Dynamics of a fractional three-species food chain system with mixed functional responses and fear effect

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Dynamics of a fractional three-species food chain system with mixed functional responses and fear effect

Yangling Wang · Jinde Cao · Chengdai Huang

Abstract: In this paper, a novel fractional three-species food chain system with Ivlev-type and ratio-dependent functional responses is proposed. Also the fear effect of the middle predator on the prey is taken into account. First the boundedness of the solutions and the existence of positive equilibrium point are investigated for the non-delayed system, and then the delay-induced Hopf bifurcation is dealt with for the delayed system via feedback control method. Some sufficient Hopf bifurcation conditions are obtained based on Hopf bifurcation theorem and stability theorem for fractional dynamical systems. A simulation example is given to support our theoretical results, and especially the effect of the feedback gains and the level of fear on the stability and bifurcation behavior are discussed by illustrations.

Key Words: Fractional three-species food chain system · Ivlev-type and ratio-dependent functional responses · Fear effect · Stability and Hopf bifurcation · Feedback control

1 Introduction

Food chain models formulate the sophisticated relationships among species in natural ecosystems effectively and play a significant role in the fields of ecology and biology. Since Lotka and Volterra originally proposed a mathematical model in [1,2] to describe the interaction between predator populations and their prey populations, more and more modified food chain models have been developed successively by introducing some realistic factors such as sophisticated functional responses [3,4,5] time delay [6] stage structure [7] fear effect [8].

Functional response is one of the key factors involved in predator-prey models to describe the capture rate of a predator to its prey. So far, many kinds of functional responses have been proposed and most of them are prey-dependent. For example, the extensively researched Holling type I, I-II and III functional responses [4] Ivlev type functional response [3] Monod-Haldane type (so called Holling type IV) functional response [5] are all only dependent on the density of preys. However, practically there are competition or sharing of food and other complex relationship among predators in many real biological systems, so the capture rate of a predator would not be determined only by its prey. Arditi et al. put forward in [9] a class of ratio-dependent functional response which is described as a function of the ratio of prey to predator and has been shown more effective in certain cases by some empirical results [10,11]. The dynamics of prey-predator systems with ratio-dependent functional response has attracted great interest of many scholars and plentiful remarkable work has been reported in the past few decades (see [12-15] and the references therein). More recently, some attention has been paid to the study of prey-predator models with mixed functional responses. Roy et al. [16] proposed a class of prey-predator model consisting of prey, predator and generalist predator, and incorporated Holling type II and III functional responses into different prey-predator interaction. The dynamics of discrete prey-predator model with Holling type I and II functional responses were discussed in [17, 18].

A certain experimental study on sparrows [19] showed that the fear of predators led to a 40% reduction in the number of the sparrows. This is because the presence of predators would change the habitats and foraging behaviors of the preys and thus affect their birth rate and survival. Therefore, Wang et al. [20] first incorporated the fear effect term into the prey’s growth. Also they further verified that the level of fear has a significant effect on the stability and Hopf bifurcation of the relevant predator-prey model. Since then, more and more researchers have considered the fear effect of predators in the dynamical study on prey-predator models [21-26]. For instance, Zhang et al. [21] investigated the impact of the fear effect on the stability and Hopf bifurcation for a prey-predator model with Holling type II functional response by selecting the level of fear as the bifurcation parameter. Cong et al. [24] proposed a three-species food chain system consisting of prey, middle predator and top predator with mixed functional responses as follows:

\[
\begin{align*}
\frac{dx(t)}{dt} &= \frac{h_1(t)}{1+\alpha y(t)} - \alpha x(t) - \alpha x^2(t) - \frac{a_1 x(t) y(t)}{1+a_1 h_1 z(t)+a_2 h_2 y(t)} + \frac{a_1 x(t) y(t) h_1 z(t)}{1+a_1 h_1 z(t)+a_2 h_2 y(t)} \\
\frac{dy(t)}{dt} &= -d_1 y(t) + \frac{c_1 a_2 y(t) z(t)}{1+a_1 h_1 z(t)+a_2 h_2 y(t)} - \frac{a_2 h_2 y(t)}{1+a_1 h_1 z(t)+a_2 h_2 y(t)} \\
\frac{dz(t)}{dt} &= c_2 a_2 y(t) z(t) \\
\end{align*}
\]

in which \( \frac{1}{1+\alpha y(t)} \) is the fear effect term and the parameter \( \alpha \) is called the level of fear that drove anti-predator behaviours of the prey. The existence and stability of equilibria as well as the effect of the parameter \( \alpha \) on the species dynamics were systematically investigated for this system in [24].
On the other hand, as is well known to us that time delays universally exist in many practical dynamical systems, especially for biological dynamical systems due to the unavoidable delay during the processes of reaction, maturation, gestation, resource regeneration, and so on. Moreover, it has been manifested that time delay may cause the instability of equilibrium and lead to the fluctuation and even extinction of species populations [27]. In existing relevant literatures, the involved time delays are generally classified into two categories. One is gestation delay, which was proposed by Wangersky et al. [28] in 1957 in the following delayed prey-predator model

\[
\begin{align*}
\dot{x}(t) &= bx(t) - cx^2(t) - k_1 x(t)y(t) \\
\dot{y}(t) &= -d_y(t) + k_2 x(t - \tau)y(t - \tau).
\end{align*}
\]

This model assumes that the growth rate of predator at time \(t\) depends on the number of its prey and of the predator itself at previous time \(t - \tau\) because of the gestation of the predator. The dynamical study on various prey-predator systems with gestation delay has been arousing more and more attention and numerous remarkable results have been achieved (see, for example, [29-31]). The other generally considered time delay is the so-called feedback delay, which reflects the feedback time delay of a species to the growth of the species itself. In 1973, May [6] first introduced feedback delay into the density of the prey species in Lotka-Volterra model and discussed the dynamics of the corresponding delayed system. Afterwards Meng et al. [32] further consider two different feedback delays in a three-species system with Holling type II functional response and investigated the stability and Hopf bifurcation of the following system

\[
\begin{align*}
\dot{x}(t) &= rx(t)[1 - a\frac{y(t)}{k_1} - \frac{a_1 x(t)y_1(t)}{1 + b_1 x(t)} - \frac{a_2 x(t)y_2(t)}{1 + b_2 x(t)}] \\
\dot{y}_1(t) &= y_1(t)[-d_1 + \frac{c_1 x(t)}{1 + b_1 x(t)} - \frac{c_2 x(t)}{1 + b_2 x(t)}] \\
\dot{y}_2(t) &= y_2(t)[-d_2 - G y_2(t - \tau_2) + \frac{c_2 x(t)}{1 + b_2 x(t)}],
\end{align*}
\]

where \(\tau_1\) and \(\tau_2\) represent the feedback delay of the prey and the second predator, respectively. More recently, both gestation delay and feedback delay have been taken into account by many scholars to make the corresponding prey-predator models more realistic [33-37]. For example, Chen et al. [34] investigated the dynamics of the following stage-structured prey-predator system with two time delays and Monod-Haldane response function:

\[
\begin{align*}
\dot{x}(t) &= \rho x(t) - \frac{\rho x(t)}{k_1} - \frac{\beta x(t)}{1 + r x(t)} - d_1 x(t) \\
\dot{y}_1(t) &= -a + d_2 y_1(t) + e_1 \frac{\beta x(t)}{1 + r x(t)} y_1(t - \tau_2) \\
\dot{y}_2(t) &= a y_1(t) - d_3 y_2(t),
\end{align*}
\]

which incorporated the feedback delay \(\tau_1\) and gestation delay \(\tau_2\).

