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Exact solutions of flow and pressure variation inside a horizontal filter chamber using Lie symmetry analysis

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Abstract

Physical model exact solutions improve crucial industrial production processes by giving operators and designers a greater grasp of how systems operate in practice. The current case study aims to find exact momentum and pressure variation solutions during the unsteady-state filtration process to advance fluid purification. Lie symmetry analysis is used to transform a system of PDEs representing the flow and pressure variation into solvable ODEs without changing the dynamics of the case study. The velocity (momentum) and pressure solutions are then determined by integrating the obtained solvable ODEs. The effects of physical parameters resulting from the process dynamics are examined using the exact solutions acquired to identify the parameter combination that results in the maximum permeates outflow. Graphical representations of the flow velocity and pressure are presented and analysed as well as the skin friction tabular. The results indicate that as time evolves, permeate outflow decreases because internal momentum, work done, and pressure diminish over time due to the clustering of particles. In addition, low wave speed, small porosity, more chamber space, high injection rate, and stronger magnetic effects are ideal for minimising the effects of no-slip condition at the bottom wall during operation.

Keywords: Unsteady filtration process, Horizontal filter chamber, Porous medium, Magnetic effects, Lie symmetry analysis, Exact solutions, Pressure driven flow

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### NOMENCLATURE

- $t$: time
- $x$: axial coordinate
- $y$: normal coordinate
- $u$: axial velocity
- $v$: normal velocity
- $P$: internal pressure
- $l$: chamber length
- $h$: chamber height
- $\bar{t}$: non-dimensional time
- $\bar{x}$: non-dimensional axial coordinate
- $\bar{y}$: non-dimensional normal coordinate
- $\bar{u}$: non-dimensional axial velocity
- $\bar{v}$: non-dimensional normal velocity
- $\bar{P}$: non-dimensional internal pressure
- $\bar{L}$: non-dimensional length
- $\bar{H}$: non-dimensional height
- $U_w$: velocity at wall
- $R_e$: Reynolds number
- $K$: non-dimensional permeation parameter
- $k$: permeability
- $B_0$: magnetic field

### Greek symbols

- $\nu$: kinetic viscosity
- $\sigma$: electrical conductivity
- $\phi$: porosity parameter
- $\rho$: fluid density
- $\psi$: stream function

### Subscripts

- $w$: wall condition
- $h$: filter height
1. Introduction

There are different fundamental elements to consider when planning a piece of machine for industrial use. The defilements and contamination found in numerous modern liquids shift in boiling temperatures, heaps, and sizes. Unrefined petroleum, for instance, includes both short and long hydrocarbons with various breadths and boiling temperatures. When manufacturing such ointments, the most significant errand is to isolate contaminations from the liquid to accomplish ideal execution, which prompts the best distillate liquids that satisfy guideline necessities for an explicit liquid nature while guaranteeing the most minimal conceivable expense and energy use. Thus, it is ideal for further fostering a mathematical model that gives superior information on the framework’s elements to manufacture such purified liquids.

To advance the application of modern liquid flow, research has been led to explore the effects of stream elements and intensity understanding of various factors on the number of flow and heat transfer systems. For example, Poiseuille and Couette-type flows have applications in developing modern liquids in different channels. Such flows (Poiseuille and Couette) are fundamental when the industry cleans liquid during the liquid sanitisation and sifting processes. To have the best plan that ideally benefits the modern requirements is basic. However, it can be challenging to accomplish. Different researchers expect to feature logical proofs that help engineers and designers have a viable assembling interaction of the machines and work on ensuring well-informed plans. One significant plan is to understand various channel flows and utilise them in designing a few modern liquid stream machines.

In the past and recent, various research works have been done to understand flow, mass and heat transfer in different channels to obtain scientific evidence which improves real-world phenomena. Experimental studies of the pipe flow go back to the 1840s when Poiseuille established the relationship between the volumetric efflux rate of fluid from the tube, the driving pressure differential, the tube length and the tube diameter, see the following references [1–3] for more information. The distinction between laminar, transitional and turbulent regimes for fluid flows was done by Stokes [4] and later was popularised by Reynolds [5].

