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Higher-order expansions of powered beta-normal extremes

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Abstract: Higher-order asymptotic expansions of powered beta-normal extremes are derived in this paper. Furthermore, we establish the rates of convergence of distributions of normalized extremes. It is shown that with optimal normalizing constants the convergence rates of powered beta-normal extremes depend on the power index.

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1 Introduction

The beta-normal distribution due to Eugene et al. (2002) can be defined as follows. Let \(X\) be a random variable with cumulative distribution function (CDF). The CDF for a generalized class of distribution \(X\) is given by

\[
G(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}dt, \tag{1.1}
\]

where \(0 < \alpha, \beta < \infty\), \(\Gamma(\cdot)\) denotes the gamma function.

The probability density function (PDF) of the generalized class of distribution \(G(x)\) is

\[
g(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} [F(x)]^{\alpha-1}[1 - F(x)]^{\beta-1}F'(x). \tag{1.2}
\]

When \(F(x)\) is the CDF of the standard normal distribution, the random variable \(X\) is said to have beta-normal distribution \(\text{BN}(\alpha, \beta)\), with probability density function

\[
g_{\alpha, \beta}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[\Phi(x)\right]^{\alpha-1}\left[1 - \Phi(x)\right]^{\beta-1}\phi(x), \tag{1.3}
\]

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where \(-\infty < x < \infty, 0 < \alpha, \beta < \infty, \Phi(\cdot)\) and \(\phi(\cdot)\) denote the standard normal cumulative distribution function and the standard normal probability density function, respectively.Obviously, \(BN(1, 1)\) is a standard normal distribution.

Eugene et al. (2002) systematically studied the shape properties of the beta-normal distribution and discussed its great flexibility in modeling skewed and bimodal distributions. Extensions and applications of beta-normal distribution families, see Gupta and Nadarajah (2005), Famoye and Lee (2004), \(\text{Rêgo et al.}(2012)\) and others.

The aim of this paper is to deduce the higher-order expansions of powered beta-normal extremes. Hall (1980) studied the asymptotic distribution behavior of normalized powered extremes for standard normal distribution. Hall (1980) also obtained the convergence rates of distributions of powered extremes which depend on the power index. Zhou and Ling (2016) deduced the higher-order expansions of distributions of powered extremes for standard normal samples. Xiong and Peng (2020) considered the higher-order distributional expansions and convergence rates of powered skew-normal extremes. Other papers studying the higher-order expansions of distributions of the extremes are: Liao et al. (2013) for the log-skew-normal distribution; Liao et al. (2014) for the skew-normal distribution; Du et al. (2016) for generalized gamma distribution; Huang (2018) for generalized Maxwell sample; Beranger et al. (2019a, 2019b) for the extended skew-normal distribution and multivariate skew-normal distribution, and references therein.

Throughout the paper, let \(\{X_n, n \geq 1\}\) be a sequence of independent and identically distributed (i.i.d.) random variables with the beta-normal pdf given by (1.3), and let \(M_n = \bigvee_{i=1}^{n} X_i\) denote the largest of the first \(n\). In order to obtained the higher-order expansions of normalized \(|M_n|^t\) of beta-normal random variables, we define the optimal normalized constants as follows:

for \(t \neq 2\), let

\[
c_n = \beta^{-1} t b_n^{-2}, \quad d_n = b_n^t,
\]

and for \(t = 2\), set

\[
c'_n = 2\beta^{-1} (1 - b_n^{-2}), \quad d'_n = b_n^2 - 2b_n^{-2},
\]

where \(b_n > 0\) is the solution of the following equation

\[
\frac{(2\pi)^{-\frac{\alpha}{2}} \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} b_n^{-\beta} e^{-\frac{\beta}{2} b_n^2} = \frac{1}{n}.
\]

The contents of this paper are arranged as follow: Section 2 presents main results for the asymptotic behavior of normalizing powered beta-normal extremes. All the proofs are given in section 3.

