve contextual situations; graphical or geometrical interpretation; and reasoning, proof, and justification. We also provide detailed recommendations for the addition of vector-related items that address topics or use representations that are missing or in limited supply in the WeBWorK OPL.

Introduction

Vectors are situated at the crossroads from secondary to post-secondary mathematics playing a foundational role in more advanced mathematics. While basic vector concepts and vector arithmetic are often presented in secondary mathematics, students often do not experience vector dot and cross products until their postsecondary calculus courses. Most postsecondary research regarding vectors has been related to physics or engineering (e.g., Barniol & Zaval, 2014; Flores et al., 2003; Nguyen & Metzler, 2003). There is little research on student understanding of vectors in postsecondary mathematics and what does exist tends to focus on linear algebra or geometry (e.g., Stewart & Thomas, 2009; Kwon, 2013). Previous research findings suggest that students’ concept images of vectors develop in a rather fragmented manner, and students fail to develop the rich conceptual knowledge required to work with different representations of vectors (Van Deventer & Wittmann, 2007) or with different key ideas related to vectors (Martinez-Planell et al., 2015, 2017). Despite the regular occurrence of vectors throughout the curriculum, students have significant conceptual difficulties with vector concepts and manipulations. For instance, Knight (1995) found that ~40% of students in a calculus-based, college physics course had no idea what a vector was, and not one was able to evaluate a vector cross product.

The type of mathematical tasks that are assigned to students matter (Stein et al., 2009) since tasks influence how and what students learn (Chapman, 2013). Limited exposure to tasks that promote conceptual discussion is one of the reasons reported by calculus students for switching out of STEM majors (Johnson et al., 2014). Students should be assigned a wide range of tasks that help develop their understanding (Breen & O’Shea, 2010) as well as their representational fluency, since understanding is associated with a synergy of knowing and being able to connect multiple ways of representing a mathematical idea (Moore-Russo & Viglietti, 2012; Duval, 1999, 2017). Ellis and colleagues (2015) found that calculus students felt that online homework helped their confidence by providing instant feedback and allowing multiple chances to complete tasks correctly. Halcrow and Dunnigan (2012) report that online homework positively impacts calculus students’ motivation, for both high and low achieving students. Others have studied the ways that certain online homework platforms and improved in order to foster productive mathematical dispositions (Dorko, 2021; Willhoite, 2020). Many post-secondary mathematics instructors use WeBWorK (The WeBWorK Project, n.d.) as an online platform to assign tasks (Mathematical Association of America, 2021). WeBWorK is intended as a platform where instructors can submit and select items that go beyond textbook exercises, which often simply mimic previous examples (El Turkey et al., 2020). WeBWorK has great potential as a platform to provide a wide array of tasks, but it has largely been unstudied.

In line with White and Mesa (2014), who studied instructor-assigned tasks in calculus, this study uses WeBWorK items from its Open Problem Library (OPL) to determine “instructors’ intentions and ... students’ opportunities to learn” (p. 676) about vectors. When studying items, it is common to consider both the topics required to solve the tasks as well as the cognitive load required on the part of the learner to complete the tasks. There is a difference between tasks that can be answered by memorization versus tasks that require novel, non-routine thinking for their solutions. In addition to the cognitive complexity of a task, it is also important to consider how the information in the task is presented to the students. The representations used in the prompt of a task may impact if a student is able to decipher the information presented and what is being asked. In a similar manner, the representations required for a task, especially in online platforms where items are often automatically graded, can impact if a student is able to succeed in a task.

The overarching goal of this study is to investigate multivariable calculus vector-related items in the WeBWorK OPL. We consider the usage and nature of these items. We study this by looking at the following research questions.
1. How many WeBWorK vector-related items are there? What is their usage; in other words, how often are these items assigned, how many attempts are made for these items, and how successful are individuals in solving these items?

2. What vector-related items appear in WeBWorK OPL?

   1. How many dimensions are the vectors in the items and what item types appear?
   2. What is the cognitive complexity of these items?
   3. What key vector ideas do items address? Which are most and least prominent? Which appear together in items?
   4. What representations of vectors are used in the item prompts and expected for responses in these items?
   5. How do the above classifications correspond to WeBWorK usage?

3. Do certain key vector ideas appear more frequently in items with higher cognitive complexity?

4. Are key vector ideas addressed in items with a variety of vector representations?

**Framing For The Study**

Marton and Booth (1997) put forward variation theory by suggesting that variation is a necessary component in pedagogical decisions (Lo & Marton, 2012) so that students are able to notice, focus on, and discern exactly what is to be learned (Marton, 2015; Marton & Pang, 2006). According to variation theory, students learn a concept when they are able to perceive the invariant relationships between the critical features of the concept. Critical features are aspects or conditions of a concept that are necessary for understanding that concept (Runesson, 2006).

Previous research first identified the critical features of the vector cross product as: magnitude, direction, angle between two vectors, and orientation of the cross product to the two vectors that form it (Moore-Russo et al., 2017, VanDieren et al., 2017, 2018). More recently a study (VanDieren & Moore-Russo, 2019b) of the Test of Understanding of Vectors (Barniol & Zavala, 2014) and as well as a study of how multivariable students communicate and connect their ideas related to their understanding of vectors (VanDieren & Moore-Russo, 2019a), have been used to identify topics in vector instruction at the multivariable calculus level. The authors have synthesized this data as well as their experiences in teaching and leading workshops for multivariable calculus instruction to create the following list of key ideas, including critical features of vectors and vector operations, related to vectors displayed in Table 1. In multivariable calculus, students who know how to visualize, represent, identify, describe, compute, distinguish, and make comparisons of each key idea of vectors below would be considered to have a solid understanding of vectors. To have a robust concept image (Tall & Vinner, 1981) of vectors, students need to both understand each of the key vector ideas, recognize how they are connected to each other to be able to call upon the key idea needed, or to move fluidly between keys ideas if needed, in any context, and using the representation that is most efficient to the task at hand.
Table 1  
*Key Vector Ideas*

<table>
<thead>
<tr>
<th>Key Ideas by Type</th>
<th>Description of Task Requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vector (V)</strong></td>
<td>Student must identify vectors in their component forms without emphasis on any of the critical features of vectors or vector operations listed below</td>
</tr>
</tbody>
</table>

**Critical Features of Vectors**

<table>
<thead>
<tr>
<th>Direction (VD)</th>
<th>Student must consider, use, compute, or compare vector direction, the feature related to the angle the vector makes with a horizontal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Magnitude (VM)</strong></td>
<td>Student must consider, use, compute, or compare vector magnitude, the feature that assigns a quantity or length to a vector</td>
</tr>
</tbody>
</table>

**Special Vectors (VS)**

- zero vector with a magnitude of zero and an undefined direction
- unit vector with a magnitude of one
- unit basis vector (e.g., $\vec{i}, \vec{j}, \vec{k}$)