In pursuit to have a theoretical understanding of mass and heat transfer inside different channels Srinivas et al. [6] have studied the influence of chemical reaction on the MHD flow of a nanofluid in an expanding or contracting porous pipe in the presence of heat source/sink. The authors found that the temperature increases with an increase in Hartmann number and Prandtl number for the case of a heat source, while it decreases for the case of a heat sink. Their work also found that as the temperature significantly increases with an increase in the Brownian motion parameter and the thermophoresis parameter for both the cases of wall expansion and contraction combined when fluid is injected into the system. The works of Cheviakov [7] have analytically studied the dynamics of two non-mixing fluids inside a horizontal channel under shallow water approximation. They obtained exact solutions corresponding to periodic, solitary and kink-type bidirectional travelling waves, which, from the Reynolds equation with a Dirichlet boundary condition for the pressure, were found to
be in better agreement with engineering practice. The study of asymptotic pressure-driven flow in a thin domain was studied by Fabricius et al. [8]. They discovered that by prescribing the external pressure as normal pressure, the Dirichlet condition for a limit pressure was discovered.

Lie group method was used by Onanugaa and Oyetolab [9] to investigate the convective heat and mass transfer of pressure-driven hydromagnetic flow past an inclined fixed permeable surface under the influence of a magnetic field. It was perceived from their work that an increase in the magnetic field parameter, Hartmann number or degree of inclination of the magnetic field decreases the flow velocity and pressure. It was also observed that the velocity, pressure, and temperature distributions gradually increase as the heat source increases. Numerical analysis of a three-dimensional pressure-driven flow inside a channel bounded by one and two parallel porous walls was studied by Liu and Prosperetti [10]. The authors found that the values of the slip coefficient for pressure-driven and shear-driven flow were different and depended on the Reynolds number. Analysis of unsteady pulsatile pressure-driven flow through an elliptic cylindrical microchannel with the Navier slip condition was examined by Chuchard et al. [11] using the Mathieu and modified Mathieu functions. Recently analytical solutions to the pressure-driven flow of the Oldroyd-B fluid inside a variable narrow-shaped channel were obtained by Boyko and Stone [12] using the perturbation expansion method to investigate the relationship between the flow rate and pressure drop. The study found that pressure decreases for narrow contracting channels due to shear stress.

The Lie symmetry theory of infinitesimal transformations is a traditional method to determine a class of similarity reductions of non-linear differential equations. Various scientists and applied mathematicians [13–19] have concentrated their efforts on showing, proving and explaining the advantages of using the Lie symmetry technique. In their works, they have shown how the Lie symmetry technique could be used to reduce the mathematical model and transform the governing equations to those that can be solved easily, particularly in fluid dynamics. The works of Lekoko et al. [20, 21] used Lie group method along with the spectral method to study the filtration process.

Dissecting mathematical models representing different flow streams, and heat and mass transfer issues have been an area of logical interest. These interests have been especially for scientists and engineers aiming to figure out the elements which lead to the best possible filtration process, thus seeking to find a channel that accomplishes the ideal filtration. The investigation of heat and mass transfer dynamics has prompted logical investigations that have assisted engineers with creating successful optimal channels for various industrial use. Motivated by the literature above, the current study aims to advance the filtration process by further investigating the unsteady state regime of the filtration process studied by [20, 21].

The literature above shows that none of these studies have found exact solutions in pursuit of understanding unsteady internal flow and pressure variation during filtration. Thus, the present case study uses the Lie symmetry approach to compute exact solutions to study internal flow and pressure variation inside a filter chamber. Thereafter, use the obtained exact solutions to analyse the dynamics of an unsteady state filtration regime to advance
the filtration process. This study further investigates the filtration process to enhance fact-finding that improves the purification of fluids. The study’s main focus is to address the knowledge gap in the literature by using the Lie symmetry analysis to transform the PDEs representing flow and pressure variation into a more manageable and solvable system of ODEs that can be solved analytically. The significance of the findings of this current case study will aid the engineers and designers with better knowledge of which essential parameter leads to an optimal filter design.

2. Mathematical formulation of the case study

Figure 1 illustrates the schematic representation of the filter design under investigation. The filter design is such that it is horizontal and it injects contaminated incompressible fluid through the top permeable wall with a porous medium embedded inside the filter chamber as the second line of defence. The magnetic force is integrated as the third line of defence to restrict particles which react to magnetic force not to enter inside the chamber. The permeable top wall moves at constant velocity $U_w$ to minimise the potential of filter cake formation. In contrast, the impermeable bottom wall is fixed. The system is such that fluid injection into the chamber happens when $t > 0$, and this constant vertical injection is perpendicular to the top wall. The work done by injection and pulling effects of magnet inside the chamber is such that the top wall creates pressure at the wall $P_w$, which pushes the internal fluid so that the system maintains either maximum or minimum pressure at the bottom of the chamber.