2 Main results

In this section, we first state that the limit distribution of normalized powered extreme of beta-normal distribution is the Gumbel extreme value distribution \(\Lambda(x)\).
Theorem 2.1. Let \( \{x_n, n \geq 1\} \) be a sequence of independent and identical random variables with the pdf given by (1.3), and let \( M_n = \sqrt[n]{X_i} \) denote its partial maximum. We have
(i) For \( 0 < t \neq 2 \), then
\[
P(|M_n|^t \leq c_n x + d_n) \to \Lambda(x), \quad x \in R
\] (2.1)
as \( n \to \infty \), where the normalized constants \( c_n \) and \( d_n \) are given by (1.4).
(ii) For \( t = 2 \), then
\[
P(|M_n|^t \leq c_n' x + d_n') \to \Lambda(x), \quad x \in R
\] (2.2)
as \( n \to \infty \), where the normalized constants \( c_n' \) and \( d_n' \) are given by (1.5).

In the following Theorem 2.2, we present the higher-order expansions for the cdf of \( |M_n|^t \) under the norming constants \( c_n \) and \( d_n \) given by (1.4).

Theorem 2.2. Under the assumptions of Theorem 2.1, for \( 0 < t \neq 2 \), we have as \( n \to \infty \)
\[
b_n^2 \left[ b_n^2 \left( P(|M_n|^t \leq c_n x + d_n) - \Lambda(x) \right) - \kappa(t, x) \Lambda(x) \right] \to \omega(t, x) \Lambda(x),
\] (2.3)
where
\[
\kappa(t, x) = e^{-x} \left[ (1 - \frac{1}{2} t) \beta^{-1} x^2 + x + \beta \right]
\] (2.4)
and
\[
\omega(t, x) = e^{-x} \left[ -\frac{1}{8} (t - 2)^2 \beta^{-2} x^4 + \frac{1}{6} (t - 2)(2t + 3\beta - 2) \beta^{-2} x^3 + \frac{1}{2} (\beta - 1) t \right. \\
-3\beta \left. \right] \beta^{-1} x^2 - (\beta + 2)x - \frac{1}{2} \beta (\beta + 5) \right] + \frac{1}{2} e^{-2x} \left[ (\frac{1}{2} t - 1) \beta^{-1} x^2 - x - \beta \right]^2.
\] (2.5)

Remark 2.1. Theorem 2.2 shows that the optimal convergence rate of \( |M_n|^t \) to Gumbel extreme value density function \( \Lambda(x) \) is proportional to \( 1/\log(x) \) for all \( 0 < t \neq 2 \) since \( b_n^2 \sim 2\beta^{-1} \log n \) as \( n \to \infty \) by (1.6).

Theorem 2.3 establishes the asymptotic expansion of the distribution of \( |M_n|^t \) in order to gain the rate of convergence as \( t = 2 \).

Theorem 2.3. For norming constants \( c_n' \) and \( d_n' \) given by (1.5), we have as \( n \to \infty \)
\[
b_n^2 \left[ b_n^4 \left( P(|M_n|^2 \leq c_n' x + d_n') - \Lambda(x) \right) - \kappa'(x) \Lambda(x) \right] \to \omega'(x) \Lambda(x),
\] (2.6)
where
\[
\kappa'(x) = -e^{-x} \left[ \beta^{-1} x^2 + 3x + \frac{7}{2} \beta \right],
\] (2.7)
and
\[
\omega'(x) = e^{-x} \left[ \frac{4}{3} \beta^{-2} x^3 + 6\beta^{-1} x^2 + 14x + \frac{43}{3} \beta \right].
\] (2.8)

Remark 2.2. Theorem 2.3 presents accurate convergence rate of \( |M_n|^2 \) to \( \Lambda(x) \) is proportional to \( 1/(\log(x))^2 \), which is rather faster than that for the other cases.
3 proof

In order to prove the main results, we need some auxiliary results. Lemma 3.1 derives the distributional tail representation of the beta-normal distribution, which is the further conclusion from Jiang et al. (2022).

**Lemma 3.1.** Let $G_{\alpha, \beta}(x)$ and $g_{\alpha, \beta}(x)$ denote respectively, the cdf and pdf of $BN(\alpha, \beta)$, then

$$1 - G_{\alpha, \beta}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} \frac{\phi^\beta(x)}{x^\beta} \left[ 1 - \beta x^{-2} + \frac{1}{2} \beta(\beta + 5)x^{-4} - \frac{1}{6} \beta(\beta^2 + 15\beta + 74)x^{-6} + O(x^{-8}) \right]$$

for large $x$.

