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Mohammad Aqil Sahil  
South Asian University

Meenakshi Kaushal (meenakshikaushal247@gmail.com)  
South Asian University

Q.M. Danish Lohani  
South Asian University

Research Article

Keywords: Fuzzy Sets, Data Envelopment Analysis, Fuzzy Data Envelopment Analysis, Banking Sector

Posted Date: December 16th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-2373131/v1

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Additional Declarations: No competing interests reported.
A Novel Pythagorean Fuzzy Data Envelopment Analysis Model: An Assessment of Indian Public Sector Banks

Mohammad Aqil Sahil\textsuperscript{1}, Meenakshi Kaushal\textsuperscript{1}† and Q.M. Danish Lohani\textsuperscript{1}

\textsuperscript{1}Applied Mathematics, South Asian University, Chanakyapuri, New Delhi, 110021, Delhi, India.

Contributing authors: mohammadaqilsahil@gmail.com; meenakshikaushal247@gmail.com; danishlohani@cs.sau.ac.in;
†These authors contributed equally to this work.

Abstract
Fuzzy data envelopment analysis (FDEA) is an efficient modeling technique to rank decision-making units (DMUs) with imprecise inputs/outputs. It is a linear programming problem that constructs an optimal frontier line, known as an efficient frontier. The role of the efficient frontier is to distinguish between the efficient DMUs with inefficient DMUs such that all efficient DMUs lie on the frontier line. Whereas the inefficient DMUs are enveloped by the frontier line. This optimization problem aims to improve the efficiency score of each inefficient DMU, that is, to move them to the efficient frontier. In this study, we propose a novel approach called the Pythagorean approach, which is both input and output oriented. The proposed approach is implemented in the CCR model, and a new version of the BCC model is introduced, namely a novel Pythagorean BCC model. The deterministic form of the Pythagorean BCC model is extended to a fuzzy environment to deal with the vagueness of the given dataset. In the paper, a new form of a non-linear fuzzy number, namely a sine-shaped fuzzy number, has been introduced to display the higher order of uncertainty. Consequently, the Pythagorean BCC model is further extended to a novel Pythagorean fuzzy BCC model. The model’s efficacy is finally tested in the Indian public sector banks.

Keywords: Fuzzy Sets, Data Envelopment Analysis, Fuzzy Data Envelopment Analysis, Banking Sector

1 Introduction
Data envelopment analysis (DEA) is a well-known performance evaluation technique to measure the relative efficiency of decision-making units (DMUs) with multiple inputs and outputs. The traditional DEA, known as the CCR model proposed by Charnes et al. \cite{1} is a ratio of virtual output to the virtual input called efficiency score. The virtual output/input is deduced by a weighted sum of given outputs/inputs. The efficiency score of each DMU is computed as maximization of the linear programming problem subject to the constraint that the ratio of virtual output to the virtual input is less than or equal to one. The model further, extended by Banker et al. \cite{2} in variable return to scale or BCC model produces a convex curve depicting piece-wise efficient frontier passing through all efficient DMUs. The curve of the efficient frontier is due to its unrestricted variable used in the constraints of the linear programming optimization problem. The DEA model measures the relative
efficiency of given DMUs based on given outputs (i.e., maximization problem) or given inputs (i.e., minimization problem).

The traditional DEA models are considered for the crisp inputs and outputs. Although, in real-world problems, various factors such as incomplete or unavailable information and non-quantifiable results cause fluctuation in the dataset. The uncertainty present in the dataset is described in terms of intervals, ordinal relations, or fuzzy numbers. The fuzzy set [3] is a suitable tool to handle impreciseness and uncertainty in real-world problems. In the literature, the DEA model with imprecise inputs and imprecise outputs has been extended to fuzzy DEA (FDEA). Hatami-Marbini et al. (see [4]) classified the FDEA models into four group approaches and later expanded into six group approaches by Emroujnezad et al. [5] which are the tolerance approach, the $\alpha$-cut approach, the fuzzy ranking approach, the fuzzy arithmetic approach, the possible approach, and type-2 fuzzy sets approach. Wang, Liu, and Liang constructed the efficiency modeling of FDEA with the help of fuzzy arithmetic operations (see [6]). Kao and Liu [7] formulated a pair of parametric FDEA models to derive the lower and upper bound on the membership function of efficiency score by using the $\alpha$-cut approach. Based on the $\alpha$-cut approach, Sahil et al. derived an FDEA model to evaluate the efficiency score of DMUs in the presence of parabolic fuzzy inputs and outputs (see [8]). Peykani et al. [9] developed an FDEA model with the help of possibility, necessity, and credibility measures. Nasseri et al. [10] constructed a fuzzy stochastic DEA model based on probability possibility, necessity, and probability credibility with undesirable outputs. The other studies on the FDEA model have been done by Fukuyama et al. [11], Puri, and Yadav [12], Dotoli M et al. [13], Arya A, and Yadav, S.P (see [14], [15]), Arya A, and Singh S [16].