<table>
<thead>
<tr>
<th>Angle (VA)</th>
<th>Student must consider, use, compute, or compare angles between vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Orientation (VO)</strong></td>
<td>Student must consider, use, compute, or compare the relative orientation between vectors according to the right (or left) hand rule</td>
</tr>
</tbody>
</table>

**Vector Operations**

<table>
<thead>
<tr>
<th>Scalar Multiplication (S)</th>
<th>Student must consider, use, compute, or compare the multiplication of a vector by a scalar</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Addition (A)</strong></td>
<td>Student must consider, use, compute, or compare the addition of two or more vectors</td>
</tr>
<tr>
<td><strong>Subtraction (B)</strong></td>
<td>Student must consider, use, compute, or compare the subtraction of two or more vectors</td>
</tr>
<tr>
<td><strong>Dot Product (D)</strong></td>
<td>Student must consider, use, compute, or compare the dot products of two vectors that results in a single scalar quantity</td>
</tr>
<tr>
<td><strong>Projection (P)</strong></td>
<td>Student must consider, use, compute, or compare the projection of one vector onto another vector</td>
</tr>
<tr>
<td><strong>Cross Product (X)</strong></td>
<td>Student must consider, use, compute, or compare the cross products of two vectors that results in a single vector that is orthogonal to the two initial vectors as determined by the right-hand rule</td>
</tr>
</tbody>
</table>

**Item Complexity**

Cognitive complexity refers to the level of thinking required to answer a task. Mathematical tasks that vary from requiring only memorization to non-algorithmic problem-solving requiring creativity can be assigned to students. Both extremes, as well as levels in between these extremes, can be a part of mathematics instruction. "The challenge is to achieve some mixture of routine skill and understanding and, even more difficult, to integrate procedural skills with high-level thinking" (Stein et al., 1996, p. 473).

Complexity has been measured in many ways through the years (Holmes, 2012). There have been many frameworks used for cognitive complexity, such as Skemp's (1976) framework, Hiebert and Carpenter's (1992) procedural and conceptual understanding, Porter's (2002) cognitive complexities, and Webb and Hess's (2005) depth of understanding. Stein and colleagues’ (1996) model for the cognitive demands of tasks was later adapted to a four-level cognitive complexity framework by Stein and Smith (1998), which has been widely used. Tasks have often been divided into lower order and higher order thinking tasks in the aforementioned frameworks (Park, 2011), and researchers typically promote higher order thinking abilities as an instructional goal in order for students to become independent problem solvers. For example, Ellis et al. (2015) studied calculus instructors’ reports of the content of their assignments and assessments to see what types of tasks were reported to determine if the tasks extended beyond a “plug and chug” level.

**Representations of Vectors**

It is important to consider the ways that the critical features are communicated and represented, since mathematical representations influence how a person comes to understand a mathematical concept (Alibali et al., 2009). The interplay and connections between different mathematical representations impact how individuals develop understanding and are able to send and receive communication about the
Methods

Data Set and Coding

WeBWorK is a free, open-source, online homework system that is supported by the Mathematical Association of America and the National Science Foundation. WeBWorK is used at over 700 colleges and universities. It was created in 1994 and has been maintained primarily by the postsecondary mathematics community. Its library of over 30,000 homework items primarily addresses lower division undergraduate mathematics topics (The WeBWorK Project, n.d.).

A WeBWorK advanced search in the Open Problem Library (OPL) using the keyword “vectors” on January 19, 2018 resulted in the research team identifying 148 possible items for the study. Note that the OPL contains only items vetted by the editorial board. Those 148 items had been sorted into 119 groups, with a single group being the items deemed by the editorial board as being similar. Since items with 2-dimensional vector tasks and items with 3-dimensional vector tasks were grouped as being similar, the research team decided to start with all 148 items. Several passes of the tasks were conducted to eliminate duplicates and non-relevant tasks. The data set of the 106 tasks that remained after the initial pass along with additional data generated and analyzed in this study is available in the github repository (see https://github.com/vandieren/webwork-vector-analysis). The following paragraph details the passes that resulted in the 76 analyzed in this study.

Criteria for the research team to exclude items from the study in the first pass included: 1) if the task or its solution involved matrices or other linear algebra content, since the study focused on vectors in multivariable calculus; 2) if the task or its solution involved planes, lines or vector fields, since the study considered material that only involves student understanding of vectors and their operations; or 3) if the item prompt did not invoke any vector terminology and the solution to the task could be found using basic algebra formulas. After the first pass, 40 of the 148 items were eliminated, leaving 108 items for the study. On a second pass, it was determined that item #52 referenced planes and item #58 did not require the use of vectors; so, these two items were removed leaving 106 items. The criterion for the research team to exclude items from the study in the third pass was if the task duplicated another task. Items were considered duplicates if they pointed to the same source file and only differed by randomizations of the same task. After the third pass, 30 items were eliminated, leaving 76 items for the study. Of the 76, 32 items had more than one part; with a total of 112 parts. We coded each part of the 32 multi-part items but determined after reviewing the data to report findings primarily by item since subsequent parts in a single multi-part item built on previous parts of the item or all parts of an item were similar with only numeric changes.

We also wanted to study the item usage statistics. Of the 76, 4 did not have any usage statistics available. We report on all 76 items when possible, and only on the 72 when reporting usage statistics. We used the following WeBWorK usage statistics collected from the WeBWorK Problem Library from the year that WeBWorK collected this data. WeBWorK usage statistics were collected from many, but not all, institutions that use WeBWorK. These statistics include global usage, attempts, and status (Pizer, 2019). Global usage for an item represents the number of times that an item has been attempted at least once. This might be thought of as a proxy for how frequently an item is assigned. Attempts is the “global average of the number of attempts (both correct and incorrect) individuals take on [the] problem” (Pizer, 2019). Each submission would count as an attempt. Therefore, a high attempt value may indicate a difficult item. Status is the global average of the percentage correct recorded for an item. Since many instructors allow multiple submissions, status is generally high because students will continue to attempt the item until they receive full credit. Therefore, a low status value would likely indicate a difficult item.

WeBWorK items reviewed were those 72 whose selection was described previously. As previously noted, we coded each part of each item; however, the data are primarily reported with the item being the unit of analysis. Joint coding of items (and their parts) took place over an extended period of almost two years, due to pandemic interruptions. Multiple passes were made through the data using the coding schemes now outlined.

We now consider the dimensions of vectors coding. Vectors are typically introduced in two dimensions. However, they are needed in three dimensions for multivariable calculus and often needed in more than three dimensions for linear algebra. So, the first coding pass was for the
number of dimensions for the vector(s) in the item with the codes being 2D, 3D, and 4D or more.

Next, we consider item type coding. Ellis and colleagues (2015) used five codes to describe the types of tasks assigned by calculus instructors. These included: a) skills and methods for carrying out computations, b) graphical interpretation of central ideas, c) solving standard word problems, d) solving complex or unfamiliar word problems, and e) proofs or justifications. For this study, we collapsed the two categories involving word problems into a single category, since we used the items themselves as data, rather than the instructor reports of the tasks assigned as Ellis et al. did in their study. We also added a category when a justification was required to complement their work; this allowed the four item type codes used, which as listed in Table 2, to be mutually exclusive.

