ON SOME INEQUALITIES FROM POSITIVE MATRICES TO SECTOR MATRICES

LEILAN ASIRI* AND CHAOJUN YANG
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FACULTY OF SCIENCE,
UNIVERSITY OF LORESTAN, KHORRAMABAD, IRAN
EMAIL:LEILANASIRI468@GMAIL.COM
EMAIL:NASIRI.LE@FS.LU.AC.IR
DEPARTMENT OF MATHEMATICS, CHANGZHOU UNIVERSITY, CHANGZHOU, 213000, P. R.
CHINA.
EMAIL:MAILCJYANGMATH@163.COM

Abstract. In the present paper, we discuss some inequalities related to sector matrices such as relative entropy, the Kantorovich inequality and Heron inequality. Among other results, we prove that if $A \in M_n$ be such that $W(A) \subseteq S_{\theta}$ and $0 < m \leq \Re A \leq M$. Then

$$\Re \Phi(A^{-1}) \leq K(h) \sec^2(\theta) \Re \Phi^{-1}(A),$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

1. Introduction

Let $M$ and $m$ be scalars and $I$ be the identity operator. Let $\mathcal{B}(\mathcal{H})$ denote $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We say $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, if it satisfies $A = A^*$. An operator $A$ is said to be positive and is denoted by $A \geq 0$ if $< Ax, x > \geq 0$ for all $x \in \mathcal{H}$, and $A$ is said to be strictly positive and is denoted by $A > 0$, if $< Ax, x > > 0$ for all $x \in \mathcal{H}$. For two self-adjoint operators $A$ and $B$, $A \geq B$ means $A - B \geq 0$. We say that a linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. For $A, B \in \mathcal{B}(\mathcal{H})$ such that $A, B > 0$ and $0 \leq \nu \leq 1$, we use the notations $A_{\#}\nu B$, $A\triangledown\nu B$ and $A!\nu B$, to define the geometric mean, the arithmetic mean and the harmonic mean respectively are defined in the following form:

$$A_{\#}\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}} \quad \text{and} \quad A\triangledown\nu B = (1 - \nu)A + \nu B$$

$$A!\nu B = ((1 - \nu)A^{-1} + \nu B^{-1})^{-1}.$$

2010 Mathematics Subject Classification. Primary 15A45, Secondary 15A42; 47A30.
Key words and phrases. Sector matrices; Relative entropy; Heron mean; Kantorovich inequality.

*Corresponding author.
The noncommutative AM-GM-HM inequalities for two strictly positive operators $A, B$ and $0 \leq \nu \leq 1$ have been proved in following form:

$$A\nu B \leq A^\#\nu B \leq A\triangledown \nu B,$$

where the second inequality is famous as the operator Young inequality. For more information, see Bhatia [2]. For special case, when $\nu = \frac{1}{2}$, we have the following inequality:

$$\left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1} \leq A^\# B \leq \frac{A + B}{2}.$$  

Let $\mathcal{M}_n$ denote the space of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$, the Cartesian decomposition of $A$ is presented as

$$A = \Re A + i\Im A,$$

where $\Re A = \frac{A + A^*}{2}$ and $\Im A = \frac{A - A^*}{2i}$ are the real and imaginary parts of $A$, respectively. The matrix $A \in \mathcal{M}_n$ is called accretive, if $\Re A$ is positive definite. Also, The matrix $A \in \mathcal{M}_n$ is called accretive-disipative, if both $\Re A$ and $\Im A$ are positive definite. Here, we recall that the numerical range of $A \in \mathcal{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\theta \in [0, \frac{\pi}{2})$, we define a sector as follows:

$$S_\theta = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z) \tan \theta\}.$$

The matrix $A \in \mathcal{M}_n$ is called sectorial, if whose numerical range is contained in sector $S_\theta$. In other words, $W(A) \subset S_\theta$. Clearly, a sector matrix is accretive with extra information about the angle $\theta$. Since $W(A) \subset S_\theta$ implies that $W(X^*AX) \subset S_\theta$ for any nonzero $n \times m$ matrix $X$, thus $W(A^{-1}) \subset S_\theta$, that is, inverse of every sector matrix is a sector matrix. For more information on sector matrices, the interested reader can refer to [5, 9, 10, 11, 16]. Liu et. al [9] and Lin [10] extended the inequalities (1.2) to sector matrices as follows:

$$\cos^4(\theta)\Re \left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1} \leq \Re (A^\# B) \leq \sec^2(\theta)\Re \left(\frac{A + B}{2}\right),$$  

where $\theta \in [0, \frac{\pi}{2})$.