![Figure 1: Schematic representation of the filter design.](image)
2.1. Governing equations and boundary conditions

Based on the presumptions which arise from the filter design, the system of equations describing the flow and pressure variation inside a filter chamber is given by

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] - \frac{\nu \phi u}{k} - \frac{\sigma B_0^2}{\rho} u, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= \frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] - \frac{\nu \phi v}{k},
\end{aligned}$$

where $u(t, x, y)$ is axial-velocity, $v(t, x, y)$ is normal velocity, $\rho$ is density, $P(t, x, y)$ is pressure, $\nu$ is kinetic viscosity, $\sigma$ is the electrical conductivity, $B_0$ is the induced magnetic field, $\phi$ is the porosity parameter and $k$ is the permeability of the porous medium.

The appropriate boundary conditions according to the filter design (Figure 1) are as follows:

(i) $u = U_w, \quad v = -U_w, \quad P = P_w,$ at $y = h,$

(ii) $u = 0, \quad v = 0,$ at $y = 0.$

3. Solution method

This section aims to represent the dimensional system of equations (1) and boundary conditions (2) describing the problem at hand into the simpler and solvable dimensionless representation of the case study using Lie symmetry analysis. Thereafter, we integrate the solvable system of equations together with boundary conditions to obtain exact solutions of momentum and pressure variation inside the filter chamber.

3.1. Dimensional analysis

This subsection represents the system of equations describing the current case study in dimensionless form. The non-dimensional variables are

$$\begin{aligned}
\bar{u} &= \frac{u}{U_w}, \quad \bar{v} = \frac{v}{U_w}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{P} = \frac{P}{\rho U_w^2}, \\
\bar{t} &= \frac{t U_w}{L}, \quad N = \frac{\sigma h B_0^2}{\rho U_w^2}, \quad \frac{1}{K} = \frac{\nu \phi L}{k U_w},
\end{aligned}$$

where $\bar{u}$ and $\bar{v}$ are the dimensionless velocity components, $\bar{P}$ is dimensionless pressure, $\bar{t}$ is dimensionless time, $\bar{x}$ and $\bar{y}$ are the dimensionless space coordinates, $L$ is the characteristic length, $U_w$ is the upper wall velocity.

Substituting non-dimensional quantities (3) into a system of equations (1) and boundary equations (2) and dropping the bars for simplicity yields the following non-dimensional representation of the current case study.
where $R_e = \frac{hU_w}{\nu}$ is the Reynolds number, $K = \frac{kU_w}{\nu \phi h}$ is the permeation parameter and $N = \frac{\sigma h B_0^2}{\rho U_w}$ is the Stuart number.

The boundary conditions take the following form

\begin{align*}
(i) & \quad u = 1, \quad v = -1, \quad p = 1, \quad \text{at} \quad y = \frac{h}{L} = H, \\
(ii) & \quad u = 0, \quad v = 0, \quad \text{at} \quad y = 0.
\end{align*}

According system model (4), there exists a non-dimensional stream velocity $\psi(t, x, y)$ such that

\begin{align*}
u_x = \frac{\partial \psi}{\partial y}, \quad \nu_y = -\frac{\partial \psi}{\partial x},
\end{align*}

which satisfies continuity equation from system (4) identically. Thus, dimensionless system (4) becomes

\begin{align*}
\psi_y t + \psi_y \psi_{xy} - \psi_x \psi_{yy} + P_x - \frac{1}{R_e} [\psi_{xxy} + \psi_{yy}] + \frac{1}{K} \psi_y + N \psi_y = 0, \\
\psi_x t + \psi_y \psi_{xx} - \psi_x \psi_{xy} + P_y - \frac{1}{R_e} [\psi_{xy} + \psi_{xxx}] + \frac{1}{K} \psi_x = 0,
\end{align*}

The dimensionless boundary conditions become

\begin{align*}
(i) & \quad \psi = y - x + 1, \quad p = 1, \quad \text{at} \quad y = \frac{h}{L} = H, \\
(ii) & \quad \psi = 0, \quad \text{at} \quad y = 0.
\end{align*}

The stream flow $\psi$ above satisfies the internal positive flow stream according to dimensionless model representing the current case study in a region $x, y > 0$ when $y - x + 1 > 0$ during operation when $t > 0$.

### 3.2. Symmetry reduction

In this subsection, we use Lie groups of symmetries to map momentum and pressure variations equations during the filtration process into a solvable system of equations.