During a long period of time, the dynamical study of predator-prey systems has been focused on integer-order systems. Fractional calculus as a generalization of ordinary integer-order differential and integral, has been proved to be more suitable to describe the genetic properties of real materials and processes in comparison with integer-order system. Nowadays, fractional dynamical systems have been widely applied to biological [38], electromagnetic wave [39], economy [40], system control [41] just name a few. Over the past several years, numerous important research results have been reported for delayed fractional predator-prey models (see 23,25,37,42-45 and the references therein). For example, a fractional three-species food chain system with fear effect and prey refuge was presented in [23], where the authors discussed the dynamics including the existence and stability of the equilibrium point, the non-negativity and boundedness of the solutions for non-delayed model and the delay-induced Hopf bifurcation for delayed model. Huang et al. [42] also investigated the delay-induced Hopf bifurcation for a fractional predator-prey system with ratio-dependent functional response and one single time delay by applying feedback control method.

Motivated by above discussion, we aim to investigate the dynamics of a novel fractional three-species food chain system with mixed functional responses and fear effect. First the boundedness of the solutions and the existence of positive equilibrium point are discussed for the non-delayed system. Then both feedback delay and gestation delay are incorporated into the considered system and the disparate delays-induced Hopf bifurcation are further investigated via feedback control method. Two classes of sufficient Hopf bifurcation conditions are obtained based on stability theorem and Hopf bifurcation theorem. The remainder of this paper is organized as follows. In Section 2, some preliminaries including the definition of Caputo fractional derivative and some lemmas are given for later analysis. In Section 3, the boundedness of the solutions and the existence of positive equilibrium point are studied for the non-delayed system. The Hopf bifurcation conditions induced by the involved two time delays are established in Section 4. In Section 5, a numerical example is provided to illustrate the effectiveness and applications of our theoretical results. Finally, the main content and contributions of this paper are summarized in Section 6.

2 Preliminaries

In this section, we will give the definition of Caputo fractional derivative based on which our prey-predator model is formulated, and then some lemmas required for later analysis.

**Definition 1** [46] Caputo's fractional derivative of order \(\alpha\) for a function \(f(t) \in C^\alpha([t_0, +\infty), R)\) is defined by

\[
D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s) ds,
\]

where \(t_0 \geq t\), \(m - 1 < \alpha \leq m\) (\(m \in N^+\)), \(\Gamma(\cdot)\) is the Gamma function. Especially, when \(0 < \alpha \leq 1\),

\[
D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} f'(s) ds.
\]

When \(f^{(l)}(0) = 0\), \(l = 1, \cdots, m\), the Laplace transform of \(D^\alpha f(t)\) is given by \(L[D^\alpha f(t); s] = s^\alpha F(s)\).

**Lemma 1** (Non-negativity) For a Caputo fractional differential system

\[
D^\alpha X(t) = f(t, X(t)), \quad X(t_0) = X_0,
\]
where $\alpha \in (0,1]$, $f : R \times R^n \rightarrow R^n$ satisfying $f(t,0) = 0$. If $X_0 = 0$, then $X(t) = 0$ is satisfied for $\forall t \geq t_0$.

**proof** If there exists a constant $t_1 > t_0$ such that $X(t_1) = 0$, $X(t_2^+) \leq 0$ and $X(t) > 0$ for $\forall t \in [t_0, t_1)$. Then it follows from $f(t,0) = 0$ that $D^\alpha X(t)_{|t=t_1} = 0$, i.e.,

$$\frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{-\alpha} X'(s) ds = -X(t_0) \frac{(t_1 - t_0)^{-\alpha} - \alpha}{\int_{t_0}^{t_1} (t_1 - s)^{-\alpha+1}} ds = 0,$$

which yields $\int_{t_0}^{t_1} \frac{X(s)}{(t_1 - t_0)^{-\alpha}} ds = -\frac{X(t_0)}{\alpha(t_1 - t_0)^{\alpha}} < 0$. However, it’s obvious that $\int_{t_0}^{t_1} \frac{X(s)}{(t_1 - t_0)^{\alpha}} ds \geq 0$ under the assumption $X(t) > 0$ for $\forall t \in [t_0, t_1)$. Thus, $X(t) > 0$ is satisfied for $\forall t \geq t_0$ by contradiction. This completes the proof.

**Lemma 2** [47] Assume that $D^\alpha W(t) \leq \lambda W(t) + d$, $\lambda, d \in R$, $\lambda \neq 0$, then

$$W(t) \leq \frac{d}{\lambda} + W(0) + \frac{d}{\lambda}E(\lambda^{\alpha}), \; t \geq 0.$$

**Lemma 3** [48] For an autonomous fractional dynamical system

$$D^\alpha x(t) = f(x(t)),$$

in which $\alpha \in (0,1]$, $f : R^n \rightarrow R^n$. Assume $\bar{x}$ is an equilibrium point of it (i.e., $f(\bar{x}) = 0$), denote $J(\bar{x}) = \frac{\partial f}{\partial x}$ the Jacobian matrix of $f(\cdot)$ evaluated around $\bar{x}$. If all eigenvalues $\lambda$ of $J(\bar{x})$ satisfy $|\arg(\lambda)| > \frac{\alpha \pi}{2}$, then the equilibrium point $\bar{x}$ is locally asymptotically stable.

### 3 Dynamical analysis for non-delayed system

In this section, the following fractional three-species prey-predator model with fear effect and mixed functional responses will be considered. The boundedness of the solutions and the existence of positive equilibrium points will be investigated for it.

$$\begin{cases}
D^\alpha x(t) = \frac{sx(t)}{1+py(t)} + a_1 x(t) - ax^2(t) \\
D^\alpha y(t) = (1 - e^{-\beta x(t)}) y(t) - b_1 y(t) - by^2(t) \\
D^\alpha z(t) = \frac{c_1 y(t) z(t)}{y(t) + \gamma z(t)} - cz z(t),
\end{cases}$$

where $q \in (0,1]$, and the biological meaning of the involving variables and parameters are defined in Table 1.

First, the following assumptions (A1) - (A3) are needed for later dynamical analysis.

(A1) $\alpha > \beta, \; c_1 > c_2, \; \delta > a_1$.

(A2) $b_1 + \frac{(c_1 - c_2)}{\gamma c_1} < \min\{\frac{d}{\lambda}, 1\}$.

(A3) $-\frac{b_1}{\gamma c_1} \ln\left(1 - b_1 - \frac{(c_1 - c_2)}{\gamma c_1}\right) < \frac{\delta - a_1}{a_2}$.