In today's market competition, commercial banks have to spend huge budgets on new technologies to facilitate customer services to remain efficient and productive. Therefore, performance analysis has been selected as a major topic for decision-makers or policymakers in the banking industry. The decision maker wants to know the underlying cause of the inefficiency. To deal with this problem in the banking industry, two approaches, the traditional ratio approach, and the frontier approach, have been used to evaluate the measurement of performance analysis for the DMUs. Further, the frontier approach is classified into five categories given as, Stochastic Frontier Analysis (SFA), Distribution Free Approach (DFAP), Thick Frontier Approach (TFAP), Data Envelopment Analysis (DEA), and Free Disposal Hull (FrDH) [17]. DEA is one of the most used approaches in the banking industry, which measures the relative efficiency of banks with crisp inputs and outputs [18]. However, the inputs and outputs of every bank fluctuate due to various factors; for instance, some of the operational expenses are uncontrollable inputs like taxes, utility charges, cost of insurance, etc. Hence, the above-mentioned inputs are linguistic or imprecise and could be represented more realistic by fuzzy sets [3] rather than crisp sets. Consequently, fuzzy DEA is a suitable method for dealing with the such dataset. In the literature, some researchers have applied fuzzy DEA to measure the relative efficiency of banks. Kao and Liu [19] proposed the fuzzy DEA models to analyze the upper and lower performance of 24 commercial banks in Taiwan based on their financial forecast. Tlig and Ben Hamid [20] proposed fuzzy DEA models to evaluate the performance analysis of Tunisian commercial banks in the presence of non-financial inputs and outputs in both crisp and fuzzy form. Puri and Yadav [21] proposed a fuzzy DEA model to measure the relative efficiency of Indian commercial banks in the presence of undesirable outputs and imprecise inputs and outputs.

The efficiency of each DMU is depicted through a graphical representation to ascertain how far an inefficient DMU is from the efficient frontier. This two-dimensional graphical representation has been used by Charnes et al. [1] mainly for three cases (1) one input and one output; (2) two inputs and one output, and (3) one input and two outputs. In literature, e Costa, C.A.B., et al. [22] have computed the efficiency for more than three variables. Based on inputs weights or outputs weights normalization, a new approach has been proposed in [22], giving a bi-dimensional representation of DEA based on virtual inputs versus virtual outputs. The graphical representation of
DEA is illustrated such that the perpendicular line provides the shortest distance between an inefficient DMU with the efficient frontier while considering both the inputs and outputs. Therefore, instead of an input- or output-oriented approach, the Pythagorean approach to the CCR model is used in the paper. The CCR model based on the Pythagorean approach is further modified into a new version of the BCC model, namely a novel Pythagorean BCC model. Meanwhile, the proposed model is extended into the fuzzy environment to deal with imprecise inputs/outputs. In the literature of FDEA, the imprecise inputs and outputs are represented by triangular fuzzy numbers (TFNs) (see [12], [23] and trapezoidal fuzzy numbers, [24]). The membership function of TFNs is a piecewise function of linear equations that can represent the vague dataset with the linear property. The higher-order polynomial functions could be displayed by the sine function appropriately. Therefore, the membership function of a sine-shaped fuzzy number is a piecewise function of non-linear equations, which is a suitable tool to represent the higher-order impreciseness in the dataset. In this paper, we represent the uncertain inputs and outputs by a sine-shaped fuzzy number and extend the novel Pythagorean approach-based BCC model from a deterministic to a fuzzy environment. Further, we name our modified model as a novel Pythagorean fuzzy BCC model. In addition, we use the $\alpha$-cut approach to convert inputs and outputs from sine-shaped fuzzy numbers to an interval. Further, the convex combination formula [25] is used as a defuzzification technique to transform the input and output interval to their respective parametric crisp number for each value of $\alpha$. Consequently, we apply our proposed method to the Indian public sector banks to assess their performances.

