Sales-Dominance Relations And Disclosure Of Product Information

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SALES-DOMINANCE RELATIONS AND DISCLOSURE OF PRODUCT INFORMATION

Hee Yeul Woo*

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Abstract

In a framework allowing for both vertical and horizontal product differentiation, the seller persuades the consumer of unknown taste by sending verifiable product information. The standard unraveling argument does not extend well under horizontal differentiation mainly because it is indefinite how to rank seller types. By defining price-dependent sales-dominance relations over seller types, we provide a generalized unraveling argument that can be applied no matter how products are differentiated. We show that if a sales-dominant seller type exists at a price over some seller types, an outcome at which all and only these seller types pool at this price does not emerge regardless of the prior. We argue that sales-dominance relation is complete at every price if and only if full disclosure is the unique outcome regardless of the prior. We find the condition under which a worst-case type is well-defined regardless of the prior for every observable deviation of the seller. Finally, we explore product information the consumer could receive when sales-dominance relations are incomplete.

Keywords: persuasion games, verifiable information, horizontal differentiation, full disclosure, sales-dominance relations

1 Introduction

In many markets, consumers are in need of product information the seller privately knows, and the seller has a variety of ways of disclosing this information in a verifiable manner: for instance, free samples, informative advertisements, third-party test results, and technical reports by independent laboratories. Laws against fraud may force the seller to provide truthful information on product characteristics. Lots of literature has dealt with the disclosure of verifiable information under vertical differentiation, where consumers agree on the ranking of valuations for products as if products are differentiated only in terms of quality. However, consumers may differ in their tastes. For instance, some prefer Windows, some prefer Mac OS, and others prefer Linux as an operating system.

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In the context of vertical differentiation, the standard unraveling argument runs as follows. Firstly, the seller with the highest quality exists for every set of seller types. Secondly, this seller with the highest quality would not pool with other seller types in this set because full disclosure would provide him a chance to raise the price (while not decreasing the demand) or increase the demand (while not lowering the price). Since the seller with the highest quality would deviate for every hypothetical pooling outcome identified with a non-singleton set of seller types, full disclosure should be the unique outcome regardless of the prior.\textsuperscript{1} The standard unraveling argument does not extend well under horizontal differentiation mainly because it is not clear how to rank products. Moreover, “worst-case types” are not necessarily well-defined, while they are conventionally used to characterize the consumer’s belief off the path of a full-disclosure equilibrium. Sun [2011] and Celik [2014] report that full disclosure may fail under horizontal differentiation. However, it may be misleading if one believes that a non-disclosure outcome arises whenever products are horizontally differentiated. Koessler and Renault [2012] (henceforth, KR) provide pairwise monotonicity, which accommodates the horizontal tastes of consumers, as a necessary and sufficient condition for full disclosure to be the unique outcome regardless of the prior.

We study the disclosure of verifiable information in a framework allowing for both vertical and horizontal differentiation and aim to provide a generalized unraveling argument that can be applied no matter how products are differentiated. How can we identify the seller with an incentive to deviate from a hypothetical pooling outcome as the seller with the highest quality does under vertical differentiation? Under what condition does a “deviator” exist for every hypothetical pooling outcome? How is pairwise monotonicity related to this condition? Under what condition is a worst-case type well-defined for every observable deviation of the seller? To answer, we would define price-dependent dominance relations over seller types.

We consider a setting similar to KR for direct usage and comparison to their results. The seller sets a price and provides verifiable product information to persuade the consumer whose preference is privately known. Given the type-dependent preference of the consumer, we define, for each price, a dominance relation in terms of “sales”. For a concrete illustration, consider an arbitrary price $p$ and two differentiated products $X$ and $Y$. Sales for $X$ at $p$ refer to a set of consumer types who strictly prefer to purchase $X$ at $p$. Analogously, sales for $Y$ at $p$ are defined. We say that $X$ sales-dominates $Y$ at $p$ if the consumer, regardless of her type, strictly prefers to purchase $X$ at $p$ whenever she does for $Y$.\textsuperscript{2} Under vertical differentiation, sales-dominance relation is complete at every price since a product sales-dominates another product with lower quality independently of prices.\textsuperscript{3} Not surprisingly, if products are vertically differentiated, sales-dominance relation can be incomplete at some prices. We show that the consumer’s preference is pairwise monotone if and only if sales-dominance relation is complete at every price.

To justify sales-dominance relations as appropriate rankings over seller types, we need to show that if a product (say, $X$) is dominant in sales at price $p$, the seller of $X$ should not pool


\textsuperscript{2}$X$ sales-dominates $Y$ at $p$ if sales for $Y$ at $p$ are a subset of sales for $X$ at $p$.

\textsuperscript{3}We say that sales-dominance relation is complete at a price if for every two seller types, one seller type sales-dominates the other at this price.
at $p$ with other sales-dominated products (say, $Y$). Indeed, we prove that the consumer must value the sales-dominant product $X$ strictly higher than $p$ whenever she is willing to purchase at $p$ with whatever full-support belief on $X$ and $Y$. Therefore, sales for $X$ never reduce below the sales at this pooling outcome, even if the seller of $X$ fully discloses and raises the price marginally. We show that under complete sales-dominance relations, full disclosure uniquely obtains regardless of the prior because at least one seller type is dominant in sales for every hypothetical pooling outcome identified with a pair of a price and a set of seller types. Under pairwise monotonicity, the generalized unraveling argument applies subject to prices. Given any price, the most sales-dominant seller type would not pool with other seller types because his payoff is sure to improve through full disclosure. Analogously, the seller of the second highest sales would distinguish himself from other sales-dominated. The inductive reasoning unravels down to the seller type with the lowest sales.

Alternatively, we define price-dependent dominance relations in terms of perfect-information sales. At a price, perfect-information sales to a seller type refer to a set of all consumer types whose willingness to pay for this seller type is at least as high as the price.\footnote{The consumer is assumed to purchase whenever indifferent between purchasing and not. Hence, for every seller type, sales at a price are a subset of perfect-information sales at the same price.} We show that sales-dominance relations and perfect-information sales-dominance relations are similar in nature. Namely, if one dominance relation is complete at every price, so is the other. This simple result comes in with two implications. Firstly, we explore the condition under which a worst-case type is well-defined for every off-the-equilibrium path and show that under pairwise monotonicity, the seller type with the lowest perfect-information sales is a worst-case type regardless of the prior for every observable deviation of the seller. Secondly, it provides an alternative notion of the dominant seller types. Under pairwise monotonicity, an alternative unraveling argument runs as follows. For every price and set of seller types, some seller type is perfect-information sales-dominant. Among these perfect-information sales-dominant, at least one type has no incentive to pool with other seller types in this set because he can enlarge the sales while not lowering the price or he is sales-dominant.

According to KR, if the consumer’s preference is not pairwise monotone, there exists an independent prior that induces a non-disclosure outcome. However, what product information could be provided if the preference is not pairwise monotone is still an open question. Under a mild condition, we show that if no sales-dominant type exists at a price over some seller types, an independent prior exists that induces a non-disclosure outcome at which all and only these seller types pool and each pooling seller achieves an equilibrium payoff strictly higher than the full-disclosure payoffs.

Section 2 formalizes the model. In section 3, we define price-dependent sales-dominance relations over seller types in terms of sales and perfect-information sales and provide the generalized unraveling argument. Moreover, we show that a worst-case type should be well-defined for every observable deviation if sales-dominance relations are complete. In section 4, we explore the equilibria under incomplete sales-dominance relations. In section 5, we conclude.

\textbf{Related literature:} Recently, several papers investigated the disclosure of private product information allowing for horizontal differentiation. Sun [2011] analyze the disclosure of multiple product attributes. Products differs in vertical quality and a horizontal attribute. She shows that when the product’s quality is common knowledge, the monopolist chooses to disclose
the horizontal attribute only if he is located in a central region of the Hotelling product line. Therefore, at a non-disclosure equilibrium, the consumer can not infer at which extreme the monopolist is located. Using a spatial model in which the product is characterized by a horizontal attribute, Celik [2014] shows that a fully revealing equilibrium exists if and only if buyers’ preferences for ideal tastes are strong. Otherwise, the seller partially reveal the horizontal attribute, and the consumer learns how far the horizontal attribute is far from her expected ideal taste, but not necessarily on which side. All these papers are set under an independent prior distribution and a specific form of utility function is assumed for the consumer. Anderson and Renault [2006] analyzes the seller’s choice of advertising content in a setting where the buyer determines whether to learn precisely the price and product type at some positive search cost. They show that the monopolist only informs the buyer whether her utility is above a threshold. Janssen and Teteryatnikova [2016] analyze the disclosure of horizontal product attributes in a competitive environment. Only in the case of comparative and price advertising, fully disclosure is the unique outcome. Otherwise, if advertising is non-comparative or sellers set prices after they observe information disclosed, a non-disclosure outcome may arise. Among them, a work by KR is the most closely related. They show that pairwise monotonicity is necessary and sufficient for full disclosure to be the unique outcome regardless of the prior. We characterize pairwise monotonicity as the completeness of sales-dominance relations at all prices. We deepen the intuition about why full disclosure should be the unique outcome regardless of the prior if the consumer’s preference is pairwise monotone by identifying the seller type with an incentive to deviate from a hypothetical pooling outcome. According to KR, if sales-dominance relations are incomplete, a non-disclosure outcome may emerge. We extend this result and explore the product information the seller could provide at a non-disclosure outcome.