**Proof.** By integration by parts, we have

$$1 - G_{\alpha, \beta}(x) = \frac{1}{\beta} \frac{\Phi(-x)}{\phi(x)} g_{\alpha, \beta}(x) \left[ 1 + \frac{\alpha - 1}{\beta + 1}(\Phi^{-1}(x) - 1) + \frac{(\alpha - 1)(\alpha - 2)}{(\beta + 1)(\beta + 2)}(\Phi^{-1}(x) - 1)^2 ight.$$ 

$$+ \ldots + \frac{(\alpha - 1)(\alpha - 2)\ldots(\alpha - n)}{(\beta + 1)(\beta + 2)\ldots(\beta + n)}(\Phi^{-1}(x) - 1)^n(1 + o(1)) \right]$$

for large $x$. It is easy to check that for all $r$,

$$x^r(\Phi^{-1}(x) - 1) \to 0,$$

and

$$x^r(\Phi^{\alpha-1}(x) - 1) \to 0,$$

as $x \to \infty$.

For large $x$, we have

$$1 - \Phi(x) = \frac{\phi(x)}{x}(1 - x^{-2} + 3x^{-4} - 15x^{-6} + O(x^{-8})).$$

Thus, by (3.2), (3.3), (3.4) and (3.5), we have

$$1 - G_{\alpha, \beta}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} \frac{\phi^\beta(x)}{x^\beta} \left[ 1 - \beta x^{-2} + \frac{1}{2} \beta(\beta + 5)x^{-4} - \frac{1}{6} \beta(\beta^2 + 15\beta + 74)x^{-6} + O(x^{-8}) \right]$$

for large $x$, which is (3.1). The proof is complete.

**Proof of Theorem 2.1.** For large $n$ and $0 < t \neq 2$, $c_n x + d_n > 0$, where $c_n$ and $d_n$
are normalizing constants given by (1.4). We have \((c_nx + d_n)^\dagger = [\beta^{-1}tb_n^{-2}x + b_n^\dagger] = b_n[1+\beta a_n^2 x + O(a_n^4)] = a_n x + b_n + O(a_n^3)\), where \(a_n = \beta^{-1}b_n^{-1}\) with \(b_n\) is the solution of (1.6).

Utilizing (1.6) and (3.1), for large \(n\) we have

\[
\begin{align*}
n \left[1 - \gamma_{\alpha,\beta}(a_n x + b_n + O(a_n^3))\right] &\sim n \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} \phi^\beta(a_n x + b_n + O(a_n^3)) \\
&= n \frac{(2\pi)^{-\frac{\alpha}{2}} \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} (a_n x + b_n + O(a_n^3))^{-\beta} \exp \left\{ -\frac{\beta}{2} (a_n x + b_n + O(a_n^3))^2 \right\} \\
&= n \frac{(2\pi)^{-\frac{\alpha}{2}} \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} b_n^{-\beta} (1 + \beta a_n^2 x + O(a_n^4))^{-\beta} \exp \left\{ -\frac{\beta}{2} b_n^2 (1 + 2\beta a_n^2 x + O(a_n^4)) \right\} \\
&= n \frac{(2\pi)^{-\frac{\alpha}{2}} \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} b_n^{-\beta} e^{-\frac{\beta}{2} b_n^2} (1 - \beta a_n^2 x + O(a_n^4)) \exp \left\{ -x + O(a_n^3) \right\} \\
&= (1 - \beta^2 a_n^2 x + O(a_n^4)) \exp \left\{ -x + O(a_n^3) \right\} \\
&\to e^{-x}
\end{align*}
\]
as \(n \to \infty\). Then

\[
P(|M_n|^t \leq c_n x + d_n) = P(M_n \leq (c_n x + d_n)^\dagger) - P(M_n \leq -(c_n x + d_n)^\dagger)
\]

\[
= \gamma_{\alpha,\beta}(a_n x + b_n + O(a_n^3)) + o(1)
\]

\[
\to \Lambda(x)
\]
as \(n \to \infty\). For \(t = 2\), note that

\[
\frac{c'_n}{c_n} \to 1, \quad \frac{d'_n - d_n}{c_n} \to 0
\]
as \(n \to \infty\). So, the result follows by Theorem1.2.3 in Leadbetter et al.(1983). The proof is complete.