The main contributions of the paper are as follows:

- The deterministic form of the Pythagorean BCC model is extended to the fuzzy environment to deal with the vagueness and impreciseness of the dataset, named as a novel Pythagorean fuzzy BCC model.
- A new form of non-linear fuzzy number known as the sine-shaped fuzzy number is introduced to validate the performance of the proposed model over Indian public banks.

The rest of the paper is organized into five sections. Section II outlines the essential preliminaries used in the proposed work. Section III gives the proposed theoretical work in the name of a novel Pythagorean fuzzy BCC model. In section IV, the experiment analysis of the proposed model has been discussed to verify the efficacy of the proposed model and the utility of the proposed defuzzification method over some examples. Section V assesses the performance of the Indian public sector banks using the proposed methodology. Finally, section VI states the conclusion.

2 Preliminaries

This section includes some basic definitions and mathematical formulation of data envelopment analysis.

2.1 Definition 1

Let $X$ be a universe discourse set [3]. Then a fuzzy set $\tilde{A}$ in $X$ is defined as follows:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\} \quad (1)$$

Here $\mu_{\tilde{A}} : X \to [0, 1]$ allocate the membership value and for every element $x \in X : \mu_{\tilde{A}}(x) \in [0, 1]$.

2.2 Definition 2

The fuzzy set, $\tilde{A}$ with its membership $\mu_{\tilde{A}}$, defined on set $\mathbb{R}$ of real numbers is called fuzzy number (FN) if it is normal and convex [3]. The mathematical representation of FN $\tilde{A} = (a_1, a_2, a_3, a_4)$ is given below:

$$\mu_{\tilde{A}}(x) = \begin{cases} f_A(x) & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ g_A(x) & a_3 < x \leq a_4 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$
Where $a_1 \leq a_2 \leq a_3 \leq a_4 \forall a_i \in \mathbb{R}$, $i = 1, 2, 3, 4$. The, $f_A$ is non-decreasing piecewise continuous functions, and $g_A$ is non-increasing piecewise continuous functions on their respective defined domain.

2.3 Definition 3

The $\alpha -$cut of a fuzzy set is a crisp set $\tilde{A}_\alpha$ [26], defined as follows:

$$\tilde{A}_\alpha = \{ x : \mu_{\tilde{A}}(x) \geq \alpha \} \quad \forall \alpha \in [0,1]$$ (3)

2.4 Convex combination formula

Let $[A, B]$ be an interval [25], then for $0 \leq \rho \leq 1$, we get $x_A$, $x_B \in [A, B]$ using the basic convex combination method which is as follows:

$$x_A = (1-\rho) \cdot A + \rho \cdot B \quad (4)$$

$$x_B = (1-\rho) \cdot B + \rho \cdot A \quad (5)$$

Where $x_A$ and $x_B$ are the points close to $A$ and $B$ respectively. The middle point of interval is obtained when $\rho = 0.5$.

2.5 Data envelopment analysis

Suppose there are $n$ decision making units (DMUs) with $m$ different inputs to produce $s$ different outputs. Let $X \in \mathbb{R}^{m \times n}, Y \in \mathbb{R}^{s \times n}$ be the input matrix and output matrix, respectively. Consequently, for the $k^{th}$ DMU, (where $k = 1, 2, \cdots , n$), $x_{ik}$ (where $i = 1, 2, \cdots , m$) and $y_{rk}$ (where $r = 1, 2, \cdots , s$) are the non-negative inputs and outputs, respectively [1]. Hence, the CCR and BCC models are given as follows:

2.5.1 The CCR model

The mathematical formulation of the CCR model is defined as follows:

$$\max E_k = \frac{\sum_{r=1}^{s} u_{rk} y_{rk}}{\sum_{i=1}^{m} v_{ik} x_{ik}} \quad (6)$$

$$s.t : \sum_{j=1}^{m} v_{ij} x_{ij} \leq 1, \quad \forall j = 1, 2, \cdots , n. \quad (7)$$

$$u_{rk} \geq 0 \quad \forall r, \quad v_{ik} \geq 0 \quad \forall i$$

2.5.2 The BCC model

The mathematical formulation of the BCC model is defined as follows:

$$\max E_k = \frac{\sum_{r=1}^{s} u_{rk} y_{rk} + u_\ast k}{\sum_{i=1}^{m} v_{ik} x_{ik}}$$ (8)

$$s.t : \sum_{j=1}^{m} v_{ij} x_{ij} \leq 1, \quad \forall j = 1, 2, \cdots , n. \quad (9)$$

$$u_{rk} \geq 0 \quad \forall r, \quad v_{ik} \geq 0 \quad \forall i, \quad u_\ast k \in \mathbb{R}$$

Where $u_{rk}$ is the $r^{th}$ weight of output and $v_{ik}$ is the $i^{th}$ weight of input, and $u_\ast k$ is the initial weight which is free from the sign.

3 A novel Pythagorean fuzzy BCC model

This section contributes to the theoretical part of the paper with four subsections. The first subsection describes the derivation of a novel sine-shaped fuzzy number and two related corollaries. The second subsection explains the derivation procedure of a novel Pythagorean BCC model. The third subsection introduces the novel Pythagorean fuzzy BCC model and the methodology of its solution. The fourth subsection describes the relationship between upper and lower parametric efficiency models. Consequently, we derive the upper parametric and lower parametric efficiency models using the $\alpha -$cut approach and the convex combination formula.
3.1 Definition: A sine-shaped fuzzy number

The fuzzy number, \( \tilde{A} = (a_1, a_2, a_3) \) is called a sine-shaped fuzzy number if its membership function is given by the following formula:

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\sin \left( \frac{x-a_1}{a_2-a_1} \cdot \frac{\pi}{2} \right) , & a_1 < x \leq a_2 \\
\sin \left( \frac{x-a_3}{a_2-a_3} \cdot \frac{\pi}{2} \right) , & a_2 < x < a_3 \\
0 , & \text{otherwise}
\end{cases}
\]

The sine-shaped fuzzy number is convex and normal such that \( \mu_{\tilde{A}}(a_2) = 1 \) describes the normality property of the fuzzy number. The graphical representation of a sine-shaped fuzzy number is shown in Fig. 1.

Fig. 1 A graphical representation of sine-shaped fuzzy number

Corollary 1 The \( \alpha \)-cut of FN can be defined as nested closed intervals. The set \( \tilde{A}_\alpha = \{ x : \mu_{\tilde{A}}(x) \geq \alpha \} = [a_L(\alpha), a_U(\alpha)], \alpha \in [0, 1] \) gives the \( \alpha \)-cut set for membership value. Here, \( a_L(\alpha) \) is increasing functions of \( \alpha \) and \( a_U(\alpha) \) is decreasing functions of \( \alpha \). Moreover, for a sine-shaped fuzzy number, the \( \alpha - cut \) have been derived as follows:

\[
\tilde{A}_\alpha = [a_1(1 - \frac{2}{\pi} \cdot \arcsin \alpha) + a_2 \cdot \frac{2}{\pi} \cdot \arcsin \alpha, \\
\hspace{2cm} a_3(1 - \frac{2}{\pi} \cdot \arcsin \alpha) + a_2 \cdot \frac{2}{\pi} \cdot \arcsin \alpha] \\
\forall \alpha \in [0, 1]
\]