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation/Calculation</td>
<td>Emphasis on students on using an algorithm, procedure, or process to determine a specified result; focus on skills and methods for carrying out computations</td>
</tr>
<tr>
<td>Graphical/Geometrical interpretation</td>
<td>Emphasis on students interpreting geometric or graphical information to solve, might involve the use of an algorithm but there is graphical/geometrical solutions</td>
</tr>
<tr>
<td>Contextual Situation</td>
<td>Emphasis on students interpreting a word (“real life”) problem with either a physical context or some storyline where the vectors have meaning beyond their geometrical meaning</td>
</tr>
<tr>
<td>Reasoning, Proof, Justification</td>
<td>Emphasis on students justifying their reasoning or using proofs and logical deduction to complete an item</td>
</tr>
<tr>
<td>Justification (complementary)</td>
<td>Requirement that students provide explanations or to justify their reasoning to complement previous work in earlier parts of an item</td>
</tr>
</tbody>
</table>

### Cognitive Complexity Coding

The four cognitive demand levels from Stein and Smith (1998), which have been used in many studies and which are described in Table 3, were used to code cognitive complexity in this study. They included: very low, low, high, and very high.

<table>
<thead>
<tr>
<th>Coding Levels</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Low - Memorization Items</td>
<td>Simple information recall without any computation required; reproduction of facts or a definition; immediate recognition of the answer; exact reproduction of previously seen, memorized facts</td>
</tr>
<tr>
<td>Low - Procedural Items without Connections</td>
<td>Recipe-style, memorized algorithmic steps using routine explanations or well-rehearsed procedures where the use of procedures is immediately evident; focus on item solution process using procedure rather than developing or demonstrating understanding</td>
</tr>
<tr>
<td>High - Procedures with Connections</td>
<td>Knowledge of how and why procedures are to be used to complete item required; completion of item not possible by only following rehearsed steps even though the procedures may (or may not) be recognized; some degree of conceptual understanding of the item complements any algorithmic thinking but focus is on developing or demonstrating understanding; item completion might require connections to multiple procedures, concepts, or definitions and involve use multiple representations</td>
</tr>
<tr>
<td>Very High - Doing Mathematics</td>
<td>Non-routine thinking required with no obvious solution path; item requires exploration into mathematical relationships and context; deep degree of conceptual understanding with focus on demonstration or development of understanding through analysis; complex item involving consideration of affordances and constraints of certain mathematical representations and strategies; solution goes well beyond procedures used to include connections, interpretations, and applications that make apparent how concepts or strategies are used in challenging, novel ways</td>
</tr>
</tbody>
</table>

### Key Vector Idea Coding

Each item was coded using the categories in Table 1 to capture the vector content that the item emphasized. Note that since all items involved vectors, we only coded an item as V (vector) if it did not require the student to use any of the other key ideas. Items could have been coded as having more than one key idea present. For each item that was coded as high or very high cognitive complexity and that expected students to draw connections between key ideas, the item was coded to indicate the connecting key ideas with a hyphen between the two connecting codes (e.g., D-VA represents a connection between dot product and angle between two vectors).

### Vector Representation Coding
Vectors can be represented in many forms. To consider which representations were used and if students were expected to move between representations, item prompts and responses were coded by the type of vector representation. The codes used, shown in Table 4, included: graphed vectors, basis vectors, column vectors, n-tuples, verbal descriptions, and implicit description by endpoints. There were responses that were coded as other; these included responses that were explanations and equations.

**Response Type Coding**

While all item prompts involved vectors in one of the six representations listed in Table 4, item responses varied. We coded for the four different response types that included: angles; scalars; vectors (i.e., graphed vectors, basis vectors, column vectors, n-tuple vectors, verbal descriptions of vectors or endpoints of vectors); and other (e.g., points, equations, explanations).

**Findings And Discussion**

**Number and Usage of Vector-Related WeBWorK OPL Items**

When considering online platforms, it is often possible, and important, to consider outcomes assessment reports (VanDieren et al., 2020) and usage to explore more about how often tasks are assigned and student performance on the tasks. To consider the first research question, we now report on the usage statistics for the WeBWorK OPL vector-related items, which are displayed in Fig. 1. The mean global usage for the 72 items was 1061. However, there were 7 outlier items, which are described in Table 5. The global usage ranged from 1 to 9767 with a median of 231. Note that these represent lower bounds for usage in one year since only a subset of WeBWorK users report their statistics. The mean number of attempts for the 72 items was 3.13 ranging from 1.35 to 9.77 with a median of 2.65. The mean status for the 72 items was 94 ranging from 69 to 100 with a median of 97.

We now consider the outliers for the usage statistics as displayed in Table 5. Considering the global usage statistics, there seems to be high demand for certain problem types. When considering the seven outliers for global usage, four are of the graphical/geometrical interpretation item type. Of the global usage outliers, five involve dot product, four involve angle, and three involve direction. All four tasks that were outliers for attempts require multiple responses. It could be the case that students were checking on the correctness of their responses after different parts; however, recall that there were 32 multipart items.
Of the four attempts outliers, all were either graphical/geometrical interpretation or contextual situation items. Of these outliers, three involve vector angle and three involve direction. When considering the three outliers for status, all three were contextual situation items whose prompts involved verbal descriptions of vectors but whose responses required non-vector answers.

Three of the outliers in Table 5 were outliers for more than one statistic. Item 34 was a multipart item involving rowing from north to south across a river, and one of the parts asks about which bank is reached first (but the required response is the north bank, which seems incongruous with the given information). Item 47 was an outlier for all three statistics; it was one of the few items that involved projection; so, students may not have had the opportunity to practice routine problems involving projection before attempting this item. Also, the physical work in the response to Item 47 may be negative (since the coefficients are randomized), which might have confused students. Note that item 50 had four parts, one of which required seven responses. There could be syntax issues in the required responses for the students typing the pairs of vectors that were required for the responses to this item.