2 Model

A male seller persuades a female consumer to purchase a single unit of his product. The seller has complete and private information about the characteristics of his product, while the consumer is completely and privately informed about her preference for differentiated products. Let $G$ denote a finite set of seller types and $T$ a finite set of consumer types. A seller type summarizes product characteristics and information he has about consumer types. Likewise, a consumer type describes her preference and information she has about seller types. A full-support prior probability distribution over the type profiles is denoted by $\mu \in \Delta(G \times T)$. The players’ types are allowed to be correlated. Hence, different types of one player may form different beliefs about the other player’s types.

The game proceeds through three stages. At the information stage, nature chooses the type of each player according to the prior, and the players learn their respective types only. At the persuasion stage, the seller sets a binding price and informs the consumer of his product through a costless but possibly vague message. At the purchasing stage, the consumer decides whether to purchase given the price and information provided by the seller. After the purchasing stage, both players receive payoffs.

Let $P$ denote a set of prices. We assume that $P$ is finite but fine enough.\footnote{If $P$ is infinite (for example, $P$ is a set of non-negative real numbers), we cannot define completely mixed strategies of the seller as in Kreps and Wilson [1982]. We can avoid this problem by assuming that $P$ is finite but fine enough. One can easily find that if $P$ is not fine enough, a non-disclosure outcome may appear even if} Let $M =$
\[ \cup_{g \in \mathcal{G}} M(g), \text{ where } M(g) = \{ G' \subseteq G : g \in G' \} \text{ for every } g \in G. \] A pure strategy \( \sigma_s \) of the seller is a mapping \( \sigma_s : G \rightarrow P \times M \) such that \( \sigma_s(g) \in P \times M(g) \) for every \( g \in G \). Namely, seller type \( g \) selects a price in \( P \) and provides only a verifiable message though it is not necessarily totally detailed.\(^6\) A pure strategy of the consumer is a mapping \( \sigma_c : T \times P \times M \rightarrow \{1, 0\} \), where her choice 1 means purchasing, while 0 non-purchasing. A belief function of the consumer is given by a mapping \( \beta : T \times P \times M \rightarrow \Delta(G) \).

A money metric utility function is given by \( u : G \times T \rightarrow \mathbb{R}_{++}, \) where \( \mathbb{R}_{++} \) is a set of positive real numbers. Attention is restricted only to the consumer of generic preference. Formally, the consumer’s preference is generic if \( u(g, t) \neq u(g', t) \) for every \( t \in T \) and \( g \neq g' \) in \( G \),\(^7\) The marginal cost to the seller is uniformly zero. Hence, unless the price is set unreasonably high, the seller receives a positive expected profit whenever he fully discloses. If consumer type \( t \in T \) purchases from seller type \( g \in G \) at price \( p \in P \), her payoff is \( u(g, t) - p \), and the seller’s payoff is \( p \). If she decides not to purchase, both players’ payoffs are zero.

The above mentioned are common knowledge between the players. We use a sequential equilibrium as a solution concept.\(^8\) Attention is laid only on pure strategy equilibria. To formally define an equilibrium, a set \( W(p, m) \) of worst-case types is defined for every pair \((p, m) \in P \times M\) of price and message. Roughly, given an unexpected observation \((p, m)\), a seller type \( w \in m \) is a worst-case type if every seller type \( g \in m \) does not gain in the perfect-information demand when being believed as type \( w \).

\[ W(p, m) = \left\{ w \in m : \mu \left( \{ t \in T : u(g, t) \geq p \} | g \right) \geq \mu \left( \{ t \in T : u(t, w) \geq p \} | g \right) \text{ for every } g \in G \right\} \]

**Definition 1** (Equilibrium). An equilibrium is a triple \((\sigma_s, \sigma_c, \beta)\) satisfying the below conditions.

1. For every \((t, p, m) \in T \times P \times M, \sigma_c(t, p, m) = 1\) if and only if \( \sum_{g \in G} \beta(t, p, m)(g) u(g, t) \geq p \), where \( \beta(t, p, m) \) is the probability assigned to seller type \( g \in G \) by belief \( \beta(t, p, m) \).

2. For every \( g \in G \), \( \sigma_s(g) \in \arg \max_{(p, m) \in P \times M(g)} p \cdot \mu \left( \{ t \in T : \sigma_c(t, p, m) = 1 \} \right) \).

3. For every \((t, p, m) \in T \times P \times M\), belief \( \beta(t, p, m) \in \Delta(G) \) is obtained by Bayes rule on the equilibrium path. Off the equilibrium path, \( \beta(t, p, m) \) assigns probability one to a type in \( W(p, m) \) if it is non-empty.

Part (1) implies that the consumer purchases whenever her expected payoff is non-negative, which is evaluated with the belief described in (3). Notice that the consumer is assumed to purchase whenever indifferent between her decisions. (2) requires that a message be truthful. Moreover, the seller makes an optimal choice fully anticipating the consumer’s responses to all prices and messages available to him. The consumer’s belief off the equilibrium path is specified

\(^6\)Every message \( m \in M \) is verifiable in that \( m \) can be sent by all and only seller types \( g \in m \). Using the term in Seidmann and Winter \([1997]\), the message space is a “rich language”. A relatively strong assumption is made about the message space. As long as the seller can verify his real type, full disclosure is the unique outcome regardless of the prior whenever sales-dominance relation is complete at every price. With a rich message space, any product information that prices may signal can be directly provided through a costless message.

\(^7\)We implicitly assume that the products of different seller types are not homogeneous. An in-depth inspection of the consumer on these heterogeneous products would make her strictly prefer one to the others.

\(^8\)We use the same solution concept as in KR for direct usage of their results. Alternatively, we may take perfect Bayesian equilibrium as a solution concept, though it is usually used when types are independent (see Osborne and Rubinstein \([1994]\)). Even with this alternative solution concept, our results remain unchanged.
with worst-case types whenever they exist. For an unexpected observation \((p, m)\), she believes a type in \(W(p, m)\) with probability one. If the selection of a worst-case type is independent of consumer types, the consistency requirement of a sequential equilibrium is trivially satisfied whatever the prior.

An equilibrium \((\sigma_s, \sigma_c, \beta)\) is called a full-disclosure equilibrium if for every \((p, m) \in P \times M\), at most one seller type \(g \in G\) exists such that \(\sigma_s(g) = (p, m)\). A non-disclosure equilibrium refers to an equilibrium which does not fully discloses product information.

3 Complete sales-dominance relations

KR provide pairwise monotonicity as a necessary and sufficient condition for full-disclosure to be the unique outcome regardless of the prior.\(^9\)

**Definition 2** (Pairwise Monotonicity). The preference of the consumer is pairwise monotone if for every consumer types \(t \neq t'\) in \(T\) and every seller types \(g \neq g'\) in \(G\), it is either (1) statewise monotone with respect to seller types: if \(u(g, t) > u(g', t)\), \(u(g, t') > u(g', t')\), or (2) statewise monotone with respect to consumer types: if \(u(g, t) > u(g, t')\), \(u(g', t) \geq u(g', t')\).

For every pair of non-empty sets \(G' \subseteq G\) and \(T' \subseteq T\), let a saddle point be a type pair \((g^*, t^*)\) in \(G' \times T'\) such that \(u(g^*, t^*) = \max_{G'} \min_{T'} u(g, t) = \min_{T'} \max_{G'} u(g, t)\). Given the assumption of generic preference, pairwise monotonicity is equivalent to the existence of a saddle point for every pair of doubleton sets \(G' \subseteq G\) and \(T' \subseteq T\). We start by defining, for each price, a binary relation over seller types, which we call sales-dominance relation. Then, we will show that under pairwise monotonicity, sales-dominance relation is complete at every price.