3.2 A novel Pythagorean BCC Model

Suppose there are \( n \) decision-making units (DMUs) with \( m \) different inputs to produce \( s \) different outputs. Let \( X \in \mathbb{R}^{m \times n}, Y \in \mathbb{R}^{s \times n} \) be the input matrix, and output matrix, respectively. Consequently, for the \( k^{th} \) DMU, where \( (k = 1, 2, \cdots, n) \), the \( x_{ik} \) and \( y_{rk} \) are the non-negative inputs and outputs, respectively. Hereupon the virtual input and virtual output are denoted by \( XX \) and \( YY \), respectively, which are defined as follows:

\[
XX = \sum_{k=1}^{n} \sum_{i=1}^{m} u_{ik} x_{ik} \tag{14}
\]

\[
YY = \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} y_{rk} \tag{15}
\]

Based on virtual input versus virtual output, the bi-dimensional graphical representation is illustrated in Fig. 2 The green arrow shows the perpendicular line, which implies the maximization of virtual output and the minimization of virtual input. Suppose the maximized virtual output is denoted by \( YY \) and minimized virtual input by \( XX \) respectively; then, by using the law of Pythagoras in the right triangle, we get.

\[
\cos \alpha = \frac{XX - XX}{w} \tag{16}
\]

\[
\sin \alpha = \frac{-YY + YY}{w} \tag{17}
\]

\[
\theta = \frac{\pi}{2} - \alpha \tag{18}
\]
Fig. 2 A graphical derivation of proposed Model 3

Model 3: Pythagorean BCC model
The objective function for model 3 can be defined as follows:

\[
\max E_k = \sum_{k=1}^{n} \sum_{r=1}^{s} u_r y_r + w \cos \theta 
\]  
subject to:

\[
\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik} - w \sin \theta = 1
\]
\[
\sum_{j=1}^{n} \sum_{r=1}^{m} v_{rj} y_{rj} - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij} + w (\sin \theta + \cos \theta) \leq 0
\]
\[
u_{rk} \geq 0 \ \forall r, \ v_{ik} \geq 0 \ \forall i, \ w \in \mathbb{R}
\]

In Model 3, \(w\) is the perpendicular oriented, which is free from any sign, i.e.,

- If \(w = 0\), then the model is a constant return to scale model.
- If \(w \leq 0\), then the model is a decreasing return to scale model.
- If \(w \geq 0\), then the model is an increasing return to scale model.

In the mentioned above model, inputs and outputs are considered crisp numbers. Therefore, to investigate the imprecise inputs and outputs in the fuzzy area, the crisp data is represented in terms of sine-shaped fuzzy numbers and defined in the following subsection.

3.3 A novel Pythagorean fuzzy BCC model

Suppose there are \(n\) decision-making units (DMUs) with \(m\) different sine-shaped fuzzy inputs to produce \(s\) different sine-shaped fuzzy outputs. Let \(\tilde{x}_{ik} = (x^L_{ik}, x^M_{ik}, x^U_{ik})\) be the sine shaped fuzzy inputs of \(k^{th}\) DMU, \(\tilde{y}_{ik} = (y^L_{ik}, y^M_{ik}, y^U_{ik})\) be the sine shaped fuzzy outputs of \(k^{th}\) DMU. Then its mathematical formulation is obtained in the form of model 4.

Model 4: Pythagorean fuzzy BCC model
The objective function of model 4 is defined as:

\[
\max \tilde{E}_k = \sum_{k=1}^{n} \sum_{r=1}^{s} u_r \tilde{y}_r + w \cos \theta 
\]
subject to:

\[
\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik} - w \sin \theta = 1
\]
\[
\sum_{j=1}^{n} \sum_{r=1}^{m} v_{rj} y_{rj} - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij} + w (\sin \theta + \cos \theta) \leq 0
\]
\[
u_{rk} \geq 0 \ \forall r, \ v_{ik} \geq 0 \ \forall i, \ w \in \mathbb{R}
\]
\[
\begin{align*}
\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik} - w \sin \theta &= 1 \\
\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij} + w(\sin \theta + \cos \theta) &\leq 0
\end{align*}
\]