Main aim of this paper is to give extend the relative entropy and the Kantorovich inequality for sector matrices.

2. RELATIVE OPERATOR ENTROPY

Let $A, B \in \mathcal{M}_n$ be accretive and let $\nu \in (1,2) \cup (-1, 0)$. Then

$$\Re (A^\# \nu B) \leq (\Re A)^\# \nu (\Re B).$$  

(2.1)
Let $A, B \in M_n$ be such that $A \subset S_\theta$ and $B > 0$ and let $\nu \in (1, 2) \cup (-1, 0)$. Then
\[
\Re(A^\lambda B) \geq \cos(\theta)(\Re A)^\lambda (\Re B).
\] (2.2)

**Lemma 2.1.** ([10, 5]) Let $A \in M_n$ with $W(A) \subset S_\theta$. Then we have $\Re(A^{-1}) \leq \Re^{-1}(A) \leq \sec^2(\theta)\Re(A^{-1})$. The first inequality holds for an accretive matrix $A \in M_n$.

**Lemma 2.2.** ([13]) If $A, B \in M_n$ be accretive and $0 < \lambda < 1$. Then
\[
(\Re A)^\lambda (\Re B) \leq \Re(A^\lambda B).
\]

**Lemma 2.3.** [20] Let $A \in M_n$ be such that $W(A) \subseteq S_\alpha$ for $0 \leq \alpha < \frac{\pi}{2}$ and let $\| \|$ be any unitarily invariant norm on $M_n$. Then
\[
\cos(\alpha)\|A\| \leq \|\Re A\| \leq \|A\|.
\]

The numerical radius $\omega(A)$ of $A \in M_n$ is defined by
\[
\omega(A) = \sup\{< Ax, x >: x \in C^n, \|x\| = 1\}.
\]

Kittaneh et al. [21] proved that
\[
\omega(\Re A) \leq \omega(A) \leq \sec^2(\theta)\omega(\Re A).
\] (2.3)

Bedrani et al. [4] showed that if $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$, then
\[
\cos^3(\alpha)\omega^{-1}(A) \leq \omega(A^{-1}).
\] (2.4)

\[
\cos(\alpha)\|A\| \leq \omega(A) \leq \|A\|.
\] (2.5)

### 2.1. Relative operator entropy.
For two strictly positive operators $A$ and $B$, the relative operator entropy is defined by Fujii et. al [21]
\[
S(A|B) := A^\frac{1}{2} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^\frac{1}{2}.
\]

Raissouli et. al [13] recently extended the definition above and defined the relative operator entropy of two accretive operators $A$ and $B$ via the following formula:
\[
S(A|B) = \int_0^1 \frac{A^t B - A}{t} dt.
\]

In the same time, they obtain following remarkable property about the relative operator entropy of two accretive operators:
\[
\Re(S(A|B)) \geq S(\Re A|\Re B).
\] (2.6)

The next Lemma give a reverse of (2.6).