**Proof of Theorem 2.2.** Note that for \(0 < t \neq 2\) and sufficiently large \(n\), \(c_n x + d_n > 0\).

We have by (1.4) that \(z_n = (c_n x + d_n)^\dagger = (\beta^{-1}tb_n^{-2}x + b_n^\dagger) = b_n(\beta a_n^2 x + 1)^\dagger\) with \(a_n = \beta^{-1}b_n^{-1}\). For large \(n\), we have the following Taylor’s expansion

\[
z_n^k = b_n^k \left[ 1 + k\beta a_n^2 x + \frac{1}{2} k(k - t)\beta^2 a_n^4 x^2 + \frac{1}{6} k(k - t)(k - 2t)\beta^3 a_n^6 x^3 + O(a_n^8) \right], k \in \mathbb{R}, \text{ (3.6)}
\]

Then

\[
1 - \beta z_n^{-2} + \frac{1}{2} \beta(\beta + 5) z_n^{-4}
\]

\[
= 1 - \beta \left[ \beta^2 a_n^2 (1 - 2\beta a_n^2 x + O(a_n^4)) \right] + \frac{1}{2} \beta(\beta + 5) \left[ \beta^4 a_n^4 (1 - 4\beta a_n^2 x + O(a_n^4)) \right]
\]

\[
= 1 - \beta^3 a_n^2 + \left[ \frac{1}{2} \beta(\beta + 5) + 2x \right] \beta^4 a_n^4 + O(a_n^6).
\]  

(3.7)
Further, applying (3.6) with \( k = 2 \) and \( k = -\beta \), we have
\[
e^{-\frac{4}{3}z_n^2} = e^{-\frac{4}{3}z_n^2} \cdot e^{-x} \left[ 1 - \frac{1}{2}(2-t)\beta x^2 a_n^2 + \left( \frac{1}{8}(2-t)x - \frac{1}{3}(1-t) \right) (2-t)\beta^2 x^3 a_n^4 + O(a_n^6) \right]
\]
and
\[
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} \frac{\phi^\beta(z_n)}{z_n^\beta}
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} (2\pi)^{-\frac{3}{2}} e^{-\frac{4}{3}z_n^2} z_n^{-\beta}
\]
\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} (2\pi)^{-\frac{3}{2}} e^{-\frac{4}{3}z_n^2} z_n^{-\beta} \cdot e^{-x} \left[ 1 - \frac{1}{2}(2-t)\beta x^2 a_n^2 + \left( \frac{1}{8}(2-t)x - \frac{1}{3}(1-t) \right) (2-t)\beta^2 x^3 a_n^4 + O(a_n^6) \right]
\times \beta^2 x^3 a_n^4 + O(a_n^6)\right]
= n^{-1} e^{-x} \left[ 1 + \left( \frac{1}{2}(t-1)\beta x^2 - \beta^2 x \right) a_n^2 + \left( \frac{1}{8}(2-t)^2\beta^2 x^4 + \frac{1}{6}(2-t)(2t + 3\beta - 2)\beta^2 x^3
\right.
\left. + \frac{1}{2}(t + \beta)\beta^3 x^2 \right) a_n^4 + O(a_n^6) \right] \times \left[ 1 - \beta^3 a_n^2 + \left( \frac{1}{2}\beta(\beta + 5) + 2x \right) \beta^4 a_n^4 + O(a_n^6) \right]
:= n^{-1} e^{-x} \left[ 1 + A_1(t,x) a_n^2 + A_2(t,x) a_n^4 + O(a_n^6) \right],
\]
where (3.7) and (3.8), have
\[
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta + 1)} \frac{\phi^\beta(z_n)}{z_n^\beta} \left( 1 - \beta z_n^{-2} + \frac{1}{2} \beta(\beta + 5) z_n^{-4} \right)
= n^{-1} e^{-x} \left[ 1 + \left( \frac{1}{2}(t-1)\beta x^2 - \beta^2 x \right) a_n^2 + \left( \frac{1}{8}(2-t)^2\beta^2 x^4 + \frac{1}{6}(2-t)(2t + 3\beta - 2)\beta^2 x^3
\right.
\left. + \frac{1}{2}(t + \beta)\beta^3 x^2 \right) a_n^4 + O(a_n^6) \right] \times \left[ 1 - \beta^3 a_n^2 + \left( \frac{1}{2}\beta(\beta + 5) + 2x \right) \beta^4 a_n^4 + O(a_n^6) \right]
\]
where
\[
A_1(t,x) = \left( \frac{1}{2}(t-1)\beta x^2 - \beta^2 x - \beta^3 \right)
\]
and
\[
A_2(t,x) = \frac{1}{8}(2-t)^2\beta^2 x^4 + \frac{1}{6}(2-t)(2t + 3\beta - 2)\beta^2 x^3 + \frac{1}{2}(3\beta + (1 - \beta)t)\beta^3 x^2
\left. + \beta^4(\beta + 2)x + \frac{1}{2}\beta^5(\beta + 5). \right]
\]
Note that \( P(M_n \leq -(c_n x + d_n) \beta) = o(a_n^r), r > 6, \) and \( \log(1-x) = -x + O(x^2) \) as \( x \to 0, \)
so by (3.9), we have