For every price \(p \in P\) and every non-empty set \(G' \subseteq G\) of seller types, set \(T(p, G')\) includes all consumer types who strictly prefer to purchase some seller type \(g \in G'\) at \(p\) if \(g\) were to be perfectly disclosed.

\[
T(p, G') = \{ t \in T : u(g, t) > p \text{ for some } g \in G' \} = \bigcup_{g \in G'} T(p, \{g\})
\]

Suppose that the consumer faces price \(p\) and believes all and only seller types in non-singleton set \(G'\).\(^{10}\) Firstly, regardless of the prior, the set of consumer types who would purchase cannot be larger than \(T(p, G')\). Assuming that \(T \setminus T(p, G')\) is non-empty, pick an arbitrary consumer type \(t \in T \setminus T(p, G')\). Purchasing any seller type in \(G'\) yields at most a payoff of 0 (i.e., \(u(g, t) \leq p\) for every \(g \in G'\)). Since the consumer’s preference is generic and \(G'\) is non-singleton, at least one seller type in \(G'\) is not worth purchasing but believed with a positive probability (i.e., \(u(g, t) < p\) for some \(g \in G'\)). Hence, whatever the prior, consumer type \(t\) receives a negative expected payoff by purchasing. Secondly, all consumer types in \(T(p, G')\) would purchase at \(p\) with their respective posteriors on \(G'\) if for every \(t \in T(p, G')\), the prior assigns a sufficiently high probability to some

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9 The definition of pairwise monotonicity in KR is slightly extended as for statewise monotonicity with respect to the consumer types in order to cover the following preference of the consumer. Let \(G = \{g_1, g_2\}\) and \(T = \{t_1, t_2\}\). The willingness to pay of the consumer is given by \(u(g_1, t_1) = u(g_1, t_2) = 2, u(g_2, t_1) = 1,\) and \(u(g_2, t_2) = 3\). According to the definition in KR, this preference of the consumer is not pairwise monotone. However, only a full disclosure equilibrium exists regardless of the prior. At a pooling price \(p < 2\), seller type \(g_1\) has an incentive to disclose, while seller type \(g_2\) would do at a pooling price \(p \geq 2\).

10 In this article, a non-singleton set refers to a non-empty set containing with more than one elements.
seller type \( g_t \in G' \) with \( u(g_t, t) > p \). We call \( T(p, G') \) the highest possible pooling sales when all and only seller types in non-singleton set \( G' \) pool at price \( p \).

For every \( p \in P \) and \( g \in G \), \( A(p, \{ g \}) \) denotes perfect-information sales at price \( p \) for seller type \( g \), that is, \( A(p, \{ g \}) = \{ t \in T : u(g, t) \geq p \} \). Under the assumption that the consumer purchases whenever indifferent, \( T(p, \{ g \}) \) is a subset of \( A(p, \{ g \}) \), though not necessarily a proper subset. To delineate the possible difference in sales, \( T(p, \{ g \}) \) is called floor sales at price \( p \) for seller type \( g \). Notice that \( T(p, \{ g \}) = A(p', \{ g \}) \) for every price \( p' \in (p, \min_{u \in T(p, \{ g \})} u(g, t)] \). Hence, floor sales \( T(p, \{ g \}) \) can be interpreted as the highest possible perfect-information sales when seller type \( g \) fully discloses and raises the price above \( p \). If we refer to sales without qualifying as floor or perfect-information, the default interpretation will be floor sales.

**Definition 3** (Weak sales-dominance relations). Pick an arbitrary price \( p \in P \). For every two seller types \( g \) and \( g' \) in \( G \), \( g \) weakly sales-dominates \( g' \) at \( p \) (denoted by \( g \succsim_p g' \)) if \( T(p, \{ g \}) \supseteq T(p, \{ g' \}) \). For every non-empty set \( G' \subseteq G \) of seller types, \( g \in G' \) is sales-dominant at \( p \) over \( G' \) if \( g \succsim_p g \) for every \( g \in G' \).

At every price \( p \in P \), sales-equivalence relation and strict sales-dominance relation are defined over seller types in conventional ways. Seller types \( g \) and \( g' \) are sales-equivalent at \( p \) if \( g \succsim_p g' \) and \( g' \succsim_p g \), i.e., \( T(p, \{ g \}) = T(p, \{ g' \}) \). A seller type \( g \) strictly dominates another type \( g' \) at \( p \) if \( g \succ_p g' \) but not \( g' \succ_p g \), i.e., \( T(p, \{ g \}) \supset T(p, \{ g' \}) \). Whenever we refer to sales-dominance without qualifying as strict or weak, the default interpretation will be weak dominance.

We simply say that sales-dominance relations are complete if it is the case at every price. Under vertical differentiation, a product sales-dominates another product with lower quality independently of prices since the consumer strictly prefers to purchase the former at whatever prices she strictly prefers to purchase the latter. Namely, sales-dominance relations are complete under vertical differentiation. In contrast, sales-dominance relations are not necessarily complete under horizontal differentiation. Consider two utility functions \( u^1 \) and \( u^2 \) given below. Products are horizontally differentiated. With \( u^1 \), neither seller type \( g \) sales-dominates the other at price \( p \in \{2, 3\} \) since both \( T(p, \{ g_1 \}) \setminus T(p, \{ g_2 \}) \) and \( T(p, \{ g_2 \}) \setminus T(p, \{ g_1 \}) \) are non-empty. With \( u^2 \), however, sales-dominance relation is well-defined over \( g_1 \) and \( g_2 \) at every price. At price \( p \notin \{1, 2\} \cup \{3, 4\} \), \( T(p, \{ g_1 \}) = T(p, \{ g_2 \}) \), hence \( g_1 \) and \( g_2 \) are sales-equivalent. At a price in the range \( \{3, 4\} \), \( g_1 \) strictly sales-dominates \( g_2 \), while \( g_1 \) is strictly sales-dominated at a price in \( \{1, 2\} \). Note that sales-dominance relations are price-dependent in that the dominance relations can be “reversed” as is observed when a price in the range \( \{3, 4\} \) falls to one in \( \{1, 2\} \).

\[
\begin{array}{c|cc}
\text{u}^1(g, t) & t_1 & t_2 \\
g_1 & 4 & 1 \\
g_2 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{c|cc}
\text{u}^2(g, t) & t_1 & t_2 \\
g_1 & 4 & 1 \\
g_2 & 3 & 2 \\
\end{array}
\]

What restriction would complete sales-dominance relations impose on the ranking of the consumer’s valuations in the pairwise comparison of seller types? Consider two arbitrary seller types \( g_1 \) and \( g_2 \). For price \( p \in P \), let \( T_1(p) = T(p, \{ g_1 \}) \setminus T(p, \{ g_2 \}) \) and \( T_2(p) = T(p, \{ g_2 \}) \setminus T(p, \{ g_1 \}) \).

\begin{footnotesize}
\footnote{Since sales refer to a set of consumer types, the adjectives “large” or “small” may be appropriate usages of language. However, we would use “high” or “low” as they are normally used to describe the sales volume.}
\footnote{As the floor function takes a real number \( x \) as input and returns the greatest integer less than or equal to \( x \), floor sales take perfect-information sales then return a subset of perfect-information sales by excluding the indifferent consumer types.}
\end{footnotesize}
The completeness of sales-dominance relation at $p$ requires that either $T_1(p)$ or $T_2(p)$ be empty. Namely, the two products should be “partially” vertically differentiated in that the valuations for $g_1$ and $g_2$ are identically ranked to the consumer types in $T_1(p) \cup T_2(p)$, while such an identical ranking is not necessary to the ones in $T \setminus (T_1(p) \cup T_2(p))$. Now, assume that consumer type $t_1$ values $g_1$ over $g_2$ (see a solid point in Figure 1). At every price $p \in [u(g_2, t_1), u(g_1, t_1))$, $T_1(p)$ is non-empty, which implies $T_2(p) = \emptyset$ under complete sales-dominance relations. If consumer type $t_2$ strictly prefers $g_2$ over $g_1$, the consumer’s preference should be represented at a point on the areas shaded in blue. Namely, their valuations should be monotone with respect to consumer types in that $u(g_1, t_1) \leq u(g_1, t_2)$ or $u(g_2, t_2) \leq u(g_2, t_1)$. In Proposition 1, we show that sales-dominance relations are complete if and only if the consumer’s preference is pairwise monotone.

![Figure 1: Ranking of the valuations for two seller types under complete sales-dominance relations](image)

**Proposition 1.** Sales-dominance relation is complete at every price $p \in P$ if and only if the preference of the consumer is pairwise monotone.