**Theorem 2.4.** Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\theta$. Then
\[
\Re(S(A|B)) \leq \sec^2(\theta)S(\Re A|\Re B).
\] (2.7)
Proof. By the first inequality of Lemma 2.1,
\[
\Re(tA^{-1} + (1 - t)B^{-1})^{-1} \leq \Re^{-1}(tA^{-1} + (1 - t)B^{-1}) = (t\Re A^{-1} + (1 - t)\Re B^{-1}))^{-1}. \tag{2.8}
\]
Now, the second inequality of Lemma 2.1 ensure us that
\[
t\Re^{-1}(A) + (1 - t)\Re^{-1}(B) \leq \sec^2(\theta)(t\Re A^{-1} + (1 - t)\Re B^{-1}).
\]
Taking reverse from the latter inequality, we get
\[
(t\Re^{-1}(A + (1 - t)\Re^{-1}B^{-1}) \leq \sec^2(\theta)(t\Re^{-1}A + (1 - t)\Re^{-1}B)^{-1}. \tag{2.9}
\]
Applying (2.8), together with (2.9), we have
\[
\Re(tA^{-1} + (1 - t)B^{-1})^{-1} \leq \sec^2(\theta)(t\Re^{-1}A + (1 - t)\Re^{-1}B)^{-1}. \tag{2.10}
\]
Making use definition of the relative operator entropy of two accretive operators and using (2.10), it follows that
\[
\Re(S(A|B)) = \int_{0}^{1} \frac{\Re(tA^{-1} + (1 - t)B^{-1})^{-1} - \Re A}{t} dt \\
\leq \int_{0}^{1} \sec^2(\theta)\frac{(t\Re A^{-1} + (1 - t)(\Re B)^{-1})^{-1} - \Re A}{t} dt \\
= \sec^2(\theta)S(\Re A|\Re B).
\]
This complete the proof. \qed

It is well known that for two positive operators \(A, B\), the informational monotonicity property of relative operator entropy satisfies \(\Phi(S(A|B)) \leq S(\Phi(A)|\Phi(B))\) for all unital positive linear maps \(\Phi\). A sectorial operator version of the previous inequality stands below.

**Theorem 2.5.** Let \(A, B \in M_n\) be such that \(W(A), W(B) \subseteq S_\theta\). Then
\[
\Re(\Phi(S(A|B))) \leq \sec^2(\theta)\Re S(\Phi(A)|\Phi(B)).
\]

**Proof.** We have the following chain of inequalities
\[
\Re(\Phi(S(A|B))) = \Phi(\Re(S(A|B))) \\
\leq \sec^2(\theta)\Phi(S(\Re A|\Re B)) \\
\leq \sec^2(\theta)S(\Phi(\Re A)|\Phi(\Re B)) \\
= \sec^2(\theta)S(\Re \Phi(A)|\Re \Phi(B)) \\
\leq \sec^2(\theta)\Re S(\Phi(A)|\Phi(B)).
\]
\qed
2.2. **Operator Kantorovich inequality.** For $A \in \mathbb{M}_n$ such that $0 < m \leq A \leq M$, Marshall and Olkin [12] obtained an operator Kantorovich inequality as follows:

$$\Phi(A^{-1}) \leq K(h)\Phi^{-1}(A),$$

where $\Phi$ is a positive unital linear map. The next Lemma is an extension of Kantorovich operator inequality.

**Theorem 2.6.** Let $A \in \mathbb{M}_n$ be such that $W(A) \subseteq S_\theta$ and $0 < m \leq RA \leq M$. Then

$$\Re\Phi(A^{-1}) \leq K(h)\sec^2(\theta)\Re\Phi^{-1}(A),$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ with $h = \frac{M}{m}$ is Kantorovich constant.

**Proof.** The desired inequality concludes by the computation of the following chain of the inequalities:

\[
\Re\Phi(A^{-1}) = \Phi(\Re A^{-1})(\text{by [15, Lemma 1]}) \\
\leq \Phi(\Re^{-1}A)(\text{by the first inequality of Lemma 2.1}) \\
\leq K(h)\Phi^{-1}(\Re A)(\text{by the Kantorovich inequality}) \\
= K(h)\Re^{-1}\Phi(A) \\
\leq K(h)\sec^2\Re\Phi^{-1}(A)(\text{by the second inequality of Lemma 2.1}).
\]

\[\square\]

**Corollary 2.7.** Let $A \in \mathbb{M}_n$ be such that $W(A) \subseteq S_\theta$ with $m \leq RA \leq M$. Then

$$\|\Phi(A^{-1})\| \leq K(h)\sec^2(\theta)\|\Phi^{-1}(A)\|,$$

where $\Phi$ is a positive unital linear map and $\|\cdot\|$ is any unitarily invariant norm.