Let a one-parameter Lie-group of infinitesimal transformation in $(x, y, t, \Psi, P)$ be

\begin{align*}
x^* = x + \varepsilon \phi(x, y, t, \psi, P) + 0(\varepsilon^2),
\end{align*}
\begin{align*}
y^* &= y + \epsilon \zeta(x, y, t, \psi, P) + O(\epsilon^2), \\
t^* &= t + \epsilon F(x, y, t, \psi, P) + O(\epsilon^2), \\
\psi^* &= \psi + \epsilon \eta(x, y, t, \psi, P) + O(\epsilon^2), \\
P^* &= P + \epsilon \xi(x, y, t, \psi, P) + O(\epsilon^2),
\end{align*}
\quad (9)

with \( \epsilon \) as a small parameter. In view of Lie’s algorithm, the infinitesimal operators are given by the vector field
\[
X = \phi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + F \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial \psi} + \xi \frac{\partial}{\partial P},
\quad (10)
\]
if they leave, the flow dynamics of the current case study invariant under the transformation \((x, y, t, \psi, P) \rightarrow (x^*, y^*, t^*, \psi^*, P^*)\).

The exact solutions of stream velocity \( \psi = (x, y, t) \) and pressure \( P = P(x, y, t) \), are invariant under the symmetry (10), if the vector field \( X \) satisfies
\[
X[3](\Delta_j)|_{\Delta_{j=0}} = 0, \quad j = 1, 2,
\quad (11)
\]
where
\[
X[3] = \phi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + F \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial \psi} + \xi \frac{\partial}{\partial P} + \eta^x \frac{\partial}{\partial \psi^x} + \eta^y \frac{\partial}{\partial \psi^y} + \xi^x \frac{\partial}{\partial P_x} + \xi^y \frac{\partial}{\partial P_y} \\
+ \eta^{xy} \frac{\partial}{\partial \psi_{xy}} + \eta^{xt} \frac{\partial}{\partial \psi_{xt}} + \eta^{yt} \frac{\partial}{\partial \psi_{yt}} + \eta^{xx} \frac{\partial}{\partial \psi_{xx}} + \eta^{yy} \frac{\partial}{\partial \psi_{yy}} + \eta^{xy} \frac{\partial}{\partial \psi_{xy}} \\
+ \eta^{yy} \frac{\partial}{\partial \psi_{yy}} + \eta^{xx} \frac{\partial}{\partial \psi_{xx}} + \eta^{yy} \frac{\partial}{\partial \psi_{yy}}
\quad (12)
\]
is the third prolongation of \( X \) and
\[
\Delta_1 = \Psi_{yt} + \Psi_y \Psi_x - \Psi_x \Psi_{yy} + P_x - \frac{1}{Re} [\Psi_{xyy} + \Psi_{yy}] + \frac{1}{K} \Psi_y + N \Psi_y = 0, \\
\Delta_2 = \Psi_{xt} + \Psi_x \Psi_y - \Psi_x \Psi_{xx} + P_y - \frac{1}{Re} [\Psi_{xxy} + \Psi_{xx}] + \frac{1}{K} \Psi_x.
\]

Solving the resulting equations from (11) yields the following five Lie point symmetries spanned by system (7):
\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = F_1(t) \frac{\partial}{\partial \psi}, \quad X_5 = F_2(t) \frac{\partial}{\partial P}. \quad (13)
\]
For unsteady behaviour, we firstly consider the linear combination of \( X_1 \) and \( X_3 \), which gives \( \frac{\partial}{\partial t} + c \frac{\partial}{\partial y} \), where \( c \) is the permeates travelling wave speed. The resulting characteristic equations give the following group invariants:
\[
\psi = \psi(s, x), \quad p = p(s, x), \quad s = y - ct \quad \text{and} \quad x = x. \quad (14)
\]
\[8\]
Substituting the above invariants \((14)\) into system \((7)\) yields
\[
\begin{align*}
\frac{c}{Re} \psi_{ss} + \frac{\psi_{xx}}{Re} - \frac{\psi_{s}}{K} - N\psi_{s} - p_{x} - \psi_{x}\psi_{xs} + \psi_{ss}\psi_{x} &= 0, \\
\frac{c}{Re} \psi_{xs} + \frac{\psi_{xxx}}{Re} - \frac{\psi_{x}}{K} + p_{s} + \psi_{x}\psi_{xs} - \psi_{s}\psi_{xx} &= 0,
\end{align*}
\] (15)

here \(\psi\) and \(p\) are stream velocity and pressure, respectively, while \(s\) and \(x\) are new variables that affect filtration process dynamics.