\(u_{rk} \geq 0 \ \forall r, v_{ik} \geq 0 \ \forall i, \ w \in \mathbb{R}

### Methodology for solving model 4

To solve Model 4, firstly, we apply the \(\alpha\)-cut approach to each sine-shaped fuzzy input and each sine-shaped fuzzy output; then, concerning Corollary 1, the following nested closed intervals are obtained. Let \(\tilde{x}_{ik} = (x_{ik}^L, x_{ik}^M, x_{ik}^U)\) be the sine shaped fuzzy inputs of \(k^{th}\) DMU, \(\tilde{y}_{ik} = (y_{ik}^L, y_{ik}^M, y_{ik}^U)\) be the sine shaped fuzzy outputs of \(k^{th}\) DMU. Hence their respective intervals are below where \(\alpha \in [0, 1]\).

\[
\begin{align*}
(\tilde{x}_{ik})_\alpha &= \\
[&x_{ik}^L(1 - \frac{2}{\pi} \cdot \arcsin \alpha) + x_{ik}^M \cdot \frac{2}{\pi} \cdot \arcsin \alpha, \\
&x_{ik}^U(1 - \frac{2}{\pi} \cdot \arcsin \alpha) + x_{ik}^M \cdot \frac{2}{\pi} \cdot \arcsin \alpha]
\end{align*}
\]

\[
(\tilde{y}_{rk})_\alpha = \\
[y_{rk}^L(1 - \frac{2}{\pi} \cdot \arcsin \alpha) + y_{rk}^M \cdot \frac{2}{\pi} \cdot \arcsin \alpha, \\
y_{rk}^U(1 - \frac{2}{\pi} \cdot \arcsin \alpha) + y_{rk}^M \cdot \frac{2}{\pi} \cdot \arcsin \alpha]
\]

Here \((\tilde{x}_{ik})_\alpha\) and \((\tilde{y}_{rk})_\alpha\) are transformed nested closed intervals of sine-shaped fuzzy inputs and sine shaped fuzzy outputs of \(k^{th}\) DMU respectively. To obtain optimal crisp values from the lower and upper bound of eqn. (29) and eqn. (30), the convex combination formula is used, and the following equations are obtained with the help of Corollary 2.

\[
\begin{align*}
\bar{x}_{ik}^L &= (1 - \frac{2}{\pi} \cdot \arcsin \alpha)(1 - \rho)x_{ik}^L + \rho x_{ik}^U + \frac{2}{\pi} \cdot \arcsin \alpha x_{ik}^M \\
\bar{x}_{ik}^U &= (1 - \frac{2}{\pi} \cdot \arcsin \alpha)(1 - \rho)x_{ik}^U + \rho x_{ik}^L + \frac{2}{\pi} \cdot \arcsin \alpha x_{ik}^M
\end{align*}
\]

\[
\begin{align*}
y_{ik}^L &= (1 - \frac{2}{\pi} \cdot \arcsin \alpha)((1 - \rho)y_{ik}^L + \rho y_{ik}^U) + \frac{2}{\pi} \cdot \arcsin \alpha y_{ik}^M \\
y_{ik}^U &= (1 - \frac{2}{\pi} \cdot \arcsin \alpha)((1 - \rho)y_{ik}^U + \rho y_{ik}^L) + \frac{2}{\pi} \cdot \arcsin \alpha y_{ik}^M
\end{align*}
\]

Where \(\rho \in [0, 1]\) is a ratio number for the convex combination formula and \(\alpha \in [0, 1]\), is the parameter. To substitute these values in model 4 based on [7], model 4 is divided into two parametric models, i.e., upper parametric efficiency model and lower parametric efficiency model.

### Model 5: Upper parametric efficiency model

The objective function of model 5 can be defined as follows:

\[
\max E_k^U = \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} y_{ik}^U + w \cos \theta
\]

subject to:

\[
\begin{align*}
\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik}^L - w \sin \theta &= 1 \\
\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj} y_{rj}^U - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij}^L + w(\sin \theta + \cos \theta) &\leq 0 \\
\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj} y_{rj}^L - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij}^U + w(\sin \theta + \cos \theta) &\leq 0 \\
u_{rk} \geq 0 \ \forall r, v_{ik} \geq 0 \ \forall i, \ w \in \mathbb{R}
\end{align*}
\]

Where \(u_{rk}\) is the \(r^{th}\) weight of sine-shaped fuzzy output and \(v_{ik}\) is the \(i^{th}\) weight of sine-shaped fuzzy input, and \(w\) is perpendicularly oriented which is free from the sign.