**Theorem 1.** All the solutions of system (1) with positive initial conditions $x(0) > 0, y(0) > 0, z(0) > 0$ are uniformly bounded.

**Table 1:** Biological Meaning of the Variables and Parameters for System (2).

<table>
<thead>
<tr>
<th>Variable / Parameter</th>
<th>Biological Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t)$</td>
<td>The population densities of the prey at time $t$</td>
</tr>
<tr>
<td>$y(t)$</td>
<td>The population densities of the middle predator at time $t$</td>
</tr>
<tr>
<td>$z(t)$</td>
<td>The population densities of the super predator at time $t$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>The intrinsic growth rate of the prey</td>
</tr>
<tr>
<td>$\rho$</td>
<td>The level of fear of the middle predator on the prey</td>
</tr>
<tr>
<td>$a_1$</td>
<td>The natural mortality rate of the prey</td>
</tr>
<tr>
<td>$a_2$</td>
<td>The intraspecies competition coefficient of the prey</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>The rate of the conversion of consumed prey to the middle predator</td>
</tr>
<tr>
<td>$\beta$</td>
<td>The rate of trophic absorption of the middle predator</td>
</tr>
<tr>
<td>$b_1$</td>
<td>The natural mortality rate of the middle predator</td>
</tr>
<tr>
<td>$b_2$</td>
<td>The intraspecies competition coefficient of the middle predator</td>
</tr>
<tr>
<td>$b_3$</td>
<td>The capturing rate of the super predator</td>
</tr>
<tr>
<td>$\eta$</td>
<td>The conversion rate of nutrients into the reproduction of the super predator</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>The half capturing saturation constant</td>
</tr>
<tr>
<td>$c_2$</td>
<td>The natural mortality rate of the super predator</td>
</tr>
</tbody>
</table>

**proof** It follows from Lemma 1 that $x(t) > 0, y(t) > 0, z(t) > 0$ for $\forall t \geq 0$. Let $W(t) = x(t) + y(t) + \frac{b_3}{c_2} z(t)$, then combining with the assumption $\alpha > \beta$ one can get

$$D^\alpha W(t) = \frac{\delta x(t)}{1+p y(t)} + a_1 x(t) - a_2 x^2(t) + [e^{-\alpha x(t)} - e^{-\beta x(t)}] y(t) - b_1 y(t) - b_2 y^2(t) - \frac{b_3 c_2}{c_1} z(t) \leq -\frac{(\delta - a_1)}{a_2} x(t) - \frac{b_1}{a_2} y(t) - \frac{b_2}{a_2} y^2(t) - \frac{b_3 c_2}{c_1} z(t) \leq -\eta W(t) + \frac{\delta^2}{a_2},$$

where $\eta = \min\{\delta - a_1, b_1, c_2\}$. Thus it can obtained that $W(t) \leq \frac{\delta^2}{\eta a_2}$, i.e., $x(t) + y(t) + \frac{b_3}{c_2} z(t) \leq \frac{\delta^2}{\eta a_2}$, which means that $x(t), y(t), z(t)$ are all uniformly bounded. This completes the proof.

**Theorem 2.** Under the assumptions (A1) - (A3), the system (1) has at least one positive equilibrium point $(x^*, y^*, z^*)$.

**Proof** The equilibria $(x, y, z)$ of system (1) should satisfy the following equations

$$\begin{cases}
\frac{sx}{1+py} = a_1 x + a_2 x^2 + (1 - e^{-\alpha x}) y \\
(1 - e^{-\beta x}) y = b_1 y + b_2 y^2 + \frac{b_3 z}{c_2} \\
\frac{c_1 y z}{y + \gamma z} = c_2 z
\end{cases}$$

From the third equation of (2) we have $z = \frac{(c_1 - c_2) y}{\gamma c_2}$, substituting which into the second equation yields

$$y = \frac{1}{b_2} \left[1 - b_1 - \frac{b_3 (c_1 - c_2)}{\gamma c_1} - e^{-\beta x}\right].$$
Denote $k_0 = 1 - b_1 - \frac{b_2(c_1-c_2)}{\gamma_1}$ for convenience, then it follows from assumption (A2) that $0 < k_0 < 1$. Combining (3) with the first equation gives
\[
\frac{dx}{dt} = \frac{\delta}{1 + \frac{\delta}{k_0} - \alpha_1 x - \alpha_2 x^2} = \frac{1}{b_2} (1 - e^{-\alpha x})(k_0 - e^{-\beta x}).
\]

Let
\[
C_1: y = g_1(x) = \frac{1}{b_2} (1 - e^{-\alpha x})(k_0 - e^{-\beta x})
\]
\[
C_2: y = g_2(x) = -a_2 x^2 + (\delta - a_1) x
\]
\[
C_3: y = g_3(x) = \frac{dx}{dt} = \frac{\delta}{1 + \frac{\delta}{k_0} - \alpha_1 x - \alpha_2 x^2}.
\]
Obviously, $g_1(0) = g_2(0) = 0$, $g_2'(0) = \frac{(k_0-1)\alpha_1}{b_2} < 0$, $\lim_{x \to +\infty} g_1(x) = \frac{k_0}{b_2} > 0$. Furthermore, note that $\alpha > \beta$, so when $x \geq -\frac{\delta}{\alpha_1} \ln k_0 = -\frac{\delta}{\beta} \ln \frac{k_0 \alpha + \beta}{(\alpha + \beta)} > \frac{1}{b_2} \frac{k_0 \alpha + \beta}{(\alpha + \beta)}$, we can get
\[
g_3'(x) = \frac{1}{b_2} \left[ k_0 \alpha e^{-\alpha x} + \beta e^{-\beta x} - (\alpha + \beta) e^{-(\alpha + \beta) x} \right] > \frac{\alpha \beta}{b_2} k_0 \alpha + \beta - (\alpha + \beta) e^{-(\alpha + \beta) x} > 0.
\]

On the other hand, it’s well known that $C_2$ is a downward parabola intersecting with $x$-axis at two points $(0, 0)$, $k_0 \alpha$, and monotonically decreasing when $x \geq -\frac{\delta \alpha}{\alpha_1} \ln k_0$. Based on assumption (A3), i.e. $-\frac{1}{\beta} \ln k_0 < \frac{\delta \alpha}{\alpha_1}$, and according to the monotonicity and continuity of the functions $g_1(x)$, $g_2(x)$, there must exist an intersection point $(\bar{x}, \bar{y})$ of curves $C_1$ and $C_2$ with $0 < -\frac{1}{\beta} \ln k_0 < \bar{x} < -\frac{\delta \alpha}{\alpha_1}$, $\bar{y} > 0$. For the function $g_1(x)$, obviously $g_1(0) = 0$ and it follows from assumptions (A1), (A3) that
\[
g_1'(0) = \frac{\delta}{1 + \frac{\delta}{k_0} - \alpha_1} > 0,
\]
\[
g_2(0) = \frac{1}{b_2} \left[ k_0 \alpha - \alpha_1 + \frac{a_2}{\beta} \ln k_0 \right] > 0.
\]

Simultaneously it should be noted that when $x \geq -\frac{1}{\beta} \ln k_0$, we have $g_3(x) \leq g_2(x)$. As a result, the curves $C_1$ and $C_3$ must intersect at a point $(x^*, y^*)$ with $0 < -\frac{\delta \alpha}{\alpha_1} \ln k_0 < x^* < -\frac{\delta \alpha}{\alpha_1}$. Consequently it can be obtained that
\[
y^* = \frac{1}{b_2} \left[ 1 - b_2(c_1-c_2)e^{-\beta x^*} \right] = \frac{1}{b_2} (k_0 - e^{-\beta x^*}) > 0
\]
and $z^* = \frac{(c_1-c_2)y^*}{\gamma_1} > 0$. This completes the proof.