Table 5
Outliers for OPL Item Statistics (including Global Usage, Attempts, and Status)

<table>
<thead>
<tr>
<th>Item</th>
<th>Outlier</th>
<th>Dim</th>
<th>Item Type</th>
<th>Cog Comp</th>
<th>Key Vector Ideas</th>
<th>Prompt</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>Global usage (1623)</td>
<td>3D</td>
<td>Computation/Calculation</td>
<td>Low</td>
<td>VM-D, VA-D</td>
<td>Basis vectors</td>
<td>Angle</td>
</tr>
<tr>
<td>10</td>
<td>Global usage (9377)</td>
<td>3D</td>
<td>Computation/Calculation</td>
<td>Low</td>
<td>VD, VM</td>
<td>N-tuple vectors</td>
<td>N-tuple vectors</td>
</tr>
<tr>
<td>14</td>
<td>Global usage (8126)</td>
<td>3D</td>
<td>Graphical/Geometrical interpretation</td>
<td>High</td>
<td>VO-D</td>
<td>N-tuple vectors</td>
<td>Scalars</td>
</tr>
<tr>
<td>19</td>
<td>Global usage (9767)</td>
<td>2D</td>
<td>Graphical/Geometrical interpretation</td>
<td>High</td>
<td>VA-D</td>
<td>Graphed vectors</td>
<td>Words</td>
</tr>
<tr>
<td>30</td>
<td>Attempts (5.81)</td>
<td>2D</td>
<td>Graphical/Geometrical interpretation</td>
<td>High</td>
<td>VD-S</td>
<td>Basis vectors</td>
<td>Scalars</td>
</tr>
<tr>
<td>34*</td>
<td>Attempts (6.61), Status (75)</td>
<td>2D</td>
<td>Contextual Situation (distance, velocity)</td>
<td>Very High</td>
<td>VM-A, VA</td>
<td>Verbal description of vectors</td>
<td>Angles and verbal descript. of vectors</td>
</tr>
<tr>
<td>35</td>
<td>Status (72)</td>
<td>3D</td>
<td>Contextual Situation (velocity)</td>
<td>Very High</td>
<td>VM-A</td>
<td>Verbal description of vectors</td>
<td>Scalars</td>
</tr>
<tr>
<td>47</td>
<td>Global usage (8039), Attempts (9.44), Status (69)</td>
<td>2D</td>
<td>Contextual Situation (e.g., force, work)</td>
<td>High</td>
<td>VD-P, VA-P, S-P, S-B, A, D-P, D, D</td>
<td>Verbal description of vectors</td>
<td>Scalars and basis vectors</td>
</tr>
<tr>
<td>50</td>
<td>Global usage (7652), Attempts (9.77)</td>
<td>3D</td>
<td>Graphical/Geometrical interpretation</td>
<td>Low</td>
<td>VA-D, VD-S</td>
<td>Basis vectors</td>
<td>Basis vectors</td>
</tr>
<tr>
<td>95</td>
<td>Global usage (9683)</td>
<td>2D</td>
<td>Graphical/Geometrical interpretation</td>
<td>Low</td>
<td>VM, A, S-A, S-B</td>
<td>Graphed vectors</td>
<td>Scalars and n-tuple vectors</td>
</tr>
</tbody>
</table>

Note
The * indicates a multipart item with one part that does not make sense in the given context.

To consider if items with multiple parts had different usage statistics than items with a single part, we display the box and whisker plot comparisons in Fig. 2. There was a difference in medians across all three usage statistics, and the interquartile range from Q1 to Q3 was disjoint for multipart and single-part items in attempts. The attempts may be higher for a multipart item since students may submit the item
to check the first part before completing the remaining parts of the item. Therefore, a high value for attempts may not correspond to a difficult item if the item had several parts.

**Nature of Vector-Related WeBWorK OPL Items**

To address the multiple parts of the second research question, we now consider specific characteristics of the vector items in the OPL. For this, we report on five different item characteristics: dimensions of vectors in the item, type of item, cognitive capacity of item, key vector ideas addressed in the item, and representations of vectors used in the item.

We first consider the dimensions used in the vector items. Of the 112 parts of items, 44 were coded as 2D, 62 as 3D, and six as 4D or more. For the 76 items, we used the highest dimension in any part of the item for coding. There were 30 items coded as 2D, 44 as 3D, and two as 4D or more.

For the item types, we found that more computation/calculation items (43 of 76) were in the WeBWorK OPL than any other item type. Next most common (23 of 76) were the graphical/geometric interpretation items. Multivariable calculus often emphasizes analytic geometry; so, it might be expected that this would be a larger percentage of the OPL vector items. Most (7 of 10) of the contextual situation items were based in a physics context. The contextual situation items (10 of the 76) in the OPL had a mean global usage of 1174, while the mean global usage for all OPL tasks was 1061. This might suggest that the OPL would benefit from the addition of more as well as more diversified contextual situation vector items. No items were found to emphasize justification of reasoning using proofs or logical deduction over the other possible item type categories.

We compared the distribution of item types in the WeBWorK OPL to previous findings of student reports on the types of items instructors assign in single-variable calculus. Specifically, Ellis et al. (2015) reported on the means for two groups of students, the selected group involving almost 600 students at universities where they were more likely to retain "confidence, enjoyment, and interest in mathematics (p. 269)" compared to the non-selected group involving about 1410 students at universities that were less likely to retain those three. Another study by Burn and Mesa (2015) reported the median responses from a survey of calculus instructors about the item types that they include on assignments. These findings are included in Table 6.

The first noted similarity between these studies and ours is that all reported at least 40% of the tasks assigned were computation/calculation tasks. The second is that graphical/geometrical interpretation tasks ranged from 20–33% in all the studies. Differences between these studies and ours could be attributed to many reasons. First, our study looks at actual problems, the Ellis et al. (2015) study uses student reports, and the Burn and Mesa (2015) study uses instructor reports. Second, our study looks at multivariable calculus, while the Ellis et al. and the Burn and Mesa studies were for single-variable calculus. Three-dimensional graphical analysis should play an explicit role in multivariable calculus (Martínez-Planell et al., 2017; McGee & Moore-Russo, 2015; McGee et al., 2015); this might be why graphical/geometrical interpretation tasks seem to be assigned more in multivariable calculus compared to single-variable calculus, apart from Ellis and colleagues’ selected group. Another factor to consider is the limitation of the types of problems that can be programmed into WeBWorK. The essay prompt and draggable proofs are newer features in WeBWorK. Therefore, there may be fewer items in the OPL that require a proof or justification. However, instructors may augment existing OPL items by adding an essay prompt asking students to provide reasoning, proof, or justification (VanDieren, 2021a). These adaptations may be assigned by instructors but may not appear in the OPL. Additionally, it is quite possible that instructors assign tasks involving reasoning, proof, or justification outside of the WeBWorK platform.

We now look at item cognitive complexity. Of the 112 parts of items, about two-thirds (73) were coded as low cognitive complexity, and about one-fourth were coded as high complexity. These were followed by very high (7) then very low (1) cognitive complexity. When the 76 items were coded, we coded by the highest cognitive complexity noted in any part of the item. Still, the results were similar with about two-thirds (51) coded as low cognitive complexity, and one-fourth (19) coded as high complexity, followed by very high (5) then very low (1) cognitive complexity.
Table 6
Item Type Distribution

<table>
<thead>
<tr>
<th>Item Type</th>
<th>This Study (n = 76 items)</th>
<th>Ellis et al.’s Selected Students (n = ~ 590)</th>
<th>Ellis et al.’s Non-Selected Students (n = ~ 1410)</th>
<th>Burn &amp; Mesa’s Instructors (n ranged from 345 to 355)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation/Calculation</td>
<td>57%</td>
<td>40%</td>
<td>51%</td>
<td>50%</td>
</tr>
<tr>
<td>Graphical/Geometrical Interpretation</td>
<td>30%*</td>
<td>33%</td>
<td>21%</td>
<td>20%</td>
</tr>
<tr>
<td>Contextual Situation (collapsed the two categories)</td>
<td>13%</td>
<td>55%</td>
<td>39%</td>
<td>30%</td>
</tr>
<tr>
<td>Reasoning, Proof, Justification</td>
<td>0%</td>
<td>14%</td>
<td>9%</td>
<td>10%</td>
</tr>
</tbody>
</table>

Note

The * symbol represents that students were asked to provide explanations justifying their answers to earlier parts of an item for two of the 23 items deemed to be graphical/geometrical interpretation items. However, the emphasis was not on the justification but rather on the graphical/geometrical interpretation of the item.