**Proof.**

\[
\|\Phi(A^{-1})\| \leq \sec(\theta)\|\Re\Phi(A^{-1})\| \\
\leq K(h)\sec^2(\theta)\|\Re\Phi^{-1}(A)\| \\
\leq K(h)\sec^2(\theta)\|\Phi^{-1}(A)\|
\]

\[\square\]

**Corollary 2.8.** Let $A \in \mathbb{M}_n$ be accretive-dissipative with $0 < m \leq RA \leq M$. Then

$$\|\Phi(A^{-1})\| \leq 2\sqrt{2}K(h)\|\Phi^{-1}(A)\|,$$

where $\Phi$ is a positive unital linear map and $\|\cdot\|_u$ is any unitarily invariant norm.
2.3. Heron inequalities involving positive maps. Suppose that $A$ and $B$ are two strictly (invertible) positive operators in $\mathcal{B}(\mathcal{H})$. The operator version of the Heinz and heron means are defined by

$$F_\nu(A, B) = (1 - \nu)(A^\# B) + \nu(A \nabla B).$$

It is trivial that for $A, B > 0$ and $0 \leq \nu \leq 1$,

$$F_\nu(A, B) \leq A \nabla B. \quad (2.11)$$

Ando [1] proved that if $A, B \in M_n$ are positive definite, then for any positive linear map $\Phi$, it holds

$$\Phi(A^\# B) \leq \Phi(A)^{\#_\nu} \Phi(B). \quad (2.12)$$

Let $A$ and $B$ be two strictly (invertible) positive operators on a Hilbert space $\mathcal{H}$ such that $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_2 I \leq B \leq M_2 I$ and $0 < 1$, then for $0 < \nu \leq 1$

$$\Phi(A^\#_\nu B) \geq \Phi(A)^{\#_\nu} \Phi(B) K(\nu), \quad (2.13)$$

where $K(\nu) = \frac{h^\nu - h}{(\nu-1)(\nu-h)} \left( \frac{(h^\nu - 1)(\nu - 1)}{\nu(h^\nu - h)} \right)$ and $h = \frac{M_1 M_2}{m_1 m_2}$. Also, the constant $K(\nu)$ which is famous as the generalization Kantorovich constant has the following properties,

$$0 < K(\nu) \leq 1, \quad K(\nu) = K(1 - \nu).$$

Using (2.12) and (2.13), we have

$$K\left(\frac{1}{2}\right) F_\nu(\Phi(A), \Phi(B))$$

$$= (1 - \nu)K\left(\frac{1}{2}\right)\Phi(A)^{\#_\nu} \Phi(B) + \nu K\left(\frac{1}{2}\right)\Phi(A \nabla B)$$

$$\leq \Phi(F_\nu(A, B))$$

$$= (1 - \nu)\Phi(A^\#_\nu B) + \nu \Phi(A \nabla B)$$

$$= (1 - \nu)\Phi(A^\#_\nu B) + \nu \Phi(A) \nabla \Phi(B)$$

$$\leq (1 - \nu)(\Phi(A)^{\#_\nu} \Phi(B)) + \nu \Phi((A) \nabla \Phi(B))$$

$$= F_\nu(\Phi(A), \Phi(B)).$$

So,

$$K\left(\frac{1}{2}\right) F_\nu(\Phi(A), \Phi(B)) \leq \Phi(F_\nu(A, B)) \leq F_\nu(\Phi(A), \Phi(B)). \quad (2.14)$$

Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\theta$ and $\Phi$ be any positive linear map. The authors [17] obtained the following result:

$$\cos^2(\theta) \Re \Phi(A^\#_\nu B) \leq \Re(\Phi(A)^{\#_\nu} \Phi(B)). \quad (2.15)$$
The famous Choi’s inequality [2][p.41] says: if $\Phi$ is a positive unital linear map and $A > 0$, then

$$\Phi^t(A) \leq \Phi(A^t), \quad t \in [-1, 0]. \quad (2.16)$$

The first inequality in Theorem 2.9 is an extension of (2.14) with respect to Heron mean for sector matrices.

**Theorem 2.9.** Let $A, B \in M_n$ be such that $W(A), W(B) \subseteq S_\theta$ and $0 < m < \Re A, \Re B < M$. Then for any positive linear map $\Phi$ and $0 \leq \nu \leq 1$, we have

$$K(\frac{1}{2})F_{\nu}(\Phi(\Re A), \Phi(\Re B)) \leq \Re(\Phi(\Re F_{\nu}(A, B))) \leq \sec^2(\theta)F_{\nu}(\Phi(\Re A), \Phi(\Re B)),$$

$$\cos^2(\theta)\Re(\Phi(A^t B)) \leq \Re(\Phi(F_{\nu}(A, B))) \leq \sec^2(\theta)\Re(\Phi(A)\nabla\Phi(B)),$$

where $K(\frac{1}{2})$ is defined as (2.13).