By some computation, the Jacobian matrix of system (1) at the equilibrium point $(x^*, y^*, z^*)$ can be obtained as follows:
\[
J = \begin{pmatrix}
h_{11} & h_{12} & 0 \\
h_{21} & h_{22} & h_{23} \\
0 & h_{32} & h_{33}
\end{pmatrix},
\]
where $h_{11} = \frac{\delta}{1 + \frac{\delta}{k_0} - \alpha_1 - 2a_2 x^* - \alpha y^* e^{-\alpha x^*}}, h_{12} = -\frac{\delta \alpha_1}{(1 + \frac{\delta}{k_0})^2} - (1 - e^{-\alpha x^*}), h_{21} = -\frac{\delta}{1 + \frac{\delta}{k_0} - \alpha_1}, h_{22} = \beta y^* e^{-\beta x^*}, h_{23} = 1 - e^{-\beta x^*} - b_1 - 2b_2 y^* - \frac{b_2(c_1-c_2)}{\gamma_1} y^*, h_{32} = \frac{\gamma_1 c_1 y^*}{(1 + \frac{\delta}{k_0})^2}, h_{33} = -c_2 + \frac{c_1 y^*}{(1 + \frac{\delta}{k_0})^2}.
\]
The following assumption (A4) can ensure the local stability of the equilibrium point $(x^*, y^*, z^*)$.

(A4) Assume that all the eigenvalues $\lambda$ of the matrix $J$ satisfy $|\arg(\lambda)| > \frac{\pi}{2}.$

Remark 1. The characteristic equation of $J$ can be specified as
\[
\lambda^3 + M_1 \lambda^2 + M_2 \lambda + M_3 = 0,
\]
in which $M_1 = (-h_{11} + h_{22} + h_{33}), M_2 = h_{11} h_{22} + h_{12} h_{33} + h_{22} h_{33} - h_{12} h_{23} - h_{23} h_{32}, M_3 = h_{12} h_{23} h_{32} + h_{12} h_{21} h_{33} - h_{11} h_{22} h_{33}. If M_1 > 0, M_2 > 0 and M_1 M_2 - M_3 > 0$, then according to Routh-Hurwitz criterion the roots of the equation (4) satisfy $|\arg(\lambda)| \geq \frac{\pi}{2} > \frac{\pi}{2}.$, which can ensure that the equilibrium point $(x^*, y^*, z^*)$ of system (1) is locally stable.

### 4 Hopf bifurcation control for delayed system

In view of the inevitability and importance of time delays, we further take both feedback delay and gestation delay into account in system (1). Simultaneously we will apply feedback controllers to the delayed system to achieve Hopf bifurcation. In this case system (1) changes into the following controlled delayed system:
\[
\begin{align*}
D^q x(t) &= \frac{\delta x(t)}{1 + \rho x(t)} - a_2 x(t) x(t - \tau_1) \\
&= (1 - e^{-\alpha x(t)}) y(t) + U_1 \\
D^q y(t) &= (1 - e^{-\beta (t - \tau_2)}) y(t - \tau_2) - b_1 y(t) \\
&= -b_2 y(t) (t - \tau_1) - \frac{b_2(c_1-c_2)}{\gamma_1} y(t) + U_2 \\
D^q z(t) &= \frac{c_1 y(t - \tau_2) z(t - \tau_2)}{\gamma_1} + U_3 - c_2 z(t) + U_3,
\end{align*}
\]
where $\tau_1 > 0$ denotes the feedback delay of the prey and the gestation delay of the middle predator, $\tau_2 > 0$ denotes the feedback delay of the middle predator and the gestation delay of the super predator. The feedback controllers are designed as $U_1 = K_1[y(t) - x(t - \tau_1)], U_2 = K_2[y(t) - y(t - \tau_2)], U_3 = K_3[z(t) - z(t - \tau_2)]$ and $K_i < 0 (i = 1, 2, 3)$ represent the feedback gains.

Obviously, system (5) has the same equilibrium points as system (1), so $(x^*, y^*, z^*)$ is also one of positive equilibrium points of system (5). By making the variable substitutions $u(t) = x(t) - x^*, v(t) = y(t) - y^*, w(t) = z(t) - z^*$, system (5) is transformed into the following system with one zero equilibrium point $(0, 0, 0)$
\[
\begin{align*}
D^q u(t) &= \frac{\delta u(t)}{1 + \rho (u(t) + y^*)} - a_2 (u(t) + x^*) \\
&= (1 - e^{-\alpha (u(t) + x^*)}) y(t) + u(t) + y^* \\
&= -b_1 (u(t) + y^*) - b_2 (u(t) + y^*) \\
&= (1 - e^{-\beta (u(t) + x^*)}) y(t) + y^* \\
&= -b_1 (u(t) + y^*) - b_2 (u(t) + y^*) \\
&= \frac{c_1 (u(t - \tau_2) + z^*)}{\gamma_1} (t - \tau_2) + y^* \\
&= K_2 y(t) - v(t - \tau_2) \\
&= -c_2 (z(t) + z^*) + K_3 w(t) - w(t - \tau_2).
\end{align*}
\]

Consequently, the stability of the equilibrium point $(x^*, y^*, z^*)$ of system (5) is equivalent to the stability of the zero equilibrium point of system (6).
Linearizing system (6) around the origin (0, 0, 0) yields

\[
\begin{align*}
D^q u(t) &= m_{11} u(t) + m_{12} u(t - 7) + m_{13} v(t) \\
D^q v(t) &= m_{21} u(t - 7) + m_{22} v(t) + m_{23} v(t - 7) + m_{25} w(t) \\
D^q w(t) &= m_{31} v(t - 7) + m_{32} v(t) + m_{33} w(t)
\end{align*}
\]

(7)

where \( m_{11} = \frac{\dot{\delta}}{1+\rho^{\beta_1}} - a_1 - a_2 \delta^{\alpha} - \alpha y + \nu x + K_1, \)
\( m_{12} = -a_2 \delta^{\alpha} - K_1, \)
\( m_{13} = -\frac{\delta}{1+\rho^{\beta_1}} - (1 - e^{\alpha x^*}), \)
\( m_{21} = \beta y e^{-\beta x^*}, \)
\( m_{22} = b_1 - b_2 y^* \frac{\delta}{1+\rho^{\beta_1}} + K_2, \)
\( m_{23} = e^{-\beta x^*}, \)
\( m_{24} = -b_2 y^* - K_2, \)
\( m_{25} = -b_2 y^* \frac{\delta}{1+\rho^{\beta_1}}, \)
\( m_{31} = -c_2 + K_3, \)
\( m_{32} = -c_2 + K_3, \)
\( m_{33} = -c_2 + K_3. \)