\[ P(|M_n| \leq c_n x + d_n) = P(M_n \leq (c_n x + d_n)^{1/2}) - P(M_n \leq -(c_n x + d_n)^{1/2}) \]
\[ = G_{\alpha, \beta}^n(\sigma) - G_{\alpha, \beta}^n(c_n x + d_n) \]
\[ = \left\{ 1 - \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} \frac{\phi^2(z_n)}{z_n} \left[ 1 - \beta z_n^{-2} + \frac{1}{2} \beta^5 z_n^{-4} + O(z_n^{-6}) \right] \right\}^n + o(a_n^6) \]
\[ = \left\{ 1 - n^{-1} e^{-x}[1 + A_1(t, x) a_n^2 + A_2(t, x) a_n^4 + O(a_n^6)] \right\}^n + o(a_n^6) \]
\[ = \Lambda(x) \exp \left\{ e^{-x}[-A_1(t, x) a_n^2 - A_2(t, x) a_n^4 + O(a_n^6)] \right\} + o(a_n^6) \]
\[ = \Lambda(x) + e^{-x} A_1(t, x) a_n^2 + \left( \frac{1}{2} e^{-2x} A_2(t, x) - e^{-x} A_2(t, x) \right) a_n^4 + O(a_n^6) \]
\[ = \Lambda(x) + \Lambda(x) [\kappa(t, x) b_n^{-2} + \omega(t, x) b_n^{-4} + O(b_n^{-6})]. \]

The proof is complete.

**Proof of Theorem 2.3.** For \( t = 2, z_n = (c_n x + d_n)^{1/2} = b_n [1 + 2 \beta (x + \beta) a_n^4]^{1/2} \) with \( a_n = \beta^{-1} b_n^{-1} \). Hence, for large \( n \),

\[
z_n^k = b_n^k \left[ 1 + k \beta x a_n^2 + \left( \frac{1}{2} (k - 2) x^2 - \beta x - \beta^3 \right) k \beta^3 a_n^4 + \left( \frac{1}{6} (k - 4) x^3 - \beta x^2 - \beta^2 x \right) \right.
\times k(k - 2) \beta^3 a_n^6 + O(a_n^8) \].

Especially, with \( k = -2, -4 \), we obtain

\[
1 - \beta z_n^{-2} + \frac{1}{2} \beta (\beta + 5) z_n^{-4} - \frac{1}{6} \beta (\beta^2 + 15 \beta + 74) z_n^{-6} = 1 - \beta z_2^{-2} - \frac{1}{2} \beta^2 (x + \beta (5 \beta + 4)) a_n^4 - 2 \beta^5 \left(2 x^2 + \beta (5 \beta + 4) x + \frac{1}{12} \beta^2 (\beta^2 + 15 \beta + 86) \right) a_n^6 + O(a_n^8).
\]

Using \( e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + O(x^4) \), \( x \to 0 \) and the equality (3.8) with \( k = 2 \) and \( k = -\beta \), we have

\[
e^{-\frac{1}{2} \beta z_n^2} = e^{-\frac{1}{2} \beta z_n^2} \cdot e^{-x} \left(1 + \beta^2 (x + \beta) a_n^2 + \frac{1}{2} \beta^4 (x + \beta) a_n^4 + \frac{1}{6} \beta^6 (x + \beta) a_n^6 + O(a_n^8) \right),
\]

and

\[
z_n^{-\beta} = b_n^{-\beta} \left[ 1 - \beta^2 x a_n^2 + \beta^3 \left( \frac{1}{2} (\beta + 2) x^2 + \beta x + \beta^2 \right) a_n^4 - \beta^4 (\beta + 2) \left( \frac{1}{6} (\beta + 4) x^3 \right. \right.
\]
\[
+ \beta x^2 + \beta^2 x \right) a_n^6 + O(a_n^8) \].