In Table 7 below we compare our results with those of White and Mesa's (2014) study of the tasks assigned by five, two-year college, Calculus I instructors. Multivariable students may not have enough experience working challenging vector tasks, as compared to Calculus I students. They may also not have much exposure to higher cognitive complexity tasks prior to exams. This suggests the need to develop more WeBWorK vector items that are of very high cognitive complexity.

Key Vector Ideas

The items addressed all key ideas for vectors that were identified in the coding, as displayed in Table 1. This seems to suggest that the key vector idea list is comprehensive. However, some key ideas were much more prominent in the items than others as shown by the differing bar heights in Fig. 3. For example, vector magnitude stood apart as the most prominent key idea. It was addressed in 30 of the items, while cross product, subtraction, projection, and orientation only appear in 12, 6, 3, and 2 items respectively.

Table 7
Cognitive Complexity Comparison to Previous Research

<table>
<thead>
<tr>
<th>Cognitive Complexity</th>
<th>This Study (n = 76 items)</th>
<th>White &amp; Mesa’s Assignment Tasks*</th>
<th>White &amp; Mesa’s Worksheet Tasks</th>
<th>White &amp; Mesa’s Exam Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple Procedure (Very Low and Low categories)</td>
<td>68%</td>
<td>54%</td>
<td>54%</td>
<td>40%</td>
</tr>
<tr>
<td>Complex Procedure (High category)</td>
<td>25%</td>
<td>21%</td>
<td>10%</td>
<td>11%</td>
</tr>
<tr>
<td>Rich Task (Very High category)</td>
<td>7%</td>
<td>25%</td>
<td>37%</td>
<td>49%</td>
</tr>
</tbody>
</table>

Note

The * symbol refers to tasks included WeBWorK as well as textbook tasks, which had similar percentage breakdowns when separated.

Computation/calculation was prevalent in most of the key vector ideas as displayed in Fig. 3. Only vector angle, projection and orientation had three or fewer computation/calculation items. Graphical/geometric interpretation was found in most of the key vector ideas. Only special vectors, subtraction, projection, and orientation had three or fewer graphical/geometric interpretation items. Addition, vector magnitude, and scalar multiplication had several items with contextual situations. However, the key vector ideas of cross product, subtraction, and orientation had zero items with contextual situations, and only one projection item had contextual situations. We now consider the vector key ideas (subtraction, projection, cross product, and orientation) with either no graphical geometric interpretation or no contextual situations items in light of previous research.
Shaffer and McDermott (2005) reported that postsecondary students in physics classes were better able to subtract vectors when there was no physical context than when one was present. Vector subtraction is related to vector addition, which occurred in 18 items. However, while 8 of the 18 items involving addition were contextual situations, there were no contextual situation items that involved subtraction. Additionally, 4 of the 18 addition items involved a graphical or geometric interpretation while only 2 of the subtraction items involved a graphical or geometric interpretation. Paired with research studies by Heckler and Scaife (2015) and Carleschi (2016) that recommend students practice graphical problems in a variety of vector orientations, this points to the need of additional WeBWorK vector tasks involving subtraction in both contextual and graphical/geometric situations.

There is sparse research on student understanding of vector projection; yet Zavala and Barniol (2013) found that students perform poorly in both contextual and non-contextual problems. The three projection items (i.e., items 15, 47, 96) in our sample are frequently assigned (global usage: 812, 8039, 675 compared to the median 231), have high numbers of attempts (3.49, 9.44, 3.39 compared to the median 2.65), and lower success rates (status: 95, 69, 88 compared to the median 97).

Research on student understanding of the cross product indicates that students have difficulty with orientation (i.e., applying the right-hand rule) and often fail to recognize that the cross product is a noncommutative operation (Deprez et al., 2019; Kustusch, 2016; Scaife & Heckler, 2010; VanDieren et al., 2017; Zavala & Barniol, 2010). Additionally, students not only struggle with cross product and orientation in graphical problems but also in physical contexts (Deprez et al., 2019). Furthermore, Deprez and colleagues report that the context can help students understand the geometric aspects of the cross product. With no items in our sample addressing cross product or orientation in a contextual situation, and with only one item addressing both cross product and orientation, there is a need to develop additional WeBWorK vector tasks to better address these student difficulties.

Of the 76 items, 25 involved only a single key vector idea; the remaining items involved two or more key ideas. Of the 25 items that involved only a single key idea, 15 of these were assigned a vector code, and the majority of these asked a student to find the components for a vector connecting two given points. For items coded with more than one key idea, the key ideas may have been connected, or they may have been addressed by themselves in different parts of an item. Table 8 displays that total number of connections and the number of key ideas to which each idea was connected. Since the vector (V) code was used for simple identification of vectors in their component forms without emphasis on any of the critical features of vectors or vector operations, no item part was coded with both V and any other key vector idea code, and no items were coded connecting vector with any other key ideas. Therefore, the vector (V) code does not appear in Table 8.

Although a maximum of 10 connections to other key vector ideas were possible, the maximum number of connections for any key idea was 7. The top three key vector ideas in terms of connections included: scalar multiplication (34 total connections to 7 other key ideas), dot product (23 total connections to 7 other key ideas), and magnitude (24 total connections to 6 other key ideas). Note that scalar multiplication, special vectors, direction, projection, and orientation never appeared by themselves in an item. However, there were only two connections that involved orientation, one to dot product and one to cross product.