By making Laplace transform on both sides of (7), we can obtain the following characteristic equation

\[
\begin{bmatrix}
\Lambda_{11} & \Lambda_{12} & 0 \\
\Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\
0 & \Lambda_{32} & \Lambda_{33}
\end{bmatrix} = 0,
\]

(8)

where \( \Lambda_{11} = s^q - (m_{11} + m_{12} e^{-\gamma_1}), \)
\( \Lambda_{12} = -m_{13}, \)
\( \Lambda_{21} = -m_{21} e^{-\gamma_1}, \)
\( \Lambda_{22} = s^q - (m_{22} + m_{23} e^{-\gamma_1} + m_{24} e^{-\gamma_2}), \)
\( \Lambda_{23} = -m_{25}, \)
\( \Lambda_{32} = -m_{31} e^{-\gamma_1}, \)
\( \Lambda_{33} = s^q - (m_{32} + m_{33} e^{-\gamma_2}). \)

In what follows, we will discuss the stability and Hopf bifurcation of system (5) by taking \( \tau_1 \) and \( \tau_2 \) as the bifurcation parameter respectively.

First, let’s take \( \tau_1 \) as a constant, and write the characteristic equation (8) as follows

\[
T_0(s) + T_1(s) e^{-\gamma_1} + T_2(s) e^{-\gamma_2} = 0,
\]

(9)

which is the key equation to further obtain the \( \tau_2 \)-induced bifurcation criterion, and here

\[
\begin{align*}
T_0(s) &= s^q - [m_{11} + m_{12} + m_{32}] (m_{12} + m_{23}) \\
&+ m_{23} (m_{11} + m_{12}) + m_{12} (m_{12} + m_{23}) \\
&- m_{13} m_{21} + m_{12} m_{23} e^{-\gamma_1}] s^q \\
&+ m_{32} [m_{31} m_{21} - (m_{11} m_{23} + m_{12} m_{23})] \\
&\times e^{-\gamma_1} - m_{12} m_{23} e^{-\gamma_2} - m_{12} m_{23} \\
T_1(s) &= -[m_{24} + m_{34}] s^q + \left[ m_{33} (m_{11} + m_{23}) \\
&+ m_{24} (m_{11} + m_{23}) - m_{31} m_{25} + [m_{33} (m_{12} \\
&+ m_{23}) + m_{12} m_{24} e^{-\gamma_1}] s^q + [m_{33} (m_{13} m_{21} \\
&- m_{13} m_{23}) + m_{12} m_{23} m_{31} - m_{22} m_{33}] \\
&- m_{24} m_{32}] e^{-\gamma_1} - m_{12} m_{23} m_{33} e^{-\gamma_2} \\
T_2(s) &= m_{24} m_{33} s^q - m_{24} m_{23} m_{33} e^{-\gamma_1} \\
&- m_{12} m_{24} m_{33}.
\end{align*}
\]

When \( \tau_1 = 0 \), denote \( T_k(s) |_{\tau_1=0} = \tilde{T}_k(s) \) \( (k = 0, 1, 2) \), then Eq. (9) degenerates into

\[
\tilde{T}_0(s) + \tilde{T}_1(s) e^{-\gamma_1} + \tilde{T}_2(s) e^{-\gamma_2} = 0.
\]

(10)

Multiplying both sides of Eq. (10) by \( e^{\gamma_2} \) gives

\[
\tilde{T}_0(s) e^{\gamma_2} + \tilde{T}_1(s) + \tilde{T}_2(s) e^{\gamma_2} = 0.
\]

(11)

Assume that \( s = i \omega (\omega > 0) \) is a root of Eq. (11). Note that \( i = \cos \frac{\omega}{2} + i \sin \frac{\omega}{2} \), substitute which into \( \tilde{T}_k(i \omega) \) and denote \( \tilde{T}_k(i \omega) = \tilde{A}_k + i \tilde{B}_k, k = 0, 1, 2 \). Accordingly we have

\[
\tilde{A}_0 = \omega q e^{-\omega^2} = \tilde{B}_0 = \omega q e^{-\omega^2},
\]

\[
\tilde{A}_1 = -\omega q e^{-\omega^2} q \tilde{A}_0 + \omega q e^{-\omega^2} \tilde{B}_0 = \tilde{B}_1 = -\omega q e^{-\omega^2} q \tilde{A}_0 + \omega q e^{-\omega^2} \tilde{B}_0.
\]

Separating the real part and imaginary part of Eq. (11) yields

\[
\begin{align*}
\tilde{A}_0 &= \omega q e^{-\omega^2} q \tilde{A}_0 + \omega q e^{-\omega^2} \tilde{B}_0 = \tilde{B}_1 = -\omega q e^{-\omega^2} q \tilde{A}_0 + \omega q e^{-\omega^2} \tilde{B}_0,
\end{align*}
\]

follow which one can get

\[
\begin{align*}
\cos \omega \tau_2 &= -\frac{\tilde{A}_1}{\tilde{B}_1} + \tilde{B}_0, \\
\sin \omega \tau_2 &= \frac{\tilde{A}_1}{\tilde{B}_1} + \tilde{B}_0.
\end{align*}
\]

According to the trigonometric identity \( \sin^2 \omega \tau_2 + \cos^2 \omega \tau_2 = 1 \), we have

\[
\tilde{H}_1^2(\omega) + \tilde{H}_2^2(\omega) = 1.
\]

(12)

Here we assume that Eq. (12) has at least one positive root \( \omega^* \), which can be obtained by using numerical software Maple 13. Consequently, the bifurcation point \( \tau_2^* \) in the case of \( \tau_1 = 0 \) can be determined as

\[
\tau_2^* = \min_{k=0, 1, 2, \cdots} \left( \arccos \tilde{H}_1(\omega^*) + 2k\pi \right).
\]

(13)

This shows that the equilibrium point \((0, 0, 0)\) of system (6) is asymptotically stable when \( \tau_1 = 0 \), \( \tau_2 \in [0, \tau_2^*] \). In the following, we will select a specified \( \tau_2 \) in the interval \([0, \tau_2^*]\), and further take \( \tau_1 \) as the bifurcation parameter. For a fixed \( \tau_2 \), the characteristic equation (8) needs to be rewritten as

\[
Q_0(s) + Q_1(s) e^{-\gamma_1} + Q_2(s) e^{-\gamma_2} = 0
\]