**Proof.** $\Rightarrow$: Suppose that the consumer’s preference is not pairwise monotone. Then sets $G' = \{g_1, g_2\} \subseteq G$ and $T' = \{t_1, t_2\} \subseteq T$ exist for which either case in Table 1 holds. In both cases, at every price $p \in P$ such that $\max_{G'} \min_{T'} u(g, t) \leq p < \min_{T'} \max_{G'} u(g, t)$, both $T(p, \{g_1\}) \setminus T(p, \{g_2\})$ and $T(p, \{g_2\}) \setminus T(p, \{g_1\})$ are not empty, a contradiction to the completeness of sales-dominance relation at $p$.

$\Leftarrow$: Suppose that sales-dominance relation is not complete at a price $p \in P$. Then for some seller types $g_1$ and $g_2$ in $G$ and some consumer types $t_1$ and $t_2$ in $T$, $t_1 \in T(p, \{g_1\}) \setminus T(p, \{g_2\})$ and $t_2 \in T(p, \{g_2\}) \setminus T(p, \{g_1\})$. Case 1 of Table 1 holds, a contradiction to pairwise monotonicity.

\[ \begin{array}{c|c|c} \text{case 1} & t_1 & t_2 \\ \hline g_1 & u(g_1, t_1) > u(g_1, t_2) & \lor \\ g_2 & u(g_2, t_1) < u(g_2, t_2) & \land \\ \end{array} \]

\[ \begin{array}{c|c|c} \text{case 2} & t_1 & t_2 \\ \hline g_1 & u(g_1, t_1) < u(g_1, t_2) & \lor \\ g_2 & u(g_2, t_1) > u(g_2, t_2) & \land \\ \end{array} \]

Table 1: Violation of pairwise monotonicity

### 3.1 Generalized unraveling argument

KR prove that pairwise monotonicity is equivalent to the existence of a saddle point for every pair of non-empty subsets $G' \subseteq G$ and $T' \subseteq T$.\footnote{Gurvich and Libkin [1990] observes that a saddle point exists for every pair of non-empty subsets $G$ and $T$ if and only if every pair of doubleton subsets $G$ and $T$ has a saddle point. Therefore, pairwise monotonicity is} Then, they argue that if $(g^*, t^*)$ is a saddle
point for a pair of non-singleton sets $G' \subseteq G$ and $T' \subseteq T$, seller type $g^* \in G'$ has an incentive to deviate from the hypothetical pooling outcome at which all and only seller types in $G'$ pool and all and only consumer types in $T'$ purchase. However, they do not point out in what sense $g^*$ can be viewed being superior to other seller types in $G'$.

In this subsection, we will show that if a sales-dominant type exists at a price $p \in P$ over a non-singleton set $G' \subseteq G$, his payoff can strictly improve through full disclosure rather than pooling at $p$ with other sales-dominated types in $G'$. Moreover, we will argue that if sales-dominance relation is complete at every price, a sales-dominant type exists for every price and every non-singleton set of seller types. Thereby, when the preference of the consumer is pairwise monotone, we can characterize, in terms of sales, the seller type that has an incentive to deviate for every hypothetical pooling outcome identified with a pair of price and non-singleton set of seller types.

Firstly, we illustrate why, given a price, a sales-dominant type, if any, can improve his payoff through full disclosure rather than pooling with other sales-dominated ones. Consider the utility function below. Suppose that a pooling outcome exists at which the seller sets the price at $p \in [3, 6]$.$^{14}$ If $p \in [5, 6]$, $T(p, G) = \{t_1\}$, meaning that consumer types $t_2$ and $t_3$ never purchase under uncertainty over $G$. The unique sales-dominant type $g_2$ would disclose voluntarily because consumer type $t_1$ is sure to purchase even when he raises the price to 6. If $p \in [3, 4]$, $T(p, G) = \{t_1, t_2\}$. Analogously, the unique sales-dominant seller type $g_1$ has no incentive to pool at $p$ with $g_2$ because he can improve his payoff by raising the price to 4 through full disclosure. If $p \in [4, 5]$, $T(p, \{g_1\}) = T(p, \{g_2\})$, that is, $g_1$ and $g_2$ are sales-equivalent. Even though each seller type is sales-dominant only in a “weak” sense, he would voluntarily disclose and raise the price; $g_1$ raises to 5 and $g_2$ to 6.

\[
\begin{array}{|c|c|c|}
\hline
u(g, t) & t_1 & t_2 & t_3 \\
\hline
5 & 4 & 3 \\
6 & 1 & 2 \\
\hline
\end{array}
\]

Proposition 2 generalizes the above observation. Consider an arbitrary price $p \in P$ and non-singleton set $G' \subseteq G$. Indeed, we prove that if $\bar{g} \in G'$ is sales-dominant at $p$ over $G'$, $T(p, \{\bar{g}\}) = T(p, G')$. This simple result implies that as long as price $p$ is concerned, the sales-dominant type $\bar{g}$ should have no incentive pool with other seller types in $G'$ because he gets better off by fully disclosing his own type. To see why, suppose to the contrary that an outcome exists such that all and only seller types in $G'$ pool at $p$ and $\bar{g} \in G'$ is sales-dominant at $p$ over $G'$. Recall that the highest possible sales $\bar{g}$ can obtain at this hypothetical pooling outcome is $T(p, G')$. Firstly, we assume that $T(p, G') \neq \emptyset$. For every $t \in T(p, G')$, $u(\bar{g}, t) > p$. In other words, the perfect-information sales for $\bar{g}$ is the same as the highest possible pooling sales $T(p, G')$ even if he fully discloses and raises the price marginally. Otherwise, if $T(p, G') = \emptyset$, each seller type in $G'$ is sales-dominant. The hypothetical outcome cannot sustain at any equilibrium because each seller in $G'$ can improve his payoff through full disclosure by setting a sufficiently low price to obtain non-empty perfect-information sales. Such a low price is sure to exist since $u(g, t) > 0$ for every $(g, t) \in G \times T$ and the marginal cost to seller type $g \in G$ is zero. Note that the above argument runs free of the assumption on the prior. In Proposition 2, we show that equivalent to the existence of a saddle point for every pair of non-empty subsets of $G$ and $T$.

$^{14}$If $p < 3$, type $g_1$ should disclose and raise the price. If $p \geq 6$, no consumer type would purchase with whatever full support belief on $G$. Hence, all prices not in the range $[3, 6]$ can not support a non-disclosure equilibrium.
for the non-existence of an outcome at which all and only seller types in non-empty set \( G' \subseteq G \) pool at price \( p \in P \), it is sufficient to find just one sales-dominant seller type at \( p \) over \( G' \).

**Proposition 2.** Consider an arbitrary price \( p \in P \) and non-empty set \( G' \subseteq G \) of seller types. If a sales-dominant type exists at \( p \) over \( G' \), then an outcome at which all and only seller types in \( G' \) pool at \( p \) does not exist regardless of the prior.

**Proof.** Consider an arbitrary price \( p \in P \) and non-empty set \( G' \subseteq G \). It is sufficient to show that if \( \bar{g} \in G' \) is sales-dominant at \( p \) over \( G' \), then \( T(p, \{\bar{g}\}) = T(p, G') \). \( T(p, \{g'\}) \subseteq T(p, \{\bar{g}\}) \) for every \( g' \in G' \), which implies that \( T(p, G') = \bigcup_{g' \in G'} T(p, g') \subseteq T(p, \{\bar{g}\}) \). Moreover, \( T(p, \{\bar{g}\}) \subseteq T(p, G') \), by definition. \( \Box \)

**Example 1.** Proposition 2 holds at whatever price sales-dominance relation is incomplete. Let \( G = \{g_1, g_2, g_3\} \) and \( T = \{t_1, t_2\} \). Consider the utility function given below. Sales-dominance relation is not complete at any price in the range \([1, 5]\), while seller type \( g_3 \) is sales-dominant even at price \( p \in [1, 3] \). Seller type \( g_3 \) would not pool at \( p \) with other sales-dominated types \( g_1 \) and \( g_2 \) since his payoff improves by raising the price to 3 through full disclosure. In other words, if a non-disclosure outcome exists at which the seller sets the price at \( p \), the consumer would believe only types \( g_1 \) and \( g_2 \) as possible (see Proposition 5 for the related result).

\[
\begin{array}{c|cc}
\mathbf{u(g,t)} & \mathbf{t_1} & \mathbf{t_2} \\
\hline
\mathbf{g_1} & 5 & 1 \\
\mathbf{g_2} & 1 & 5 \\
\mathbf{g_3} & 6 & 3
\end{array}
\]

Proposition 2 implies that full disclosure should be the unique outcome regardless of the prior if a sales-dominant type exists for every price and every non-singleton set of seller types (i.e., for every hypothetical pooling outcome). Under what condition would it be the case? Note that sales-dominance relation is trivially reflexive and transitive at every price since it is defined using set inclusion. Additionally, if sales-dominance relation is complete at every price \( p \in P \), every non-singleton set \( G' \) of seller types is completely preordered by sales-dominance relation \( \succeq_p \).\(^{15}\) It is well known that a completely-preordered finite set has the greatest element. For instance, the most preferred alternative exists over a finite choice set if the preference relation is a complete preorder. Formally, for every price \( p \in P \), sales-dominance relation is complete at \( p \) if and only if a sales-dominant type exists at \( p \) over every non-empty subset of \( G \).