We now investigate the key ideas in terms of critical features of vectors and vector operations. There were only eight instances where critical features of vectors were connected to different critical features. This happened in the following four ways: magnitude-special vectors (3 instances), magnitude-direction (2 instances), special vectors-direction (2 instances), and angle-direction (1 instance). Variation theory points to the importance of not only presenting students opportunities to focus on critical features in isolation, but it also emphasizes that “understanding the object of learning implies understanding the object as a whole and thus involves a simultaneous discernment of the defining aspects and their relationship” (Kullberg et al., 2017, p. 560). The lack of connections between different critical features may not provide students with as many opportunities to distinguish and make comparisons that would allow them to develop a robust understanding of vectors. For example, one would expect some connections between vector orientation, angle, and direction.
Table 8
Connections between Key Vector Ideas

<table>
<thead>
<tr>
<th>Key Vector Ideas</th>
<th>Total Connections</th>
<th>Connected to ___ Other Key Ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critical Features of Vectors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Magnitude (VM)</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
<td>Angle (VA)</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>Special Vectors (VS)</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>Direction (VD)</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>Orientation (VO)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Vector Operations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scalar Multiplication (S)</td>
<td>34</td>
<td>7</td>
</tr>
<tr>
<td>Dot Product (D)</td>
<td>23</td>
<td>7</td>
</tr>
<tr>
<td>Addition (A)</td>
<td>17</td>
<td>3</td>
</tr>
<tr>
<td>Cross Product (X)</td>
<td>11</td>
<td>5</td>
</tr>
<tr>
<td>Projection (P)</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>Subtraction (B)</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

There were 21 instances where vector operations were connected to different vector operations, as displayed in the chord diagram in Fig. 4. While there was some diversity of connections between vector operations, there are certain vector operations that are not well connected. The number and diversity of connections with scalar multiplication is not surprising since addition, subtraction, dot product, projection, and cross product can each be defined in terms of scalar multiplication. In fact, scalar multiplication was connected to all other vector operations except cross product. On the other hand, apart from scalar multiplication and dot product, none of the vector operations are well connected to other operations even though there are many possible connections. For example, there are several properties related to both dot product and cross product that involve scalar multiplication and vector addition (e.g., linearity). Vector subtraction can be interpreted as a combination of vector addition and scalar multiplication. Additionally, a vector can be presented as the sum of its orthogonal component and its projection onto another vector. Therefore, there are many possibilities for additional items to be added to the WeBWorK OPL that address such connections.

There were 52 instances where critical features of vectors were connected to vector operations, as displayed in the Sankey diagram in Fig. 5. Of note, there are only five (of six possible) vector operations displayed on the right side of Fig. 5 since there were no connections between any of the critical features of vectors and vector subtraction. As might be expected, the critical feature special vectors, which can be defined in terms of scalar multiplication and vector addition, was connected to both. Also, it is no surprise that vector orientation had few connections since it was in few items. The remaining three critical features of vectors were each connected to at least three vector operations; however, there were still obvious omissions (e.g., connecting vector magnitude and projection). Despite that omission, vector magnitude was much more emphasized in the WeBWorK OPL than vector direction. Considering student confusion of a vector with its scalar (Harel, 2000), it is important for instructors to not over emphasize the length of a vector while ignoring its other component, vector direction, when assigning OPL items to students.

Vector Representations

We now consider which representations of vectors are used in the items. We first consider the representations used in the item prompts and then what was expected in the items’ responses. Note that due to the search criteria for the data set, it was not surprising that all the items’ prompts involved vectors, and 50 of the 76 items involved vectors in their responses.

Of the 76 item prompts, 58 used a single representation for vectors; 18 used 2 representations; and none had more than 2 vector representations. “Although more representations do not necessarily lead to greater understanding, numerous cognitive advantages associated with the establishment of links among various ways of representing a problem have been proposed” (Stein et al., 1996, p. 472). The most common vector representations in item prompts with a single vector representation were basis vectors and endpoints, as displayed in Table 9. However, for items with two vector representations in their prompts, the majority involved a verbal description in the prompt. When
looking across all item prompts, vectors were represented using endpoints in 23 items, as basis vectors in 21 items, using verbal descriptions in 19 items, and with n-tuples in 18 items. Table 9 shows the distribution of the item prompts.

Of the verbal descriptions, 9 were prompts describing a geometric situation; 7 described a physics concept (e.g., force, velocity, work); and 3 were contextual word problems (e.g., homework grades, production output of a factory). The vector representations that occurred least frequently in the prompts were column and graphed vectors. The lack of graphed representations is likely due to the increased difficulty of programming graphs in WeBWorK items compared to using the built-in Vector Class of Math Objects in WeBWorK which easily handles basis, n-tuple, and column vector representations (see https://webwork.maa.org/wiki/Vector_(MathObject_Class)). The lack of column vectors may be due to the fact that we considered only multivariable calculus items for this study, the column vector representation is used more widely in linear algebra and linear programming.

Table 9
Mathematical Object in Item Responses by Vector Representation Used for Item Prompt (n = 76)

<table>
<thead>
<tr>
<th>Vector Representation(s) in Item Prompt</th>
<th>Mathematical Object(s) in Item Response</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Vectors only</td>
</tr>
<tr>
<td>Single Vector Representation Used</td>
<td></td>
</tr>
<tr>
<td>Graphed (n = 5)</td>
<td>2</td>
</tr>
<tr>
<td>Basis (n = 15)</td>
<td>9</td>
</tr>
<tr>
<td>Column (n = 5)</td>
<td>2</td>
</tr>
<tr>
<td>N-tuple (n = 11)</td>
<td>4</td>
</tr>
<tr>
<td>Verbal description (n = 7)</td>
<td>2</td>
</tr>
<tr>
<td>Endpoints (n = 15)</td>
<td>9</td>
</tr>
<tr>
<td>Two Vector Representations Used</td>
<td></td>
</tr>
<tr>
<td>Basis &amp; verbal description (n = 6)</td>
<td>3</td>
</tr>
<tr>
<td>N-tuple &amp; endpoints (n = 5)</td>
<td>0</td>
</tr>
<tr>
<td>Verbal description &amp; endpoints (n = 3)</td>
<td>1</td>
</tr>
<tr>
<td>Column &amp; verbal description (n = 2)</td>
<td>0</td>
</tr>
<tr>
<td>Graphed &amp; n-tuple (n = 1)</td>
<td>1</td>
</tr>
<tr>
<td>N-tuple &amp; verbal description (n = 1)</td>
<td>1</td>
</tr>
<tr>
<td>Totals for all items</td>
<td>34</td>
</tr>
</tbody>
</table>

Note: The Other column included words (both explanations and short answers), equations, and single points (all cases of points involved endpoints of a vector)

Table 10 displays the distribution of the 50 items for which students were expected to respond with a vector in a particular representation. Of the 50 items, 28 used the same vector representation in the item response as was used in the item prompts. The most common “switch” in vector representations was from endpoints in the prompt to an n-tuple in the response of an item, which occurred in 10 items. Of note, students were never prompted to submit a vector in graphical representation. This may be because graphical inputs are a relatively new feature of WeBWorK, which requires more complex coding than
Table 10
Vector Representation Responses by Vector Representation Used for Item Prompt (n = 50)