**Proof.** Using (2.15), it follows

$$\Re(\Phi(F_{\nu}(A, B))) = (1 - \nu)\Re(\Phi(A^t B) + \nu\Re(\Phi(A\nabla B))$$

$$= (1 - \nu)\Re(\Phi(A^t B) + \nu(\Re(\Phi(\Re A))\nabla(\Re(\Phi(\Re B))))$$

$$\leq (1 - \nu)\sec^2(\theta)(\Re(\Phi(A))\nabla(\Re(\Phi(B)))) + \nu(\Re(\Phi(\Re A))\nabla(\Re(\Phi(\Re B))))$$

$$\leq (1 - \nu)\sec^2(\theta)(\Re(\Phi(A))\nabla(\Re(\Phi(B)))) + \sec^2(\theta)\nu(\Re(\Phi(\Re A))\nabla(\Re(\Phi(\Re B))))$$

$$= \sec^2(\theta)F_{\nu}(\Phi(\Re A), \Phi(\Re B)).$$

From Lemma 2.2 and (2.13), we get

$$\Re(\Phi(F_{\nu}(A, B))) = (1 - \nu)\Re(\Phi(A^t B) + \nu(\Re(\Phi(A\nabla B))$$

$$= (1 - \nu)\Phi(\Re A^t B) + \nu(\Phi(\Re A))\nabla(\Phi(\Re B))$$

$$\geq (1 - \nu)\Phi(\Re A^t B) + \nu(\Phi(\Re A))\nabla(\Phi(\Re B))$$

$$\geq (1 - \nu)K(\frac{1}{2})(\Phi(\Re A^t B) + \nu K(\frac{1}{2})(\Phi(\Re A))\nabla(\Phi(\Re B)))$$

$$= K(\frac{1}{2})F_{\nu}(\Phi(\Re A), \Phi(\Re B)).$$

Also, we have

$$\cos^2(\theta)\Re(\Phi(A^t B)) \leq \Re(\Phi(F_{\nu}(A, B)))(\text{by (1.3)}$$

$$= (1 - \nu)\Re(\Phi(A^t B) + \nu(\Re(\Phi(A\nabla B))$$

$$\leq \sec^2(\theta)\Re(\Phi(A)\nabla(\Phi(B)))(\text{by (2.15)}).$$

This proves the second inequality in Theorem 2.9. \qed
Theorem 2.10. Let $A, B \in M_n$ be such that $W(A), W(B) \subseteq S_\theta$ and $0 < m < \Re A, \Re B < M$. Then for any positive linear map $\Phi$ and $0 \leq \nu \leq 1$, we have

$$K(\frac{1}{2}) \cos^2(\theta) \Re^{-1}(F_\nu(\Phi(A), \Phi(B))) \leq \Re F_\nu^{-1}(\Phi(A), \Phi(B)) \leq \sec^2(\theta) \Re F_\nu(\Phi^{-1}(A), \Phi^{-1}(B)),$$

where $K(\frac{1}{2})$ is defined as (2.12).

Proof. Theorem 2.1 and operator convexity of the inverse function $f(t) = t^{-1}$ on positive real numbers, we deduce

$$\cos^2(\theta) K(\frac{1}{2}) \Re^{-1}(F_\nu(\Phi(A), \Phi(B)))$$

$$\leq K(\frac{1}{2}) \Re^{-1}(F_\nu(\Phi(A), \Phi(B)))$$

$$\leq K(\frac{1}{2}) \Re^{-1}(F_\nu(\Phi(A), \Phi(B)))$$

$$\leq F_\nu^{-1}(\Phi(\Re A), \Phi(\Re B))(\text{by Theorem 2.9})$$

$$= (\nu(\Phi(\Re A)\nabla \Phi(\Re B)) + (1 - \nu)(\Phi(\Re A)^{\sharp} \Phi(\Re B))^{-1}$$