Under pairwise monotonicity, the generalized unraveling argument runs as follows. Consider an arbitrary price \( p \in P \). Seller types are completely ranked according to sales-dominance relation at \( p \). A sales-dominant type \( g_1 \) over \( G \) has no incentive to pool at \( p \) with other seller types sales-dominated by \( g_1 \). The second most sales-dominant type \( g_2 \) over \( G \) would never pool at \( p \) with other seller types sales-dominated by \( g_2 \). The inductive reasoning unravels down to the seller type with the lowest sales. This is reminiscent of the standard unraveling argument under vertical differentiation. Recall that given any pair of a price and a set of seller types, the seller with the highest quality is sales-dominant if products are vertically differentiated. Hence, our argument generalizes the standard unraveling argument and alternatively explains why under vertical differentiation full disclosure should be the unique outcome regardless of the prior. Note

\(^{15}\)A binary relation is a complete preorder if it is complete, reflexive, and transitive.
that price-dependent sales are taken instead of price-independent quality as a determinant of how to rank seller types, which enables us to apply the generalized unraveling argument even if products are horizontally differentiated.

**Example 2.** Under vertical differentiation, the payoff of a low-quality seller would strictly improve if he “successfully” imitates a high-quality seller because the expected valuation of the consumer type must rise above her valuation for his product. Hence, the low-quality seller could charge a higher price while holding the demand not decreased. In contrast, if products are horizontally differentiated, a seller type may not get better off even after successfully pooling with another with higher sales. As a reference price, we consider the optimal price the seller would set if his type were to be fully disclosed.

Let \( G = \{g_1, g_2\} \) and \( T = \{t_1, t_2, t_3\} \). Given the below utility function, products are horizontally differentiated. When seller type \( g_2 \) is fully disclosed, he would obtain the payoff of 3 by setting the price at \( p = 3 \). Even though \( g_2 \) is strictly sales-dominated at \( p \), he might have no incentive to pool with sales-dominating type \( g_1 \). In particular, if a price higher than \( p \) substantially reduces the pooling sales in probability. To see why, suppose that \( g_2 \) “successfully” imitates \( g_1 \) at price \( p' \in (3, 5) \) and the consumer believes both seller types as possible. Consumer type \( t_3 \) would never purchase since her expected valuation becomes lower than \( p' \). We further assume that \( g_2 \) believes all consumer types are equally likely. Then if \( p' \in (3, 4) \), the highest profit \( g_2 \) can achieve from the hypothetical pooling outcome is \( \frac{2}{3} p' < 3 \). Otherwise, if \( p' \in [4, 5) \), the profit is at most \( \frac{1}{3} p' < 3 \). □

\[
\begin{array}{c|ccc}
  t & t_1 & t_2 & t_3 \\
  \hline
  5 & 4 & 2 \\
  3 & 3 & 3 \\
\end{array}
\]

### 3.2 Perfect-information sales-dominance relations

As alternative relations over seller types, we will consider perfect-information sales-dominance relations, which would be helpful to find out when a worst-case type is well-defined for every observable deviation of the seller. Moreover, we will provide a generalized unraveling argument in terms of perfect-information sales. Henceforth, we use PI sales as a shorthand for perfect-information sales.

Pick an arbitrary price \( p \in P \). For every seller types \( g \) and \( g' \) in \( G \), \( g \) is said to weakly PI sales-dominate \( g' \) at \( p \) if \( A(p, \{g\}) \supseteq A(p, \{g'\}) \). A seller type \( \tilde{g} \in G' \) is said PI sales-dominant at \( p \) over a non-empty set \( G' \subseteq G \) of seller types if \( \tilde{g} \) PI sales-dominates every other type in \( G' \) at \( p \). PI sales-equivalence relation and strict PI sales-dominance relation are defined in conventional ways. In the context of vertical differentiation, the two notions of dominant types coincide independently of prices since a product with the highest quality is dominant in both terms of sales for every price and every set of seller types. However, under horizontal differentiation, they may disagree. In Example 2, at the price of 3, floor sales-dominant type \( g_1 \) is strictly PI sales-dominated, while PI sales-dominant seller type \( g_2 \) is strictly floor sales-dominated. In the following lemma, we show that sales-dominance relations and PI sales-dominance relations are similar in nature. Namely, if one relations are complete at all prices, so are the other relations.

11
Lemma 1. PI sales-dominance relation is complete at every price if and only if sales-dominance relation is complete at every price.

Proof. Consider an arbitrary seller type $g \in G$ and price $p \in P$, where $p > \min P$. Let $p' \in P$ be the highest among all prices strictly lower than $p$, i.e., $p' = \max \{p \in P : p < p\}$. $p'$ is well-defined since set $P$ is finite. We claim that $A(p, \{g\}) = T(p', \{g\})$, which implies that PI sales-dominance relation at $p$ is complete if and only if sales-dominance relation is complete at $p'$. For every type $t \in A(p, \{g\})$, $u(g, t) \geq p > p'$, therefore, $A(p, \{g\}) \subseteq T(p', \{g\})$. For every type $t \in T(p', \{g\})$, $u(g, t) > p'$. Since $u(g, t) \in P$ and $p$ is the lowest among all prices strictly higher than $p'$, $u(g, t') \geq p$, $T(p', \{g\}) \subseteq A(p, \{g\})$.

$\Rightarrow$: It is sufficient to show that sales-dominance relation is complete at a price $\bar{p} = \max P$. For every $g \in G$, $T(\bar{p}, \{g\}) = \emptyset$ since $\{u(g, t) : t \in T\} \subseteq P$ implies $\max \{u(g, t) : t \in T\} \leq \max P = \bar{p}$. 

$\Leftarrow$: It is sufficient to show that PI sales-dominance relation is complete at a price $\tilde{p} = \min P$. For every $g \in G$, $A(\tilde{p}, \{g\}) = T$ since $\{u(g, t) : t \in T\} \subseteq P$ implies $\tilde{p} = \min P \leq \min \{u(g, t) : t \in T\}$.

When types are independent, a worst-case type is well-defined everywhere on the equilibrium path. Formally, for every observable deviation $(p, m) \in P \times M$ of the seller, a seller type $w \in \arg \min_{g \in m} \mu_T(A(p, \{g\}))$ is a worst-case type, where $\mu_T$ is a marginal of independent prior $\mu$ on $T$. Not surprisingly, a worst-case type is not necessarily well-defined if types are correlated under horizontal differentiation. However, for some deviation $(p, m)$ of the seller, a seller type $w \in m$ is a worst-case type regardless of the prior if $w$ is PI sales-dominated at $p$ by all other seller types in $m$. Lemma 1 implies that under pairwise monotonicity, a seller type with the lowest PI sales exists for every observable deviation $(p, m) \in P \times M$ since the finite set $m$ is completely preordered according to PI sales-dominance relation at $p$. Namely, a set of worst-case types $W(p, m) = \{w \in m : A(p, \{w\}) \subseteq A(p, \{g\})$ for every $g \in m\}$ is non-empty for every $(p, m) \in P \times M$. Proposition 3 is provided without proof.

Proposition 3. A worst-case type is well defined regardless of the prior for every observable deviation of the seller if the consumer's preference is pairwise monotone.

Example 2 (continued). According to Lemma 1, full-disclosure uniquely obtains under complete PI sales-dominance relations. We illustrate why some PI sales-dominant type can improve his payoff through full disclosure rather than pooling with other sales-dominated ones if PI sales-dominance relations are complete. Suppose that a pooling outcome exists at which the seller sets a price in the range $[3, 5]$. At every price in $(3, 5)$, the PI sales-dominant type $g_1$ is floor sales-dominant as well, hence his payoff improves through full disclosure by raising the price marginally. Now, we focus on the price of 3. Seller type $g_1$ has an incentive to deviate from the hypothetical pooling outcome since he is floor sales-dominant. The PI sales-dominant type $g_2$ also has an incentive to deviate from this outcome. All consumer types except $t_3$ would purchase at price 3 under uncertainty, i.e., $T(3, G) = \{t_1, t_2\}$. Seller type $g_2$ can improve his payoff through full disclosure while holding the price at 3 because the resulting PI sales becomes higher $(A(3, \{g_2\}) \supseteq T(3, G))$.