(14)
to obtain the $\tau_1$-induced bifurcation criterion, in which
\[
Q_0(s) = s^3 - (m_1 + m_2 + m_3)s^2 + [m_{11}m_{22} + m_{32}(m_{11} + m_{23})]s - m_{11}m_{22}m_{32}
- (m_2 + m_3)s^2 e^{-s} + m_{33}m_{11}m_{22})
+ m_{24}(m_{11} + m_{23})s^2 e^{-s} e^{-s} + m_{11}m_{22}m_{33} + m_{24}m_{32}m_{31} e^{-s} + m_{12}m_{24}m_{33} e^{-s};
\]
\[
Q_1(s) = -m_{12}m_{23}s^2 + [m_{11}m_{23} + m_{12}m_{22} + m_{32}(m_{12} + m_{23}) - m_{13}m_{23}]s + m_{12}m_{24}m_{33} e^{-s};
\]
\[
Q_2(s) = m_{12}m_{23}s^2 - m_{12}m_{23}m_{32} - m_{12}m_{24}m_{33} e^{-s}.
\]

Multiplying both sides of Eq. (14) by $e^{s\tau_1}$ gives
\[
Q_0(s)e^{s\tau_1} + Q_1(s) + Q_2(s)e^{-s\tau_1} = 0. 
\]  
(15)

Assume that $s = i\omega_0$ ($\omega > 0$) is a root of Eq. (15) and denote $Q_k(i\omega_0) = C_k + iD_k$, $k = 0, 1, 2$, then one can get
\[
C_0 = \frac{\omega^3}{2} \pi - \frac{\omega^2}{2} (m_{11} + m_{22} + m_{32}) \cos \omega = \frac{\omega^3}{2} (m_{11}m_{22} + m_{32}(m_{11} + m_{23})] \cos \omega
- m_{11}m_{22}m_{32} - m_{24}(m_{12} + m_{23}) \cos \omega \pi\tau - m_{33}m_{12}m_{23} - m_{24}m_{32}m_{31} \cos \omega \pi\tau
+ m_{33}m_{11}m_{22} + m_{24}m_{32} + m_{24}m_{32} + m_{24}m_{32} \cos \omega \pi\tau
- m_{11}m_{22}m_{32} - m_{24}m_{32} \cos \omega \pi\tau
\]
\[
D_0 = \frac{\omega^3}{2} \pi - \frac{\omega^2}{2} (m_{11} + m_{22} + m_{32}) \sin \omega
- \frac{\omega^2}{2} (m_{11}m_{22} + m_{32}(m_{11} + m_{23})] \sin \omega
- m_{11}m_{22}m_{32} - m_{24}(m_{12} + m_{23}) \sin \omega \pi\tau
- m_{33}m_{12}m_{23} - m_{24}m_{32}m_{31} \sin \omega \pi\tau
- m_{11}m_{22}m_{32} - m_{24}m_{32} \sin \omega \pi\tau
\]
\[
C_1 = -\frac{\omega^2}{2} (m_{12} + m_{23}) \cos \omega + \frac{\omega^2}{2} (m_{11}m_{22} + m_{12}m_{23} \cos \omega \pi\tau - m_{13}m_{23}]
+ \cos \omega \pi\tau
\]
\[
D_1 = -\frac{\omega^2}{2} (m_{12} + m_{23}) \sin \omega + \frac{\omega^2}{2} (m_{11}m_{22} + m_{12}m_{23} \sin \omega \pi\tau
\]
\[
C_2 = \frac{\omega^2}{2} (m_{12}m_{23} \cos \omega \pi - m_{12}m_{23} \sin \omega \pi\tau
\]
\[
D_2 = \frac{\omega^2}{2} (m_{12}m_{23} \sin \omega \pi - m_{12}m_{23} \cos \omega \pi\tau
\]

Similar to (13), we can obtain the following critical value of bifurcation point $\tau_1$ with the specified $\tau_2$ for system (5)
\[
\tau_1 = \frac{1}{\omega_0} \min_{k=0,1,2,\cdots} \{ 2\pi \cos \omega_0 \theta_0 + 2k\pi \},
\]

where $\theta_0$ is the positive root of the equation
\[
U_1^2(\theta) + U_2^2(\theta) = 1
\]

and
\[
\left\{ \begin{array}{l}
U_1(\theta) = \frac{U_1(\theta_0) + U_2(\theta_0)}{U_1^2(\theta_0) + U_2^2(\theta_0)} \\
U_2(\theta) = \frac{U_2(\theta_0) - U_1(\theta_0)}{U_1^2(\theta_0) + U_2^2(\theta_0)}
\end{array} \right.
\]

(Here we also assume that Eq. (17) has at least one positive root $\theta_0$). Differentiating both sides of Eq.(14) with respect to $\tau_1$ and making some rearrangements gives
\[
\frac{ds}{d\tau_1} = \frac{\kappa_1}{\kappa_2},
\]

in which $\kappa_1 = s e^{-s\tau_1} \left[ Q_1(s) + 2Q_2(s)e^{-s\tau_1} \right]$, $\kappa_2 = Q_0(s) + e^{-s\tau_1} \left[ Q_1(s) - \tau_1 Q_1(s) \right] + e^{-2s\tau_1} \left[ Q_1(s) - 2\tau_1 Q_2(s) \right]$. Denote $Q_k'(s) = \cos \frac{s}{2}\pi - \sin \frac{s}{2}\pi \cos \frac{s}{2}\pi$, $k = 1, 2, 3$, (the expressions of $R_k$, $V_k$ are similar to $C_k$, $D_k$, here we omit them for the sake of space-saving), then one can obtain
\[
Re \left( \frac{ds}{d\tau_1} \right) = \Phi_1 \Psi_1 + \Phi_2 \Psi_2 = \Psi_1^2 + \Psi_2^2,
\]

where
\[
\Phi_1 = C_1 \sin \tau_1 - D_1 \cos \tau_1 + 2C_2 \sin \tau_1 - 2D_2 \cos \tau_1;
\]
\[
\Phi_2 = D_1 \sin \tau_1 + C_1 \cos \tau_1 + 2D_2 \sin \tau_1 + 2C_2 \cos \tau_1;
\]
\[
\Psi_1 = R_0 + (R_1 - \tau_1 C_1) \cos \tau_1 + (V_1 - \tau_1 D_1) \sin \tau_1 + (R_2 - 2\tau_1 C_1) \cos \tau_1 + (V_2 - 2\tau_1 D_1) \sin \tau_1;
\]
\[
\Psi_2 = V_0 + (V_1 - \tau_1 D_1) \cos \tau_1 - (R_1 - \tau_1 C_1) \sin \tau_1 + (V_2 - 2\tau_1 D_1) \cos \tau_1 - (R_2 - 2\tau_1 C_1) \sin \tau_1.
\]

Let $s(\tau_1) = \zeta(\tau_1) + i\delta(\tau_1)$ be the root of Eq.(14) near $\tau_1 = \tau_{10}$ satisfying $\zeta(\tau_{10}) = 0$ and $\delta(\tau_{10}) = \theta_0$. Accordingly the following assumption ($A_5$) would ensure that the transversality condition is satisfied.
\((A_5) \frac{d\Psi + k_2 y}{d\tau} \big| \tau_1 - \tau_{10} = \omega_0 \neq 0. \)

To sum up, we can get the following \(\tau_1\)-induced bifurcation criterion based on Hopf bifurcation theorem.