<table>
<thead>
<tr>
<th>Vector Representation(s) in Item Prompt</th>
<th>Vector Representation(s) in Item Response</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Basis (n = 14)</td>
</tr>
<tr>
<td>Single Vector Representation Used</td>
<td></td>
</tr>
<tr>
<td>Basis (n = 11)</td>
<td>6</td>
</tr>
<tr>
<td>Endpoints (n = 11)</td>
<td>1</td>
</tr>
<tr>
<td>N-tuple (n = 6)</td>
<td>0</td>
</tr>
<tr>
<td>Graphed (n = 3)</td>
<td>0</td>
</tr>
<tr>
<td>Verbal description (n = 3)</td>
<td>2</td>
</tr>
<tr>
<td>Column (n = 2)</td>
<td>0</td>
</tr>
<tr>
<td>Two Vector Representations Used</td>
<td></td>
</tr>
<tr>
<td>N-tuple &amp; endpoints (n = 5)</td>
<td>0</td>
</tr>
<tr>
<td>Basis &amp; verbal description (n = 4)</td>
<td>4</td>
</tr>
<tr>
<td>Column &amp; verbal description (n = 2)</td>
<td>0</td>
</tr>
<tr>
<td>Verbal description &amp; endpts (n = 1)</td>
<td>1</td>
</tr>
<tr>
<td>Graphed &amp; n-tuple (n = 1)</td>
<td>0</td>
</tr>
<tr>
<td>N-tuple &amp; verbal description (n = 1)</td>
<td>0</td>
</tr>
</tbody>
</table>

Note

Shading indicates the same vector representation for the item prompt and response.

inputting endpoints, a column vector, an n-tuple vector, or a basis vector as a response. In light of their findings, Heckler and Scaife (2015) recommend the need to have students practice with a variety of vector orientations. They emphasize the importance of graphical representation for sense making as well as the need for using both the basis and graphed formats of vectors.

Understanding any mathematical topic, including vectors, requires an awareness of the ways the topic is represented (Duval 1999) and some flexibility in how individuals are able to move between those representations (Ainsworth, 2006; McGee & Moore-Russo, 2015). “Although more representations do not necessarily lead to greater understanding, numerous cognitive advantages associated with the establishment of links among various ways of representing a problem have been proposed” (Stein et al., 1996, p. 472), it seems important that there are sufficient WeBWorK items that require students to work with multiple representations concurrently and to switch between vector representations.

Usage Statistics by Item Characteristic

We created box and whisker plots for global usage, attempts, and status of items by the following item classifications: item types, cognitive complexity, number of key vector ideas, and vector representation. Since no patterns emerged for global usage, we only report on the findings of attempts and status by characteristic displayed in Fig. 6.

In terms of item types, contextual situation items tended to require a greater number of attempts than other items. This was followed by graphical/geometrical interpretation items then computation/calculation items. The status, which is an indicator of success on the item, was lowest for contextual situation items followed by graphical/geometrical interpretation items then computation/calculation items. This suggests that contextual situation items were the most difficult items, and computation/calculation items were the easiest. The difficulty of the contextual situation items might have been due to 7 of those 10 items requiring physics knowledge, as well as vector knowledge.

Since there was only one item that was coded as having very low cognitive complexity, a memorization item related to the general vector coding category, Fig. 6 displays only the low, high, and very high coding categories, with that one very low item included with the low category. As expected, the higher cognitive complexity items had more attempts and lower status.
As one might expect, the items with more key vector ideas (including both critical features of vectors and vector operations) had more attempts and lower status indicating these were more difficult for students. However, this was not as pronounced as was the case for item cognitive complexity.

The items with only verbal descriptions required more attempts and had lower status than items with other vector representations, including those having multiple representations. This suggests verbal descriptions are more difficult for students.

**Item Cognitive Complexity by Key Vector Ideas**

To address the third research question, we now consider if certain key vector ideas appeared more frequently in items with higher cognitive complexity. Since there was only one item that was coded as having very low cognitive complexity, a memorization item related to the general vector coding category, Table 11 displays only the counts for the low, high, and very high coding categories, with that one very low item included with the low category. For the general vector category, due to the coding definition, it was not surprising that most of the items were deemed as being of low cognitive demand (1 very low, 17 low). Recall that of the 76 items, 25 involved only a single key vector idea, and the remaining items involved two or more key ideas. Therefore, there are more than 76 entries in Table 11.

![Table 11](image)

*Note: The * signifies that the vector key idea was coded as low cognitive complexity in 17 items and very low in 1 item.*

As is noted in the ternary plot in Fig. 7, one thing stands out. None of the key vector ideas were found in more than 20% of the items, and only one key idea was in more than 15% of the items categorized as very high. White and Mesa's (2014, p. 686) analysis of the distribution of levels of cognitive complexity tasks assigned by instructors indicates that the very high cognitive complexity was noted in much greater percentages on the tasks assigned by instructors, ranging from 15–31% for bookwork, from 23–42% for worksheets, and from 26–68% for exams. This indicates a potential gap between the supply of very high items in the WeBWorK sample problem bank and the demand from instructors for all key vector ideas.

We first consider items that address critical features of vectors (i.e., all those in black except the asterisk) as graphed in Fig. 7. Except for vector orientation, there were items in our sample across all three cognitive complexity levels for the other four critical features (i.e., magnitude, angle, direction, and special vectors). However, even for these four, there were consistently more items that were of either low or high cognitive demand than there were items of very high cognitive complexity. This trend was even more pronounced when looking at the items that involved vector operations (i.e., scalar multiplication, dot product, addition, cross product, subtraction, and projection), which are
denoted in Fig. 7 with grey symbols. The vector operations were noted predominantly in items that were of low cognitive complexity. There were fewer vector operations in items with high cognitive complexity, and, apart from the dot product, almost no vector operations in items of very high cognitive complexity. As might be expected, the key ideas addressed by fewer items (i.e., vector orientation, subtraction, and projection), tended to have the least spread across complexity levels. This points to a need for more items to be developed at all levels, especially at the very high cognitive complexity level.

Orientation is a critical feature of vectors that challenges students. With only two items addressing vector orientation, both of high cognitive complexity, there are no options in this sample of WeBWorK items for instructors to assign items with low or very high cognitive complexity. Despite research pointing to student difficulty in understanding vector orientation and the right-hand rule (Kustusch, 2016; Scaife & Heckler, 2010; VanDieren et al., 2017; Zavala & Barmio, 2010), there are limited opportunities within WeBWorK for students to become “acquainted” with the critical feature of orientation before going onto more complex tasks of “contrasting,” “generalizing,” and “fusing” this feature with others (Kulberg et al., 2017, p. 560). Therefore, not only is there a need for more WeBWorK items addressing vector orientation, but new items should include those of both lower and higher cognitive complexity so that students are better equipped to discern and understand this critical feature. One explanation for the lack of items concerning vector orientation is that this critical feature relies heavily on graphical representations, and there are barriers (although not insurmountable) within the WeBWorK coding environment to create exercises with graphical prompts and/or responses.

**Vector Representations by Key Vector Ideas**

To address the fourth research question, we now consider if key vector ideas were addressed in items with a variety of vector representations. Figure 8 below displays that there was some variety of vector representations used in item prompts for all key vector ideas. It appears in Fig. 8 that item prompts for vector items are represented as only graphed, endpoints, or multiple representations. However, for the 7 items whose prompts involved multiple representations, 5 were a combination of n-tuple and endpoints while 2 were column vector and verbal description combinations. Note that there are no instances of prompts for items coded with both the general vector coding category and basis vector representation, since this combination would be given a special vectors assignment as defined in the coding.