In Proposition 4, we formalize the unraveling argument in terms of PI sales. Suppose that the consumer's preference is pairwise monotone. For every price $p \in P$ and every non-singleton set $G' \subseteq G$, at least one seller type in $G'$ is PI sales-dominant since $G'$ is a completely preordered
finite set. Among those PI sales-dominant, some type $\bar{g}$ has no incentive to pool at $p$ with other seller types in $G'$ because full disclosure (while holding the price the same as $p$) results in PI sales strictly higher than the highest possible pooling sales ($A(p, \{\hat{g}\}) \supseteq T(p, G')$, or $\hat{g}$ is floor sales-dominant at $p$ over $G'$ ($T(p, \{\hat{g}\}) = T(p, G')$).

Proposition 4. Suppose that the consumer’s preference is pairwise monotone. For every price $p \in P$ and every non-empty set $G' \subseteq G$, at least one PI sales-dominant seller type $\tilde{g} \in G'$ exists at $p$ over $G'$ such that $A(p, \{\tilde{g}\}) \supseteq T(p, G')$ or $T(p, \{\tilde{g}\}) = T(p, G')$.

Proof. Consider an arbitrary price $p \in P$ and non-singleton set $G' \subseteq G$. Since PI sales-dominance relation is complete at every price, a PI sales-dominant type exists at $p$ over $G'$. First, we claim that, $T(p, G') \subseteq A(p, \{\tilde{g}\})$ for every PI sales-dominant type $\tilde{g}$ at $p$ over $G'$. For every type $t \in T(p, G')$, a type $g_t \in G'$ exists such that $u(g_t, t) > p$, hence $t \in T(p, \{g_t\}) \subseteq A(p, \{g_t\})$. Since $\tilde{g}$ is PI sales-dominant over $G'$ at $p$, $t \in A(p, \{g_t\}) \subseteq A(p, \{\tilde{g}\})$. If there exists a PI sales-dominant type $\tilde{g}$ such that $A(p, \{\tilde{g}\}) \supseteq T(p, G')$, the proof is done. Suppose now that $T(p, G') = A(p, \{\tilde{g}\})$ for every PI sales-dominant type $\tilde{g}$. Since PI sales-dominance relations are complete at every price, a floor sales-dominance relation is complete at every price, meaning that a floor sales-dominant type $\bar{g}$ exists at $p$ over $G'$ such that $T(p, \{\bar{g}\}) = T(p, G')$. Since $A(p, \{\bar{g}\}) \subseteq A(p, \{\tilde{g}\}) = T(p, G')$ and $T(p, \{\tilde{g}\}) \subseteq A(p, \{\tilde{g}\})$, a floor sales-dominant type $\bar{g}$ is PI sales-dominant at $p$ over $G'$, which completes the proof. □

We want to highlight that none of the two desired properties in Proposition 4 may hold for some PI sales-dominant type. Firstly, it may be the case even under pairwise monotonicity. In Example 2, if the set of consumer types is restricted to $\{t_1, t_2\}$, the consumer’s preference is still pairwise monotone. Even though seller type $g_2$ is one of the PI sales-dominant types at price 3, $A(3, \{g_2\}) = T(3, G)$ and $T(3, \{g_2\}) \subseteq T(3, G)$. This implies that, unlike a floor sales-dominant type, a PI sales-dominant type may not get better off by deviating through full disclosure from a hypothetical pooling outcome. Now, recall Example 1. The consumer’s preference is not pairwise monotone. Though seller type $g_3$ is uniquely PI sales-dominant at price 3 over $G$, $A(3, \{g_3\}) = T(3, G)$ and $T(3, \{g_3\}) \subseteq T(3, G)$. Indeed, with a uniform prior, a non-disclosure outcome exists at which all seller types (including PI sales-dominant type $g_3$) set the price at 3, send a trivial message $G$, and all consumer types purchase at 3.10 This implies that, unlike a floor sales-dominant type, it is not sufficient to find a PI sales-dominant type at $p$ over $G'$ to argue the non-existence of an outcome at which all and only seller types in $G'$ pool at $P$.

We observed that if products are horizontally differentiated, a seller type may have no incentive to pool with the other superior seller type in floor sales. However, no matter how products are differentiated, a seller type could get better off if he successfully imitates the other superior seller type in PI sales. For a concrete argument, consider two seller types $g_1$ and $g_2$ such that $g_2$ is PI sales-dominated by $g_1$ at price $p_2$, where $p_2$ is the optimal price $g_2$ would set when his type is fully disclosed. Pick an arbitrary consumer type $t \in A(p_2, \{g_2\})$. Since $g_1$ PI sales-dominates $g_2$, $u(g, t) \geq p_2$ for every $g \in \{g_1, g_2\}$. Since the preference is generic, the expected valuation of type $t$ is strictly higher than $p_2$ if a pooling attempt of $g_2$ turns “successful”. In other words, if

10The consumer’s off-the-equilibrium beliefs can be specified in the following ways. Consider an arbitrary deviation $(p, m) \in P \times M$ of the seller such that $m$ is non-singleton, and $p \neq 3$ or $m \neq G$. The consumer believes a worst-case type $g_1$ regardless of her types if $g_1 \in m$. Otherwise, if $g_1 \notin m$, the consumer believes a worst-case type $g_2$ regardless of her types.
successful, $g_2$ can set a price strictly higher than $p_2$, while not reducing the pooling sales below $A(p_2, \{g_2\})$.

## 4 Product information at a non-disclosure outcome

According to KR, if the preference of the consumer is not pairwise monotone, a prior exists that induces a non-disclosure outcome. Yet, what product information the consumer could receive when the preference is not pairwise monotone is still an open question. To answer, we explore whether the converse of Proposition 2 holds. If the preference of the consumer, sales-dominance relations are incomplete, meaning that there exist some price $p \in P$ and some non-singleton sets $G' \subset G$ for which no seller type in $G'$ is sales-dominant. We show that, under a mild condition, some independent prior exists that induces a non-disclosure outcome at which all and only seller types in $G'$ pool and all and only consumer types in $T(p, G')$ purchase. Moreover, every seller type in $G'$ achieves an equilibrium payoff strictly higher than the full-disclosure payoffs. The proof of Proposition 5 can be found in Appendix A.

**Proposition 5.** Suppose that no sales-dominant type exists at price $p \in P$ over non-singleton sets $G' \subset G$ of seller types and $u(g, t) \neq u(g, t')$ for every $g \in G'$ and every $t \neq t'$ in $T(p, G')$. An independent prior exists that induces a non-disclosure equilibrium $(\sigma_s, \sigma_c, \beta)$ such that $\sigma_s(g) = (p^*, G')$ for every $g \in G'$ and $\sigma_c(t, G', p^*) = 1$ for every $t \in T(p, G')$. Moreover, every seller type $g \in G'$ achieves an equilibrium payoff strictly higher than the full-disclosure payoffs.

In Proposition 5, the “mild” condition requiring $u(g, t) \neq u(g, t')$ for every $g \in G'$ and every different $t$ and $t'$ in $T(p, G')$ is crucial for every pooling seller type to achieve an equilibrium payoff strictly higher than the full disclosure payoffs. Given the below utility function, no sales-dominant type exists over $G$ at $p \in [2, 3)$ and $T(p, G) = T$. The mild condition is not satisfied since $u(g_1, t_1) = u(g_1, t_2)$. Consider a class of non-disclosure equilibria $(\sigma_s, \sigma_c, \beta)$ in which $\sigma_s(g) = (p^*, G')$ for every seller type $g \in G$ and $\sigma_c(t, p^*, G) = 1$ for every consumer type $t \in T$. We claim that for every independent prior $\mu$ that induces an equilibrium $(\sigma_s, \sigma_c, \beta)$, seller type $g_1$ gets a payoff of 2, which is the highest payoff achievable by $g_1$ through full disclosure. Let $\mu_G$ denote a marginal of independent prior $\mu$ on $G$. Then, we must have $2 \leq p^* \leq \min_T u(\mu_G, t)$.