**Theorem 3.** Suppose that the assumptions \((A_1), (A_5)\) are satisfied, then in the case of \(\tau_2 \in [0, \tau_2^*)\), the positive equilibrium point \((x^*, y^*, z^*)\) of system (5) is asymptotically stable when \(\tau_1 \in [0, \tau_{10})\) and a Hopf bifurcation occurs near \((x^*, y^*, z^*)\) when \(\tau_1 = \tau_{10}\), where \(\tau_2^*\) and \(\tau_{10}\) are defined in (13) and (16), respectively.

Similar to the derivation of \(\tau_2^*\) and \(\tau_{10}\), it’s not difficult to get the following critical value of \(\tau_1\) in case of \(\tau_2 = 0\):

\[
\tau_1^* = \min_{\check{\theta}} \left( \text{arccos} \frac{\hat{U}_I(\check{\theta}) + 2k\pi}{\check{\theta}} \right),
\]

where \(\check{\theta}^*\) is the positive root of the equation

\[
\hat{U}_I(\check{\theta}) + \hat{U}_I(\check{\theta}) = 1,
\]

and \(\hat{U}_I(\check{\theta}) (k = 1, 2)\) can be obtained by letting \(\tau_2 = 0\) and substituting \(\check{\theta}\) for \(\check{\theta}^*\). The equilibrium point \((x^*, y^*, z^*)\) of system (5) is asymptotically stable when \(\tau_1 = \tau_1^*\).

Furthermore, select a \(\tau_1 \in [0, \tau_1^*)\) and fix its value. Denote \(T_k(s) = \frac{\cos \tau + \sin \tau}{\text{i} \omega_0}\), where \(A_k, B_k\) are dependent on \(\omega_1\) and \(\tau_1\). Obviously, \(A_k(\omega), B_k(\omega) (k = 1, 2)\) are just the expressions by letting \(\tau_1 = 0\) and substituting \(\check{\theta}\) for \(\check{\theta}^*\) in \(A_k, B_k (k = 1, 2)\), which are omitted here in view of the similarity and space saving. Accordingly, one can get

\[
\tau_{20} = \min_{\omega_0} \left( \text{arccos} \frac{\hat{H}_I(\omega_0) + 2k\pi}{\check{\theta}} \right),
\]

where \(\omega_0\) is the positive root of the equation

\[
\hat{H}_I(\omega) + \hat{H}_I(\omega) = 1,
\]

and

\[
\hat{H}_I(\omega) = -\frac{A(0) - A_0 + B(0) - B_2}{A_0 + B_1 - A_1 + B_2}, \quad \hat{H}_I(\omega) = \frac{A(0) + B_1 - A_0 + B_2}{A_0 + B_1 - A_1 + B_2}.
\]

By differentiating both sides of Eq.(9) one can get the derivative of \(s\) with respect to \(\tau_2\), which is as follows

\[
\frac{ds}{d\tau_2} = \frac{\Gamma_1 + i\Gamma_2}{\Gamma_1 + i\Gamma_2},
\]

where

\[
\Gamma_1 = A_1 \sin \omega_2 \tau - B_1 \cos \omega_2 \tau + 2A_2 \sin 2\omega_2 \tau - 2B_2 \cos 2\omega_2 \tau,
\]

\[
\Gamma_2 = B_1 \sin \omega_2 \tau + A_1 \cos \omega_2 \tau + 2B_2 \sin 2\omega_2 \tau + 2A_2 \cos 2\omega_2 \tau,
\]

\[
\Upsilon_1 = P_0 + \left( P_1 - \tau_2 A_1 \right) \cos \omega_2 \tau + \left( S_1 - \tau_2 B_1 \right) \sin \omega_2 \tau + \left( P_2 - 2\tau_2 A_2 \right) \cos 2\omega_2 \tau + \left( S_2 - 2\tau_2 B_2 \right) \sin 2\omega_2 \tau,
\]

\[
\Upsilon_2 = S_0 + \left( S_1 - \tau_2 B_1 \right) \cos \omega_2 \tau - \left( P_1 - \tau_2 A_1 \right) \sin \omega_2 \tau + \left( S_2 - 2\tau_2 B_2 \right) \cos 2\omega_2 \tau - \left( P_2 - 2\tau_2 A_2 \right) \sin 2\omega_2 \tau,
\]

and \(P_k, S_k (k = 0, 1, 2)\) are denoted as the real part and imaginary part of \(T_k(\omega) = e^{\text{i} \omega \tau_2} \text{sin} \frac{\omega_2 \tau_2}{2}\), respectively. Based on Hopf bifurcation theorem, the following transversality condition is needed to support the \(\tau_2\)-induced bifurcation criterion.

\[(A_6) \frac{\Upsilon_1 + \Upsilon_2}{\Upsilon_1 + \Upsilon_2} \big| \tau_2 = \tau_{20}, \omega = \omega_0 \neq 0. \]

Subsequently, the \(\tau_2\)-induced bifurcation criterion can be formulated as follows.

**Theorem 4.** Suppose that the assumptions \((A_2), (A_6)\) are satisfied, then in the case of \(\tau_1 \in [0, \tau_1^*)\), the positive equilibrium point \((x^*, y^*, z^*)\) of system (5) is asymptotically stable when \(\tau_2 \in [0, \tau_{20})\) and a Hopf bifurcation occurs near \((x^*, y^*, z^*)\) when \(\tau_2 = \tau_{20}\), where \(\tau_1^*\) and \(\tau_{20}\) are defined in (19) and (20), respectively.

**Remark 2.** In [14], the authors investigated the stability and Hopf bifurcation for a single integral-three-species food chain model by exerting a delayed feedback controller on the middle predator, i.e., \(K_1 = K_3 = 0, K_2 \neq 0\). Huang et al. recently incorporated two different delayed feedback controllers into a fractional-order two-species prey-predator model with one single time delay in [42]. Obviously, in this paper we make a generalization in both model and bifurcation control method.

**5 Simulation example**

In this section, we will give an example to show the application of the presented theoretical results above. Especially, the effect of the feedback gains and the level of fear on the stability is displayed through illustration.

Consider the following controlled prey-predator model

\[
D^{0.95} x(t) = \left\{ \begin{array}{l}
\frac{4x(t)}{1 + 0.085 t} - 0.4x(t) - 0.5x(t) \\
\times x(t - \tau_1) - (1 - e^{-1.1x(t)}) y(t) \\
-0.2\left[ x(t) - x(t - \tau_1) \right] \\
\end{array} \right.
\]

\[
D^{0.95} y(t) = \left\{ \begin{array}{l}
-0.1 y(t)(y(t) - 0.3y(t + 2)) \\
-0.2 \left[ y(t) - y(t - \tau_2) \right] \\
\end{array} \right.
\]

\[
D^{0.95} z(t) = \left\{ \begin{array}{l}
0.5z(t)(z(t) - 0.35z(t) - 0.35z(t) \\
-0.3 \left[ z(t) - z(t - \tau_2) \right]. \\
\end{array} \right.
\]

It can be easily verified that the model parameters satisfy the assumptions \((A_1)-(A_4)\), and thus we can obtain a positive equilibrium point \((x^*, y^*, z^*) = (1.7258, 2.8213, 0.8061)\).