As displayed in Table 9 earlier, there are few item prompts (n = 5 as a single representation, n = 1 as part of a multiple representation) in the sample that used a graphed vector representation. Moreover, we note from Fig. 8 that some key ideas have no prompts with graphed representations. None of the prompts for items coded as addressing vector direction, orientation, projection, or cross product involved graphed vector representations (either as a single representation or paired with another vector representation). Each of these key vector ideas have graphical aspects that are not being addressed in the sample of WeBWorK items in this study.

The other vector representation that occurred with less frequency in item prompts is column vector. Column vectors were used as a single representation in prompts for items involving scalar multiplication, addition, and subtraction only. Column vectors were paired with other vector representations in item prompts coded with the general vector coding category but not in items involving the other eight key vector ideas.

Finally, we noted earlier that there are differences in the number of items that address vector addition and subtraction and in the cognitive complexity of items addressing these key ideas. Furthermore, addition also had a wider variety of vector representations and more occurrences of multiple representations used in their prompts compared to subtraction.

**Summary, Conclusions, Implications**

Variation theory emphasizes designing learning activities “in a way that allows students to experience the desired variation pattern and thus discern the critical features of the object of learning” (Lo, 2012, p. 205). To do this, one must identify the critical features of the topic and their relationships (Lo, 2012; Lo & Marton, 2012), and in mathematics these relationships can be experienced through multiple representations, contexts, and simultaneous connections (Leung, 2012). This study identifies the critical features of vectors along with the ways that these critical features relate to one another and to vector operations in WeBWorK OPL items.

Since the topic of vectors spans many subtopics, it was crucial to build on previous research that classified key vector ideas as either critical features of vectors or vector operations (Moore-Russo et al., 2017; VanDieren et al., 2018; VanDieren & Moore-Russo, 2019a). For this study, we built on that existing research base and systematically adapted and employed previous coding schemes from others’ research. Our conclusion is that the list of key vector ideas appears to be comprehensive since they covered all items in this study.

Based on our findings, there appears to be a demand for vector-related items available in the WeBWorK OPL. The OPL contained several vector-related items in the study that involved both two- and three-dimensional vectors. Yet even though there were 76 distinct vector-related
items available at the time of this study, there is the need for additional items that meet certain criteria so that instructors are able to assign students a greater variety of items. In particular, there is a need for additional vector-related items in the OPL that:

- involve graphical/geometrical interpretation; contextual situations (especially situations that are outside of physics); and reasoning, proof, and justification
- are of high or very high cognitive complexity
- address the following key vector ideas in a way that allows each to have items of all levels of cognitive complexity and using items that involve multiple representations:
  
  - vector angle: computation/calculation items are needed
  - cross product: items that involve graphical/geometrical interpretation and contextual situations are needed
  - special vectors: items that involve graphical/geometrical interpretation are needed
  - orientation: items of all types are needed
  - subtraction: items that involve graphical/geometrical interpretation and contextual situations are needed
  - projection: items of all types are needed
- have connections
  
  - among all critical features (including direction, magnitude, special vectors, angle, and orientation)
  - among most vector operations (including addition, subtraction, projection, and cross product)
  - between critical features and vector operations
- address specific key vector ideas in isolation, without any connections, for the following
  
  - some critical features (including direction, special vectors, and orientation)
  - most vector operations (including scalar multiplication and projection)
- involve a variety of vector representations including items that have
  
  - prompts with graphed and column vectors
  - responses with graphed vectors, column vectors, verbal descriptions, and endpoints of vectors
  - different vector representations in the response compared to the prompt (especially involving switches between graphed vectors, column vectors, verbal descriptions, and endpoints of vectors)

**Limitations And Future Work**

We only pulled items from the curated WeBWorK OPL for our sample for this study. However, instructors may use WeBWorK items from other sources, such as the assortment of contributed problems on github as well as their own private files. Since the time of collection, more items have been added to the WeBWorK OPL, and many more items appear in contributed libraries. At the date of this submission, 474 entries were found using the keyword search vector. Furthermore, WeBWorK graphing functionality for item prompts and responses has improved. Version 2.16 of WeBWorK includes TikZ capabilities (Bonanome et al., 2021) as well as CalcPlot3D and GeoGebra embeddings (VanDieren, 2021b). These features allow for easier implementation of graphical prompts and responses.

Given the popularity of WeBWorK, other studies of the OPL items are encouraged. While this study involved OPL vector-related items, the methodology for the study could be applied to other topic areas by considering the number and usage statistics available through WeBWorK (Pizer, 2019), as well as the critical features of the topic, representations used in the items and item characteristics, such as item type and item cognitive complexity.
Declarations

Additional Information:

Data Availability: The following statement is on page 8, which explains that the data used for the study are available to all. The data set of the 106 tasks that remained after the initial pass along with additional data generated and analyzed in this study is available in the github repository (see https://github.com/vandieren/webwork-vector-analysis).

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The following statement is on page 36, just before the reference section. The authors do not have any financial, non-financial interests, or proprietary interests that are directly or indirectly related to the work or any materials discussed in this article.

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Figures
Figure 1

Box and Whisker Plots for Item Global Usage, Item Attempts, and Item Status
Figure 2

Box and Whisker Plots for Item Global Usage, Item Attempts, and Item Status (Multipart vs. Single-Part Items)

*Note.* Outliers are not included in the graph, and the scales for *global usage* and *attempts* were capped at 1500 and 7 respectively to better illustrate the differences in multipart and single-part items.
Figure 3

Stacked Bar Chart of Frequencies of Key Vector Ideas by Item Types

Note: Since items could address more than one key idea, there are more than 76 codes assigned for items.
Figure 4

*Chord Diagram of Connections between Vector Operations*
Figure 5

Sankey Diagram of Connections between Critical Features of Vectors (on Left) and Vector Operations (on Right)
Figure 6

Box and Whisker Plots of Usage Statistics by Item Characteristic

Note: Each plot on the left is displayed with coding categories listed in descending order of their medians.

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Figure 7

Ternary Plot of Key Vector Ideas by Cognitive Complexity Level Distribution

Note: The ternary plot displays the proportion of the three cognitive complexity levels that sum to 100% for each set of items involving a key vector idea with the vector operations denoted in grey. The vector addition symbol (the open circle, the ninth symbol on the key to the right, that is located towards the center of the bottom of the plot) represents 60% of the items involving vector addition being of high cognitive complexity, 40% low cognitive complexity, and 0% high cognitive complexity. The percentage amounts for the high cognitive complexity are marked on the bottom of the plot and correspond to grid lines parallel to the right side of the triangular plot. Similarly, the left side of the plot marks the percentage amounts for very high and the right side of the plot marks the percentage amounts for the low. The grid lines for the very high percentages are parallel to the bottom of the triangle while the grid lines for the low percentages are parallel to the left side of the triangular plot.)
Figure 8

Stacked Bar Chart for Vector Representation in Item Prompt by Key Vector Ideas

Note: Since items could address more than one key idea, there are more than 76 codes assigned for items.