Combining (3.9), (3.10) and (3.11), we can derive

\[
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} \frac{\phi(z_n)}{z_n^\beta} \left[ 1 - \beta z_n^{-2} + \frac{1}{2} \beta(\beta + 5)z_n^{-4} - \frac{1}{6} \beta(\beta^2 + 15\beta + 74)z_n^{-6} + O(z_n^{-8}) \right]
\]

\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} (2\pi)^{-\frac{3}{2}} \cdot e^{-\frac{\beta z_n^2}{2}} \cdot z_n^{-\beta} \left[ 1 - \beta z_n^{-2} + \frac{1}{2} \beta(\beta + 5)z_n^{-4} - \frac{1}{6} \beta(\beta^2 + 15\beta + 74)z_n^{-6} + O(z_n^{-8}) \right]
\]

\[
= n^{-1}e^{-x} \left[ 1 + \beta^3 a_n^2 + \left( x^2 + \beta x + \frac{1}{2} \beta(\beta + 2) \right) \beta^3 a_n^4 + \frac{1}{6} \left( -8x^3 + 6\beta(\beta - 2)x^2 + 6\beta^2 \right. \right. \times (\beta - 2)x + \beta^4(\beta + 6) \left. \left. \right) \beta^4 a_n^6 + O(a_n^8) \right]
\times \left[ 1 - \beta^3 a_n^2 + \frac{1}{2} \left( 4x + \beta(\beta + 5) \right) \beta^4 a_n^4 - 2(2x^2 + \beta(\beta + 6)x + \frac{1}{12} \beta^2(\beta^2 + 15\beta + 86)) \beta^4 a_n^6 + O(a_n^8) \right]
\]

\[
= n^{-1}e^{-x} \left[ 1 + \left( x^2 + 3\beta x + \frac{7}{2} \beta^2 \right) \beta^3 a_n^4 + \left( - \frac{4}{3} x^3 - 6\beta x^2 - 14\beta^2 x - \frac{43}{3} \beta^3 \right) \beta^4 a_n^6 + O(a_n^8) \right]
\]

\[
= n^{-1}e^{-x} \left[ 1 + A_1(t, x)a_n^4 + A_2(t, x)a_n^6 + O(a_n^8) \right].
\]

Similar to the proof of Theorem 2.2,

\[
P(|M_n|^2 \leq c_n' x + d_n') = P(M_n \leq (c_n' x + d_n')^{\frac{1}{2}}) - P(M_n \leq -(c_n' x + d_n')^{\frac{1}{2}})
\]

\[
= \left\{ 1 - \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} \frac{\phi(z_n)}{z_n^\beta} \left[ 1 - \beta z_n^{-2} + \frac{1}{2} \beta(\beta + 5)z_n^{-4} - \frac{1}{6} \beta(\beta^2 + 15\beta + 74)z_n^{-6} + O(z_n^{-8}) \right] \right\}^n
\]

\[
+ o(a_n^8)
\]

\[
= \left\{ 1 - n^{-1}e^{-x} \left[ 1 + A_1(t, x)a_n^4 + A_2(t, x)a_n^6 + O(a_n^8) \right] \right\}^n + o(a_n^8)
\]

\[
= \Lambda(x) \exp \left\{ -e^{-x} \left[ A_1(t, x)a_n^4 + A_2(t, x)a_n^6 + O(a_n^8) \right] \right\} + o(a_n^8)
\]

\[
= \Lambda(x) + \Lambda(x) \left[ -e^{-x} A_1(t, x)a_n^4 - e^{-x} A_2(t, x)a_n^6 + O(a_n^8) \right]
\]

\[
= \Lambda(x) + \Lambda(x) \left[ \kappa'(t, x)b_n^4 + \omega'(t, x)b_n^6 + O(b_n^8) \right],
\]

which completes the proof.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YJ: conceptualization, computation, funding acquisition, writing-original draft, writing-review and editing. BL: problem statement, supervision, writing-review, and provision of study resources. All authors read and approved the final manuscript.

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