Similarly, the above system \((15)\) admits the following four Lie point symmetries:
\[
X_{1} = \frac{\partial}{\partial s}, \quad X_{2} = \frac{\partial}{\partial x}, \quad X_{3} = \frac{\partial}{\partial P}, \quad X_{4} = \frac{\partial}{\partial \psi}.
\] (16)

Thus, the linear combination of symmetries \(X_{1}\) and \(X_{2}\), \(X = \frac{\partial}{\partial s} + \frac{\partial}{\partial x}\), yields the following invariants:
\[
\psi = f(g), \quad p = z(g) \quad \text{and} \quad g = x - s,
\] (17)

which further reduces system \((7)\) into the following system of ODEs:
\[
\begin{align*}
2c f''(g) - \frac{4f''(g)}{Re} + f'(g) + N f'(g) - z'(g) &= 0, \\
2c f''(g) - \frac{4f''(g)}{Re} + 2f'(g) + z'(g) &= 0.
\end{align*}
\] (18) (19)

Here \(f(g)\) is velocity, \(z(g)\) is the pressure, and \(g\) is the combination of chamber space and time variables during the unsteady filtration process.

3.3. Closed-form solutions

This subsection aims to establish exact solutions to study momentum and pressure variation during the unsteady filtration process.

Adding equations of momentum along the \(x\) \((18)\) and along \(y\) \((19)\) gives
\[
2c f''(g) - \frac{4f''(g)}{Re} + 2f'(g) + N f'(g) = 0.
\] (20)

The above equation \((20)\) upon integration yields internal velocity as
\[
f(g) = k_{2} e^{\frac{g(CKRe-A)}{4K}} + k_{3} e^{\frac{g(CKRe+A)}{4K}} + \frac{KK_{1}}{KN + 2},
\] (21)

where \(A = \sqrt{c^{2}K^{2}Re^{2} + 4K^{2}NR_{e} + 8KR_{e}}\).
Substituting the above velocity \( f(y) \) above into (18) yields internal pressure equation as

\[
k_2 N (c K R_e - A) e^{\frac{g(c K R_e - A)}{4 K}} + k_3 N (A + c K R_e) e^{\frac{g(A + c K R_e)}{4 K}} - 8 K z'(g) = 0. \quad (22)
\]

The above equation (22) upon integration yields the internal pressure as

\[
z(g) = \frac{1}{2} N \left( k_3 e^{\frac{gA}{2K}} + k_2 \right) e^{\frac{g(c K R_e - A)}{4K}} + k_4.
\]

Thus, the solutions of velocities along the axial and normal, and pressure are given by

\[
u(t, x, y) = k_2 (A - c K R_e) e^{\frac{(ct + x - y)(c K R_e - A)}{4K}} - k_3 (A + c K R_e) e^{\frac{(ct + x - y)(A + c K R_e)}{4K}}, \quad \text{(23)}
\]

\[
v(t, x, y) = \frac{k_2 (A - c K R_e) e^{\frac{(ct + x - y)(c K R_e - A)}{4K}} - k_3 (A + c K R_e) e^{\frac{(ct + x - y)(A + c K R_e)}{4K}}}{4K}, \quad \text{(24)}
\]

\[
P(t, x, y) = \frac{1}{2} N e^{\frac{(ct + x - y)(c K R_e - A)}{4K}} \left( k_3 e^{\frac{(ct + x - y)A}{2K}} + k_2 \right) + k_4. \quad \text{(25)}
\]

According to the current filter design, the initial and boundary conditions when \( x = L, y = H \) and \( t = 1 \) are as follows:

\[
u(1, L, 0) = 0, \quad u(1, L, H) = 1, \quad P(1, L, H) = 1, \quad \psi(1, L, H) = H - L + 1. \quad \text{(26)}
\]

The above conditions yield the following constants of integration:

\[
k_1 = \frac{1}{K} \left[ - A \coth \left( \frac{H A}{4K} \right) - c K + KN(H - L + 1) + 2H - 2L + 2 \right],
\]

\[
k_2 = - 4K e^{\frac{1}{4} \left[ (c H + L) A - c R_e(c - H + L) \right]} (c K R_e - A) \left( e^{\frac{gA}{2K}} - 1 \right),
\]

\[
k_3 = \frac{4 K e^{\frac{(c - H + L)(A + c K R_e)}{4K}}}{(A + c K R_e) \left( e^{\frac{gA}{2K}} - 1 \right)}, \quad k_4 = - \frac{N A e^{\frac{H A}{4K}}}{2 K N + 4} \coth \left( \frac{H A}{4K} \right) + (c - 2) K N - 4.
\]