$$\leq \nu(\Re^{-1}(\Phi A \nabla \Re^{-1} \Phi B) - 1 + (1 - \nu)(\Re A)^{\sharp} \Phi(\Re B))^{-1}$$

$$\leq \nu(\Re^{-1}(\Phi A \nabla \Re^{-1} \Phi B) + (1 - \nu)((\Re A)^{-1} \Phi(\Re B))^{-1}$$

$$\leq \sec^2(\theta) \nu(\Re(\Phi A)^{-1} \nabla \Re(\Phi B)^{-1}) + (1 - \nu)(\Re(\Phi A)^{-1} \Phi(\Re B)^{-1})$$

$$\leq \sec^2(\theta) [\nu(\Re(\Phi A)^{-1} \nabla \Re(\Phi B)^{-1}) + (1 - \nu)(\Re(\Phi A)^{-1} \Phi(\Re B)^{-1})](\text{by 2.16})$$

$$= \sec^2(\theta) \Re F_\nu(\Phi^{-1}(A), \Phi^{-1}(B)).$$

□

Theorem 2.11. Let $A, B \in M_n$ be such that $W(A), W(B) \subseteq S_\theta$ and $\nu \in (0, 1)$. Then for every positive unital linear map $\Phi$,

$$\Re \Phi(F_\nu^{-1}(A, B)) \leq \sec^2(\theta) \Re F_\nu(\Phi^{-1}(A), \Phi^{-1}(B)).$$
Proof. We have the following chain of inequalities

\[
\Re\Phi(F^{-1}_\nu(A, B)) \leq \Phi(\Re^{-1}(F^\nu(A, B))) \\
\leq \Phi(F^{-1}_\nu(\Re A, \Re B)) \\
= \Phi((\nu \Re A \nabla \Re B + (1 - \nu) \Re A \sharp \Re B)^{-1}) \\
\leq \Phi(\nu (\Re A \nabla \Re B)^{-1} + (1 - \nu) (\Re A \sharp \Re B)^{-1}) \\
\leq \Phi(\nu \Re^{-1} A \nabla \Re^{-1} B + (1 - \nu) \Re^{-1} A \sharp \Re^{-1} B) \\
\leq \sec^2(\theta) \Phi(\nu \Re^{-1} A \nabla \Re^{-1} B + (1 - \nu) \Re^{-1} A \sharp \Re^{-1} B) \\
\leq \sec^2(\theta)\nu \Re(A^{-1}) \nabla \Phi(B^{-1})) + (1 - \nu) \Re(\Re A^{-1}) \sharp \Re(\Re B^{-1}) \\
= \sec^2(\theta)\nu \Re(A^{-1}) \nabla \Phi(B^{-1})) + (1 - \nu) \Re(\Re A^{-1}) \sharp \Re(\Re B^{-1}) \\
\leq \sec^2(\theta)\nu \Re(A^{-1}) \nabla \Phi(B^{-1})) + (1 - \nu) \Re(\Re A^{-1}) \sharp \Re(\Re B^{-1}) \\
= \sec^2(\theta)\Re F^\nu(\Phi(A^{-1}), \Phi(B^{-1}))
\]

which completes the proof. □

Corollary 2.12. Let \( A, B \in M_n \) be such that \( W(A), W(B) \subseteq S_\theta \) and \( 0 < m < \Re A, \Re B < M \). Then for any positive linear map \( \Phi \) and \( 0 \leq \nu \leq 1 \), we have

\[
\omega(F^{-1}_\nu(\Phi(A), \Phi(B))) \leq \sec^4(\theta)\omega(F^\nu(\Phi(A)^{-1}, \Phi(B)^{-1})).
\]

Proof.

\[
\omega(F^{-1}_\nu(\Phi(A), \Phi(B))) \leq \sec^2(\theta)\omega(\Re F^{-1}_\nu(\Phi(A), \Phi(B)))(\text{by 2.3}) \\
\leq \sec^4(\theta)\omega(\Re F^\nu(\Phi(A)^{-1}, \Phi(B)^{-1})) (\text{by Theorem 2.10}) \\
\leq \sec^4(\theta)\omega(F^\nu(\Phi(A)^{-1}, \Phi(B)^{-1}))(\text{by 2.3})
\]

References