When \(\tau_1 = 0\), then by using numerical software Maple 13 we can obtain \(\check{\theta} = 0.9474\) and \(\tau_2^* = 1.7895\). Fixing the value of \(\tau_2 = 1.4\) which belongs to the interval \([0, \tau_2^*)\), we can further get \(\check{\theta} = 1.8827, \tau_{10} = 0.3276\) according to (16), and simultaneously verify the assumptions \((A_4), (A_5)\) are satisfied. Thereupon, on the basis of the Theorem 3, the equilibrium point \((x^*, y^*, z^*)\) is asymptotically stable when \(\tau_1 \in [0, 0.3216]\) and a Hopf bifurcation emerges when \(\tau \geq 0.3276\). Selecting the initial values as \((x(0), y(0), z(0)) = (1.5, 3, 0.5)\), then one can obtain the simulation plots and phase portraits of system (22) depicted in Figs. 1-2 for \(\tau_1 = 0.28\) and \(\tau_1 = 0.33\) with \(\tau_2 = 1.4\), respectively. It can be
seen from Fig. 2 that Hopf bifurcation occurs when \( \tau = 0.33 > \tau_{10} \).

\[ \begin{align*}
\text{Fig. 1: The waveform plots and phase portrait of system (22)} \\
\text{when } \tau_1 = 0.28 < 0.3276 \text{ with } \tau_2 = 1.4.
\end{align*} \]

On the other hand, if \( \tau_2 = 0 \), then we can obtain \( \hat{\theta}_0 = 0.8338, \tau^*_1 = 0.3025 \). Selecting \( \tau_1 = 0.2 \in [0, \tau^*_1] \), based on (20) we further get \( \omega_0 = 0.3882, \tau_{20} = 3.1847 \). It can be verified that the assumption \((A_4)\) is satisfied, then according to the Theorem 4 the equilibrium point \((x^*, y^*, z^*)\) is asymptotically stable when \( \tau_1 < 3.1847 \). The waveform plots and phase portrait of system (22) are illustrated in above Fig. 3 and Fig. 4 for \( \tau_2 = 2.7 \) and \( \tau_2 = 3.3 \) with \( \tau_1 = 0.2 \), respectively. It can be seen from Fig. 4 that Hopf bifurcation occurs when \( \tau_2 = 3.3 > \tau_{20} \).

\[ \begin{align*}
\text{Fig. 2: The waveform plots and phase portrait of system (22)} \\
\text{when } \tau_1 = 0.33 > 0.3276 \text{ with } \tau_2 = 1.4.
\end{align*} \]

\[ \begin{align*}
\text{Remark 3.} \text{ To explore the effect of the feedback gains on the stability, we let } K_1 = 0, K_2 = 0.2, K_3 = 0.3; \ K_1 = 0.2, K_2 = 0.3 \text{ and } K_1 = 0.2, K_2 = 0.2, K_3 = 0 \\
\text{in turn and select } \tau_1 = 0.28, \tau_2 = 1.4. \text{ From the obtained simulation plots Figs. 5-7 one can see that the equilibrium point } \text{of system (22) keeps sable when } K_1 = 0 \text{ or } K_3 = 0, \\
\text{while changes from stable to unstable when } K_2 = 0. \text{ This shows that the feedback controller exerted on the middle predator affects the stability and Hopf bifurcation of the controlled food chain system mostly.}
\end{align*} \]

\[ \begin{align*}
\text{Fig. 3: The waveform plots and phase portrait of system (22)} \\
\text{when } \tau_2 = 2.7 < 3.1847 \text{ with } \tau_1 = 0.2.
\end{align*} \]

\[ \begin{align*}
\text{Fig. 4: The waveform plots and phase portrait of system (22)} \\
\text{when } \tau_2 = 3.3 > 3.1847 \text{ with } \tau_1 = 0.2.
\end{align*} \]

\[ \begin{align*}
\text{Fig. 5: The waveform plots of } x(t) \text{ for system (22) when one of the feedback gains is equal to } 0 \text{ with } \tau_1 = 0.28, \tau_2 = 1.4.
\end{align*} \]

\[ \begin{align*}
\text{Fig. 6: The waveform plots of } y(t) \text{ for system (22) when one of the feedback gains is equal to } 0 \text{ with } \tau_1 = 0.28, \tau_2 = 1.4.
\end{align*} \]
Remark 4. The impact of fear effect on the population densities and stability has been discussed through numerical simulations or theoretical analysis in [21-26]. To get an insight into the role of fear effect, we adopt the values of parameters in (22) except changing the fear level \( \rho \) from \( \rho = 0.18 \) to \( \rho = 0.24, 0.3, 0.43, 0.62 \) successively with \( \tau_1 = 0.2, \tau_2 = 3.3 \). It can be seen from the following Figs. 8-10 that the value of the positive equilibrium point \( (x^*, y^*, z^*) \) becomes smaller as the fear level \( \rho \) increases, which is consistent with the theoretical result in [21]. On the other hand, it’s also discovered that the bigger \( \rho \) is the earlier the state trajectories achieve stability.

6 Conclusion

This paper is mainly concerned with the dynamics of a fractional three-species food chain model with fear effect and mixed functional responses. First the boundedness of solutions and the existence of positive equilibrium point are studied for the non-delayed system. Then the delay induced Hopf bifurcation is investigated for the delayed system with both feedback delay and gestation delay. By taking the involved two different delays as the bifurcation parameter respectively, we obtained two classes of Hopf bifurcation conditions based on theorem of linear autonomous system as well as Hopf bifurcation theorem. Also the effect of the feedback gains and the level of fear on the stability is discussed through numerical simulations.

The main contribution of our work contains three aspects. (1) Two different types of functional responses are incorporated into our food chain model, i.e., Ivlev type functional response of the middle predator to the prey and ratio-dependent functional response of the super predator to the middle predator, this improves the single-functional response in most of existing relevant literatures. Moreover, the fear effect of the middle predator on the prey, both feedback delay and gestation delay are all taken into account in view of the complex characteristics of real biological systems. (2) The boundedness of solutions and the existence of positive equilibrium point are discussed in detail for the considered non-delayed system, and then two disparate delays-induced conditions are presented to ensure the occurrence of Hopf bifurcation based on feedback control method for the delayed system. The critical value of bifurcation points can be easily obtained with the help of the Maple 13 computing software. (3) The impact of the feedback gains and fear effect on the stability and Hopf bifurcation are explored by illustrations. It’s discovered that the feedback gain \( K_2 \) of the middle predator has the most effect on the Hopf bifurcation for the controlled food chain system. That is, if there is no feedback controller exerted on the middle predator, Hopf bifurcation would occur earlier. On the other hand, it’s also can be found that the value of the positive equilibrium point \( (x^*, y^*, z^*) \) becomes smaller as the fear level \( \rho \) increases. Moreover, the bigger \( \rho \) is the earlier the state trajectories achieve stability.
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Declarations

Conflict of interest statement We declare that we have no conflict of any financial and personal relationships with other people or organizations.

Availability of data and material All data generated or analysed during this study are included in this article.

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