Thus, the axial and normal velocities and internal pressure are given by

\[
u(t, x, y) = e^{\frac{g(c - H + L)(A + c K R_e)}{4K}} \left( e^{\frac{1}{4} \left[ (2 A - c H + L)(A + c K R_e) \right]} + c R_e(g + c - H + L) \right) - e^{\frac{1}{4} \left[ (A + c K R_e) + c R_e(c - H + L) \right]},
\]

\[
v(t, x, y) = e^{\frac{g(c - H + L)(A + c K R_e)}{4K}} \left( e^{\frac{1}{4} \left[ (2 A - c H + L)(A + c K R_e) \right]} + c R_e(g + c - H + L) \right) - e^{\frac{1}{4} \left[ (A + c K R_e) + c R_e(c - H + L) \right]},
\]

\[
p(t, x, y) = 2 K N e^{\frac{g(c K R_e - A)}{4K}} \left( e^{\frac{2 A - c H + L)(A + c K R_e)}{4K}} \right) + \frac{1}{2} \left[ (c H + L) A - c R_e(c - H + L) \right] - \frac{N A e^{\frac{H A}{4K}}}{2 R_e(K N + 2)} \frac{(c - 2) K N - 4}{2(K N + 2)},
\]

here \( g = ct + x - y \) and \( A = \sqrt{K R_e (c^2 K R_e + 4 K N + 8)}. \)
4. Results and discussion

This section provides a graphical representation of internal speed and pressure variation and the analysis of the effect of various parameters which affect the flow and pressure variation. Also, a tabular representation of skin friction is used to study internal drag force during unstable filtration operations. The following values of physical parameters affecting the system are used beside when they varied to show their effects on the system: \( R_e = 1, t = 1, c = 3, K = 0.1, L = 1, H = 1, x = 1, N = 1 \) and \( y = 1 \).

The permeates production speed which is plotted below is given by

\[
\text{Speed} = |\mathbf{u}| = \sqrt{u^2 + v^2}.
\]

4.1. Speed and pressure variation

![Time evolution of the permeates speed.](image)

Figure 2: Time evolution of the permeates speed.
It is observed from Figure 2 that, as the time evolves from $t = 1$ (when operation starts) to $t = 3$, the permeates outflow speed decreases over time. The decrease in system speed is due to the increase in drag force due to more internal fluid mass becoming heavier when additional particles are introduced inside the filter chamber over time. This happens as the free movement of particles diminishes when the fluid bulk becomes heavier, thus decreasing permeates production. The fluid moves more towards the top wall when operation starts due to more work done at the top because of injection and available chamber space before particles become clustered. The fluid moves faster towards the bottom wall when the chamber volume becomes clustered with particles. This decrease in flow speed at the top after some time is caused by reverse flow effects at the top when fluid particles occupy the entire chamber space.

Figure 3: Time evolution of an internal pressure.

Figure 3 portrays the internal pressure variation during unstable filtration operations decreases as time evolves. The system’s internal pressure decreases over time. This decrease
in pressure is caused by the increase in mass density when more particles are injected inside the chamber over time, thus diminishing the free movement of the internal fluid bulk and decreasing internal work done, which leads to a decrease in pressure. It can be noted that when the operation starts ($t = 1$), there is more pressure towards the top of the wall due to the impact of injected fluid and magnetic pulling effects. After that, the pressure increases towards the bottom of the wall when particles push downwards each other when they become clustered.

Figure 4: Effects of Reynolds number on permeates speed and pressure.

Figure 4 illustrates that an increase in Reynolds number leads to a decrease in permeates outflow speed and an increase in internal pressure during unsteady operation. This behaviour is caused by the rise in density when particles inside the filter chamber increase, thus enhancing pressure. This pressure resulting from internal work pushes the permeates out of the chamber and causes a decline in internal pressure when pressure drives permeate out of the chamber. The top surface wall does more work due to the injection and movement of particles towards the top due to magnetic effects, thus enhancing the system’s internal pressure toward the top. It is perceived that flow and pressure are less towards the bottom wall due to the no-slip condition and increase towards the upper wall since more work is done toward the upper walls and slow towards the bottom wall due to the no-slip condition.

Figure 5 depicts that a decrease in the permeates wave speed parameter leads to a reduction in permeates outflow speed since internal particles’ movement decreases as a result of a decrease in wave speed parameter. The internal pressure increases when the system produce permeates slower since the system loses less pressure (driven force) when moving permeates slower than when permeates move higher.

According to Figure 6, it is clear that the internal flow increases when the system permits more when pore sizes are small during the unsteady filtration process. On the other hand,
this increase in internal permeation speed when the pores are small leads to a decrease in overall internal work done due to less space to allow particles to move; thus, the system loses pressure when the pore size becomes small.