The first inequality should hold since if $p^* < 2$, $g_1$ is floor sales-dominant. The second inequality holds because the expected utility should be at least as high as $p^*$ for every consumer type. Since $\max_{g \in \Delta(G)} \min_T u(\mu_G, t) = 2$, the pooling price $p^*$ must be 2 at every non-disclosure equilibrium $(\sigma_s, \sigma_c, \beta)$ induced by an independent prior. In the right table, we provide an independent prior $\mu$ that induces a non-disclosure equilibrium $(\sigma_s, \sigma_c, \beta)$.

<table>
<thead>
<tr>
<th>$u(g, t)$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$\mu$</th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_1$</td>
<td>2</td>
<td>2</td>
<td>$g_1$</td>
<td>$(1 - 2q)r$</td>
<td>$(1 - 2q)(1 - r)$</td>
</tr>
<tr>
<td>$g_2$</td>
<td>3</td>
<td>1</td>
<td>$g_2$</td>
<td>$q^r$</td>
<td>$q(1 - r)$</td>
</tr>
<tr>
<td>$g_3$</td>
<td>1</td>
<td>3</td>
<td>$g_3$</td>
<td>$q^r$</td>
<td>$q(1 - r)$</td>
</tr>
</tbody>
</table>

If the “mild” condition holds over $G$ and $T$, the corollary below directly follows from Propositions 2 and 5.

**Corollary 1.** Suppose that $u(g, t) \neq u(g, t')$ for every $g \in G$ and every $t \neq t'$ in $T$. Consider an arbitrary non-empty subset $G' \subset G$ of seller types. A sales-dominant type over $G'$ exists at
every price in \( P \) if and only if an outcome at which all and only seller types in \( G' \) pool does not exist regardless of the prior.

It is noteworthy that if the prior is independent, a full disclosure equilibrium exists even when sales-dominance relations are incomplete. Let \( \mu_T \) denote a marginal of an independent prior \( \mu \) on \( T \). A strategy \( \sigma^*_s \) of the seller is such that for every \( g \in G \), \( \sigma^*_s(g) = (p_g, \{g\}) \), where \( p_g \in \arg\max_{p \in P} p \cdot \mu_T(A(p, \{g\})) \). Every deviation \((p, m)\) available to a seller type \( g \) never improves his payoff since \( p_g \cdot \mu_T(A(p_g, \{g\})) \geq p \cdot \mu_T(A(p, \{g\})) \geq p \cdot \mu_T(A(p, \{w_{p,m}\})) \), where \( w_{p,m} \) is a worst-case type for the unexpected observation \((p, m)\).\(^{17}\)

## 5 Conclusion

The seller sends a costless and verifiable message about his product to persuade the consumer. The consumer has complete and private information about her taste for differentiated products, while the seller has complete and private information about the characteristics of his product. The products are allowed to be horizontally differentiated.

We provide dominance relations over seller types in terms of floor sales and PI sales. They are similar in nature in that if one dominance relation is complete at every price, so is the other. We find that under pairwise monotonicity, a worst-case type is well-defined for every observable deviation of the seller. Moreover, we explore the product information the consumer could receive at a non-disclosure equilibrium and show that under a mild condition, an non-disclosure outcome exists at which some seller types are believed as being possible if and only if a price exists at which a sales-dominant type does not exist over these seller types.

We generalize the standard unraveling argument by identifying the seller type who has an incentive to deviate from a hypothetical pooling outcome. The extended arguments capture the two ways of improving payoff undoubtedly. Suppose that the consumer’s preference is pairwise monotone. For every hypothetical pooling outcome, a floor sales-dominant type exists and he would get better off through full disclosure since the marginal increase in prices would not reduce the sales. For every hypothetical pooling outcome, a PI sales-dominant type exists and among the PI sales-dominant at least one has an incentive to deviate since he is floor sales-dominant or he can enlarge the sales while not lowering the price. Our approach enables us to make a price-specific prediction. In reality, the price of a product may lie in some range. For instance, the posted prices are not very flexible in the retail markets. For the uniqueness of full disclosure outcome, we need to test whether sales-dominance relations are complete over this price range. Moreover, our results suggest that prices might be used as a policy target if regulation encouraging information disclosure are required.

## References


\(^{17}\)Every deviation from a fully-disclosing strategy \( \sigma^*_s \) is observable. For every observable deviation, a worst-case type is well defined with an independent prior \( \mu \). See the discussion in a paragraph immediately preceding Lemma 1.


A Proof of Proposition 5

Indeed, we prove Lemma A.1 that covers Proposition 5 as a special case. Suppose that no sales-
dominant type exists at price \( p \in P \) over non-singleton subset \( G' \subseteq G \) of seller types. A saddle
point does not exists for \( G' \) and \( T(p, G') \). Hence, Proposition 5 directly follows from Lemma A.1.

Lemma A.1. Suppose that no saddle point exists for non-singleton sets \( G' \subseteq G \) and \( T' \subseteq T \)
and \( u(g, t) \neq u(g, t') \) for every \( g \in G' \) and every \( t \neq t' \) in \( T' \). An independent prior exists that
induces a non-disclosure equilibrium \((\sigma_s, \sigma_c, \beta)\) such that \( \sigma_s(g) = (p^*, G') \) for every \( g \in G' \)
and \( \sigma_c(t, G', p^*) = 1 \) for every \( t \in T' \). Moreover, every type \( g \in G' \) achieves an equilibrium payoff
strictly higher than the full-disclosure payoffs.

Two observations Some properties of a Nash equilibrium in a finite zero-sum game would be
useful for the proof of A.1. Consider a pair of non-singleton sets \( G' \subseteq G \) and \( T' \subseteq T \) such that
no saddle point exists for \( G' \) and \( T' \) and \( u(g, t) \neq u(g, t') \) for every \( g \in G' \), and every \( t \neq t' \)
in \( T' \). Let \( Z = \{I, \{A_i\}_{i \in I}, \{v_i\}_{i \in I}\} \) denote a two-player zero-sum game, where \( I = \{s, c\} \) is a
set of players. For every \( i \in I, A_i \) is a set of pure strategies, where \( A_s = T' \) and \( A_c = G' \), and \( v_i \) is a payoff function. For every pure strategy profile \((g, t) \in G' \times T' \), \( v_s(g, t) = u(g, t) \), where \( u \) is the utility function of the consumer. For mixed strategy profile \((\alpha_s, \alpha_c) \in \Delta(G') \times \Delta(T') \), an expected payoff to \( s \) is written by \( v_s(\alpha_s, \alpha_c) \). A Nash equilibrium \((\alpha_s^*, \alpha_c^*) \) exists such that
\( v_s(\alpha_s^*, \alpha_c^*) = \max_{\alpha_c \in \Delta(T')} \min_{\alpha_s \in \Delta(G')} v_s(\alpha_s, \alpha_c) = \min_{\alpha_s \in \Delta(G')} \max_{\alpha_c \in \Delta(T')} v_s(\alpha_s, \alpha_c) \).

Observation 1. No player chooses a pure strategy at every Nash equilibrium of game \( Z \).

Proof. Suppose to the contrary that a pure strategy \( t^* \in T' \) is played by \( c \) at some Nash equi-
librium of \( Z \). Due to the generic preference, the unique best response \( g^* \in G' \) of player \( s \)
exists such that \( v_s(g^*, t^*) = \max_{g \in G'} v_s(g, t^*) \), which in turn implies that \( t^* \) is the unique best re-
response to \( g^* \), that is, \( v_s(g^*, t^*) = \min_{t \in T'} v_s(g^*, t) \). Since \( \min_{T'} v_s(g^*, t) \leq \max_{G'} \min_{T'} v_s(g, t) \leq \max_{G'} v_s(g, t) \), \( v_s(g^*, t^*) \) is a saddle point for \( G' \) and \( T' \), a contradiction. An analogous argument applies to player \( s \). \( \square \)

Consider an mixed strategy Nash equilibrium \((\alpha_s^*, \alpha_c^*) \) of game \( Z \). For every \( g \in G' \), \( r(g) = \max_{t \in T'} v_s(g, t) h(g, t) \), where \( h(g, t) = \alpha_c^* \{t' \in T' : v_s(g, t^*) \geq v_s(g, t')\} \). \( r(g) \) would be used later as an approximation for seller type \( g \)’s full-disclosure payoff. Let \( r = \max_{g \in G'} r(g) \).

Observation 2. \( \max_{g \in G'} \min_{t \in T'} v_s(g, t) \leq r < v_s(\alpha_s^*, \alpha_c^*) < \min_{t \in T'} \max_{g \in G'} v_s(g, t) \) for every Nash equilibrium \((\alpha_s^*, \alpha_c^*) \) of game \( Z \).