Figure 7 depicts that an increase in Stuart number leads to a decrease in internal flow and a deterioration in pressure during the unsteady filtration process. These behaviours are caused by the fact that the more Stuart number binds fluid particles, thus creating drag force and decreasing particle movement. Also, internal pressure decreases when the Stuart number
Figure 7: Effects of Stuart number on permeates speed and pressure.

increases since the internal work decrease as particles become clustered.

Figure 8: Effects of height variation on permeates speed and pressure.

According to Figure 8, the system’s flow and internal pressure increase when the chamber volume becomes smaller. The flow particles turn to move in the direction of the outlet instead of moving towards the chamber walls when the volume becomes small such that more axial internal work done increases, thus enhancing outflow and pressure. The pressure increases as a result of a decrease in volume since pressure is inversely proportional to volume.
We remark here that the exact solutions obtained explicitly indicate the parameters that influence the momentum and pressure, respectively during the operation. Also, the plots satisfy speed of the system at the boundaries which are zero at the bottom wall \( y = 0 \) and \( \sqrt{2} \) at the top wall \( y = h \). Also the pressure plots satisfy the condition at the top wall \( y = h \).

4.2. Walls skin friction

This subsection studies the effects of various physical parameters on the system’s skin friction. The shear stress on the surface wall due to fluid movement is known as the skin friction coefficient, and this parameter is of physical importance since it illustrates the dynamics of internal drag within the system, and it is defined as follows:

\[
C_f = \frac{2\tau_w}{\rho V_w^2},
\]

(27)

where \( \tau_w \) is the shear stress along the chamber’s bottom and top surface walls defined as

\[
\tau_w = \mu \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0 \& \bar{y}=1},
\]

(28)

when \( y = 0 \) and \( y = 1 \), we get skin friction at the bottom and top part of the chamber, respectively. According to dimensionless quantities in this study, skin friction takes the following dimensionless form:

\[
C_f = \frac{2}{Re} \left( \frac{\partial u}{\partial y} \right)_{y=0 \& y=1},
\]

(29)

According to Table 1 and Table 2, internal drag increases and decreases at the bottom when time evolves and the system allows more injection, respectively, while it decreases at the top when time evolves as well as when the system allows more injection. This increase in drag at the bottom when time evolves is caused by the clustering of particles which increases internal mass density hence the increase in drag. Also, the decrease in drag force at the bottom when the system allows more injection is caused by the movement of particles, enhancing dynamic viscosity, thus decreasing the binding effects of fluid particles and decreasing drag force.
Table 3: Effects of wave speed parameter

<table>
<thead>
<tr>
<th>$c$</th>
<th>Top skin friction</th>
<th>Bottom skin friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.62754</td>
<td>0.960138</td>
</tr>
<tr>
<td>0.3</td>
<td>4.52956</td>
<td>1.00872</td>
</tr>
<tr>
<td>0.6</td>
<td>4.38637</td>
<td>1.08495</td>
</tr>
<tr>
<td>0.9</td>
<td>4.24769</td>
<td>1.16527</td>
</tr>
<tr>
<td>1.2</td>
<td>4.11351</td>
<td>1.24976</td>
</tr>
</tbody>
</table>

Table 4: Effect of porosity

<table>
<thead>
<tr>
<th>$K$</th>
<th>Top skin friction</th>
<th>Bottom skin friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.52956</td>
<td>1.00872</td>
</tr>
<tr>
<td>0.3</td>
<td>2.99252</td>
<td>1.53977</td>
</tr>
<tr>
<td>0.6</td>
<td>2.52821</td>
<td>1.80908</td>
</tr>
<tr>
<td>0.9</td>
<td>2.36363</td>
<td>1.88972</td>
</tr>
<tr>
<td>1.2</td>
<td>2.27931</td>
<td>1.93191</td>
</tr>
</tbody>
</table>

Table 5: Effects of Stuart number

<table>
<thead>
<tr>
<th>$N$</th>
<th>Top skin friction</th>
<th>Bottom skin friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.52956</td>
<td>1.00872</td>
</tr>
<tr>
<td>2</td>
<td>4.62959</td>
<td>0.977239</td>
</tr>
<tr>
<td>3</td>
<td>4.72795</td>
<td>0.947028</td>
</tr>
<tr>
<td>4</td>
<td>4.82472</td>
<td>0.918025</td>
</tr>
<tr>
<td>5</td>
<td>4.91996</td>
<td>0.890169</td>
</tr>
</tbody>
</table>