Proof. For every \( g \in G' \), if \( t_g \in \arg \min_{T'} v_s(g, t) \), \( \min_{T'} v_s(g, t) = v_s(g, t_g) h(g, t_g) \leq r(g) \), since \( h(g, t_g) = \alpha_c^*(T') = 1 \). The first inequality holds. For every \( t \in T' \), \( g_t \in G' \) exists such that
\( v_s(g_t, t) < \max_{G'} v_s(g, t) \) and \( \alpha^*_s(g_t) > 0 \) since the support of \( \alpha^*_s \) is not singleton by Observation 1 and the consumer’s preference is generic. Hence, \( v_s(\alpha^*_s, \alpha^*_s) \leq v_s(\alpha^*_s, t) < \max_{G'} v_s(g, t) \). The last inequality holds. Let \( r = v_s(g_r, t_r) h(g_r, t_r) \) for some \( (g_r, t_r) \in G' \times T' \). To complete the proof, it is sufficient to show that \( r < v_s(g_r, \alpha^*_s) \).

\[
v_s(g_r, \alpha^*_s) = \sum_{t \in T_r} v_s(g_r, t) \alpha^*_s(t) + \sum_{t \in T' \setminus T_r} v_s(g_r, t) \alpha^*_s(t) \\
\geq v_s(g_r, t_r) h(g_r, t_r) + \sum_{t \in T' \setminus T_r} v_s(g_r, t) \alpha^*_s(t) \geq v_s(g_r, t_r) h(g_r, t_r) = r.
\] (A.1)

, where \( T_r = \{ t \in T' : v_s(g_r, t) \geq v_s(g_r, t_r) \} \). \( r = v_s(g_r, \alpha^*_s) \) only if \( \sum_{t \in T_r} v_s(g_r, t) \alpha^*_s(t) = v_s(g_r, t_r) h(g_r, t_r) \) and \( \sum_{t \in T' \setminus T_r} v_s(g_r, t) \alpha^*_s(t) = 0 \). We argue that if \( \sum_{t \in T' \setminus T_r} v_s(g_r, t) \alpha^*_s(t) = 0 \), then \( \sum_{t \in T_r} v_s(g_r, t) \alpha^*_s(t) > v_s(g_r, t_r) h(g_r, t_r) \). Whenever \( \sum_{t \in T' \setminus T_r} v_s(g_r, t) \alpha^*_s(t) = 0 \), \( v_s(g_r, t) \geq v_s(g_r, t_r) \) for every \( t \in \text{supp}(\alpha^*_s) \). Since the support of \( \alpha^*_s \) is not singleton by Observation 1, \( t \in \text{supp}(\alpha^*_s) \) exists such that \( v_s(g_r, t) > v_s(g_r, t_r) \). Hence, \( \sum_{t \in T_r} v_s(g_r, t) \alpha^*_s(t) > v_s(g_r, t_r) h(g_r, t_r) \).

\[ \square \]

**Proof of Lemma A.1** Let \((\alpha^*_s, \alpha^*_c)\) be a Nash equilibrium of game \(Z\). The value for \(r\) is computed given \((\alpha^*_s, \alpha^*_c)\). We consider a full-support independent prior \(\mu\) on \(G \times T\) satisfying the followings. The marginal distributions of \(\mu\) on \(G\) and \(T\) are denoted by \(\mu_G \in \Delta(G)\) and \(\mu_T \in \Delta(T)\), respectively.

(a) \(\mu_T(t) = (1 - \epsilon_c)\alpha^*_c(t)\) for every \(t \in \text{supp}(\alpha^*_c)\), and \(\mu_T(T \setminus \text{supp}(\alpha^*_c)) = \epsilon_c\), where \(\epsilon_c = 0\) if \(T = \text{supp}(\alpha^*_c)\), and \(T \neq \text{supp}(\alpha^*_c)\), \(\epsilon_c\) is a sufficiently small positive number.

(b) \(\mu_G(g) = (1 - \epsilon_s)\alpha^*_s(g)\) for every \(g \in \text{supp}(\alpha^*_s)\), and \(\mu_G(G \setminus \text{supp}(\alpha^*_s)) = \epsilon_s\), where \(\epsilon_s = 0\) if \(G = \text{supp}(\alpha^*_s)\), and if \(G \neq \text{supp}(\alpha^*_s)\), \(\epsilon_s\) is a sufficiently small positive number.

Let \(p^* \in P\) be a price such that \(r < p^* < u(\alpha^*_s, \alpha^*_c)\), which exists since \(P\) is fine enough and due to Observation 2. We show that there exists a non-disclosure equilibrium \((\sigma^*_s, \sigma^*_c, \beta^*)\) such that

1. \(\sigma^*_s(g) = (p^*, G')\) if \(g \in G'\), and \(\sigma^*_s(g) = (p^*_g, \{g\})\) if \(g \notin G'\), where \(p^*_g \in \arg \max_{p \in P} p \cdot \mu_T(A(p, \{g\}))\).

2. \(\sigma^*_c(t, p^*, G') = 1\) if \(t \in T'\), and

3. For every \((p, G'')\) such that \((p, G'') \neq \sigma^*_s(g)\) for any \(g \in G\), \(\beta^*(t, p, G'')(w) = 1\) for every \(t \in T\), where \(w \in W(p, G'') = \arg \min_{g \in G'} \mu_T(A(p, \{g\}))\).

Part (1) requires that all and only seller types in \(G'\) pool at price \(p^*\) while all the other types fully disclose. (2) implies that all consumer types in \(T'\) purchase after observing \((p^*, G')\), but does not rule out the possibility that a consumer type not in \(T'\) purchases with the same observation \((p^*, G')\). (3) implies that for every deviation from the seller, the consumer’s belief assigns probability 1 to a common worst-case-type independently of her types. Note that every deviation of the seller is observable to the consumer. Moreover, a worst-case type for such a deviation is well defined since the prior is independent.

First, we claim that all consumer types in \(T'\) purchase after observing \((p^*, G')\). The consumer
believes that all and only types in the $G'$ are possible. For every type $t \in T'$,

$$
\sum_{g \in G'} \frac{\mu_G(g)}{\mu_G(G')} u(g, t) \geq \sum_{g \in G'} \mu_G(g) u(g, t) \geq \sum_{\alpha_t^*(g) > 0} \mu_G(g) u(g, t) = (1 - \epsilon_s) \sum_{\alpha_t^*(g) > 0} \alpha_t^*(g) u(g, t)
$$

$$
= (1 - \epsilon_s) u(\alpha_t^*, t) \geq (1 - \epsilon_s) u(\alpha_t^*, \alpha_t^*) > p^*.
$$

for $\epsilon_s > 0$ small enough if $G \neq \text{supp}(\alpha_t^*)$, and $\epsilon_s = 0$ if $G = \text{supp}(\alpha_t^*)$.

Now, we show that every seller type in $G'$ achieves an equilibrium payoff that is strictly higher than the ones that he can achieve with full disclosure. Consider a seller type $g \in G'$. At equilibrium $(\sigma^*_s, \sigma^*_c, \beta^*)$, $g$ achieves at least a payoff of $(1 - \epsilon_c)p^*$ since every consumer type $t \in \text{supp}(\alpha_t^*) \subseteq T'$ purchases after observing $(G', p^*)$. By fully disclosing and setting a price $p$ such that $u(g, \bar{t}) = p$ for an arbitrary $\bar{t} \in T$, the payoff to seller type $g$ is given by

$$
u(g, \bar{t}) \mu_T(T_g) \leq (1 - \epsilon_c) u(g, \bar{t}) h(g, \bar{t}) + \epsilon_c u(g, \bar{t}) \leq (1 - \epsilon_c) r(g) + \epsilon_c u(g, \bar{t}) \leq (1 - \epsilon_c) r + \epsilon_c u(g, \bar{t}) \leq (1 - \epsilon_c)p^* \quad \text{(A.2)}$$

, where $T_g = \{ t \in T : u(g, t) \geq u(g, \bar{t}) \}$. The first inequality in expression (A.2) holds since $\mu_T(T_g \cap \text{supp}(\alpha_t^*)) \leq \epsilon_c$, and the last inequality holds for $\epsilon_c > 0$ small enough if $T \neq \text{supp}(\alpha_t^*)$, or $\epsilon_c = 0$ if $T = \text{supp}(\alpha_t^*)$.

Finally, we claim that every seller type in $G$ does not have an incentive to deviate. Since the prior is independent, a worst-case type is well-defined for every deviation as is described in part (3). Hence the seller cannot achieve a payoff strictly higher than the ones with full disclosure whenever he deviates.