Table 6: Effect of chamber height variation

<table>
<thead>
<tr>
<th>$H$</th>
<th>Top skin friction</th>
<th>Bottom skin friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>44.0238</td>
<td>9.80399</td>
</tr>
<tr>
<td>0.4</td>
<td>20.1854</td>
<td>4.49524</td>
</tr>
<tr>
<td>0.6</td>
<td>11.6274</td>
<td>2.5894</td>
</tr>
<tr>
<td>0.8</td>
<td>7.16874</td>
<td>1.59646</td>
</tr>
<tr>
<td>1</td>
<td>4.52956</td>
<td>1.00872</td>
</tr>
</tbody>
</table>

As seen from Table 3 and Table 4, the internal drag decreases at the top wall and increases at the bottom wall when permeates movement and porosity increase, respectively. The decrease at the top and the increase at the bottom in drag when permeates movement increases is caused by particles’ free movement at the top and the no-slip effect, which binds particles more towards the bottom wall.

As seen from Table 5 and Table 6, internal drag increases at the top wall and decreases at the bottom wall when the magnetic effect increases, while the drag force decreases at the top and bottom walls when the volume increases. This behaviour is due to magnetic effects, which pull particles away from the bottom wall towards the top wall when magnetic force pulls internal fluid bulk upwards. The decrease in drag force when the chamber space becomes more is caused by additional space, which allows fluid to move easily, thus decreasing drag force.

Table 7: Effects of axial position

<table>
<thead>
<tr>
<th>$x$</th>
<th>Top skin friction</th>
<th>Bottom skin friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>41.1579</td>
<td>4.52956</td>
</tr>
<tr>
<td>0.25</td>
<td>23.6479</td>
<td>2.66209</td>
</tr>
<tr>
<td>0.5</td>
<td>13.5943</td>
<td>1.63796</td>
</tr>
<tr>
<td>0.75</td>
<td>7.827336</td>
<td>1.3748</td>
</tr>
<tr>
<td>1</td>
<td>4.52956</td>
<td>1.00872</td>
</tr>
</tbody>
</table>

Table 8: Behaviour of skin along the vertical

<table>
<thead>
<tr>
<th>$y$</th>
<th>skin friction along vertical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00872</td>
</tr>
<tr>
<td>0.25</td>
<td>1.13748</td>
</tr>
<tr>
<td>0.5</td>
<td>1.63796</td>
</tr>
<tr>
<td>0.75</td>
<td>2.66209</td>
</tr>
<tr>
<td>1</td>
<td>4.52956</td>
</tr>
</tbody>
</table>
As seen from Table 7 internal drag decreases at the top and bottom walls as fluid permeates move from left to right through the chamber. This decrease is due to the effects of top and bottom wall shear effects which decrease when particles move closer to the outlet. Also, it is observed from Table 8 that drag force is more at the top wall and decreases towards the bottom. This behaviour is caused by more particles at the top due to injection and the effects of magnetic force pulling the fluid away from the bottom towards the top wall.

5. Conclusion

This study presents the theoretical investigation into the dynamics of unsteady pressure-driven flow inside a horizontal filter chamber embedded with a porous medium under the influence of magnetic force. Transformation based on Lie symmetry analysis is used to transform a system of partial differential equations into a solvable system of ordinary differential equations representing internal velocity and pressure variation during filtration. The transformed system is thereafter solved to obtain solutions of velocity and pressure, which are used to study the effects of parameters that affect the filtration process. Also, analysis of skin friction is used to study system drag force. To achieve desirable permeates outflow while enhancing the system’s internal pressure and minimising internal friction during unsteady-state filtration operation, the study led to the following conclusions:

i. To increase permeates outflow velocity, it is ideal to enhance internal pressure during operation to allow pressure pushing effects to drive fluid particles out of the chamber.

ii. During unsteady operation, it is ideal to increase the chamber volume since this allows fluid particles more space to move, thus increasing permeate outflow.

iii. When the operation is unsteady, for a porous medium to restrict contaminations while increasing permeate outflow velocity, it is ideal to decrease the pore size of the medium while injecting less fluid into the filter chamber to enable free movement of particles.

iv. To enhance effects of internal pressure (driven force in the form of pressure), thus increasing outflow velocity during operation, lower Reynolds, small Stuart number, small-medium pore size, high wave speed parameter, and less chamber space are needed to increase permeates production.

v. To minimize the effects of no-slip condition at the bottom stationary wall (bottom skin friction), high injection rate and magnetic effects, low wave speed parameter, small porosity parameter, more chamber vertical space, and shorter filter chamber are ideal during operation.

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**Declarations**

**Conflict of interest:** The authors declare no conflicts of interest.

**Author Contributions:** Conceptualization, T.J.I, Conceptualization, G.M, Conceptualization, T.M, Conceptualization, L.D.M.
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