### Model 6: Lower parametric efficiency model

The objective function for model 6 can be defined as follows:

\[
\max E_k^L = \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} y_{ik}^L + w \cos \theta
\]

subject to:

\[
\begin{align*}
\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik}^L - w \sin \theta &= 1 \\
\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj} y_{rj}^L - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij}^L + w(\sin \theta + \cos \theta) &\leq 0 \\
\sum_{j=1}^{n} \sum_{r=1}^{s} u_{rj} y_{rj}^U - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_{ij}^U + w(\sin \theta + \cos \theta) &\leq 0 \\
u_{rk} \geq 0 \ \forall r, v_{ik} \geq 0 \ \forall i, \ w \in \mathbb{R}
\end{align*}
\]
\begin{equation}
\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} \mathbf{f}_{ik}^{U} - w \sin \theta = 1
\end{equation}
\begin{equation}
\sum_{j=1}^{n} \sum_{i=1}^{m} u_{ij} \mathbf{y}_{ij}^{L} - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} \mathbf{x}_{ij}^{U} + w(\sin \theta + \cos \theta) \leq 0
\end{equation}
\begin{equation}
\sum_{j=1}^{n} \sum_{i=1}^{m} u_{ij} \mathbf{y}_{ij}^{U} - \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} \mathbf{x}_{ij}^{L} + w(\sin \theta + \cos \theta) \leq 0
\end{equation}

\[ u_{rk} \geq 0 \quad \forall r, \forall i, \quad w \in \mathbb{R} \]

Where \( u_{rk} \) is the \( r^{th} \) weight of sine-shaped fuzzy output and \( v_{ik} \) is the \( i^{th} \) weight of sine-shaped fuzzy input, and \( w \) is perpendicularly oriented which is free from the sign. The parameter \( \alpha = 0 \) provides the smallest lower bound and greatest upper bound for each fuzzy input and each fuzzy output while the lower bound of each fuzzy input and each fuzzy output equals their respective upper bound when \( \alpha = 1 \). Therefore, the efficiency values for model 5 and model 6 are the same and are equal to the deterministic Pythagorean BCC model when \( \alpha = 1 \). The solution procedure of a sine-shaped fuzzy BCC model could be explained by the following algorithm I.

**Algorithm I**

Step 1: Determine the dataset (Inputs and Outputs).

Step 2: If inputs and outputs are in crisp form, go to step 3; else, if inputs and outputs are in fuzzy form, go to step 5.

Step 3: Specify the angle (\( \theta \)) between the virtual input axis and efficient frontier and go to step 4 in case of the crisp dataset and step 5 in case of the fuzzy dataset.

Step 4: Calculate the efficiency score of DMUs with Model 3.

Step 5: Represent inputs and outputs by the membership function of a sine-shaped fuzzy number, then go to step 7.

Step 6: Determine the parameter \( \alpha, \alpha \in [0, 1] \) and go to step 7.

Step 7: Determine inputs and outputs intervals using eqn. (29) and eqn. (30) respectively then go to step 8.

Step 8: Determine parametric inputs using eqn. (31), eqn. (32) and parametric outputs using eqn. (33), eqn. (34), then go to steps 9 and 10.

Step 9: Calculate the upper parametric efficiency score of DMUs using Model 5.

Step 10: Calculate the lower parametric efficiency score of DMUs using Model 6.

The architecture of Algorithm I is illustrated in Fig. 3.

3.4 Relation between upper and lower parametric efficiency models

Let \( y_{rk}^{U}, y_{rk}^{L} \) be the \( r^{th} \) upper bound and lower bound outputs and \( x_{ik}^{U}, x_{ik}^{L} \) be the \( i^{th} \) upper bound and lower bound inputs of \( k^{th} \) DMU and let \( u_{rk}, v_{ik} \) be their corresponding weights. Therefore, for \( \Delta_{rk} = y_{rk}^{U} - y_{rk}^{L} \) where \( (r = 1, 2, 3, \ldots s) \) we obtain the mathematical formulation for the upper parametric efficiency model as follows:

\[ \max E_{k}^{U} = \frac{\sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} y_{rk}^{U} + w \cos \theta}{\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik}^{U} - w \sin \theta} \]

\[ = \frac{\sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} y_{rk}^{U} + w \cos \theta}{\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik}^{U} - w \sin \theta} \]

\[ = \frac{\sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} y_{rk}^{U} + w \cos \theta + \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} \Delta_{rk}}{\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik}^{U} - w \sin \theta} \]

\[ = \frac{\sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} y_{rk}^{U} + w \cos \theta + \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} \Delta_{rk}}{\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik}^{U} - w \sin \theta} \]

\[ = \max E_{K}^{L} + \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} \Delta_{rk} \]

\[ = \frac{\sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} \Delta_{rk}}{\sum_{k=1}^{n} \sum_{i=1}^{m} v_{ik} x_{ik}^{U} - w \sin \theta} \]

Similarly, we can derive the mathematical formulation for the lower parametric efficiency model as follows:

\[ \max E_{k}^{L} = \max E_{K}^{U} - \sum_{k=1}^{n} \sum_{r=1}^{s} u_{rk} \Delta_{rk} \]

The upper and lower parametric efficiency model have the same inequality constraint, which is eqn. (38). The equality constraint of the upper parametric efficiency model is equal to one in eqn. (39) while the equality constraint of the lower parametric efficiency model is equal to one in eqn. (40).

4 Numerical Illustration

This section shows the validity and advantage of the proposed model in both deterministic and
fuzzy forms; for this reason, we consider two different examples. The input and output data are given in the crisp form in the first example, while the second example provides the fuzzy version of the dataset. These two examples conclude that the model is valid for both crisp and fuzzy datasets. Secondly, it shows its advantage that it is both output-oriented (maximization) and input-oriented (minimization) simultaneously. The proposed model maximizes the virtual output and minimizes the virtual input simultaneously, whereas the traditional BCC model either maximizes the outputs or minimizes the inputs.

**Example 1: Hospital dataset**
The crisp dataset of 14 hospitals with two inputs (doctors, nurses) and two outputs (outpatient, inpatient) are considered and has been given in Table 1. Table 2, is depicted the weights of inputs, weights of outputs, the value of a free variable, say ($w$), and the efficiency value of each hospital using model 3. The value $2.60e−05$ used in Table 2 is given by $2.60 \cdot 10^{-5}$.

**Example 2: Fuzzy Sample dataset**
Consider the analysis performance of 12 DMUs with two fuzzy inputs and two fuzzy outputs given in Table 3. The input and output data have been taken from [12]. Table 4 represent the upper and lower measurement efficiency score of model 5 and model 6, respectively, for different stages of $\alpha$–cut approach values.

### 5 Application to the Indian public sector banks

In this section, we implement the proposed methodology over the bank sector. We select the Indian public sector banks for efficiency evaluation of two financial years, 2019 and 2020. Table 5 lists the Indian public sector banks in 2019 and 2020.

**RBI Dataset:** Data for Indian public sector banks are collected from the reserve bank of India (RBI) for 2019 and 2020. It is given in Table 6 and Table 7, respectively. The selected data includes interest expended and operating expenses as inputs, interest earned, and other income as outputs.

**Interest expended:** It is the sum of three entities called (a) Interest on deposits, (b) Interest
Table 1  Dataset with two Inputs (doctors and nurses) and two Outputs (outpatient and inpatient)

<table>
<thead>
<tr>
<th>Hospitals</th>
<th>Doctors</th>
<th>Nurses</th>
<th>Outpatient</th>
<th>Inpatient</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>3008</td>
<td>20980</td>
<td>97775</td>
<td>101225</td>
</tr>
<tr>
<td>H2</td>
<td>3985</td>
<td>25643</td>
<td>135871</td>
<td>130580</td>
</tr>
<tr>
<td>H3</td>
<td>4324</td>
<td>26978</td>
<td>133655</td>
<td>168473</td>
</tr>
<tr>
<td>H4</td>
<td>3534</td>
<td>25361</td>
<td>46243</td>
<td>100407</td>
</tr>
<tr>
<td>H5</td>
<td>8836</td>
<td>40796</td>
<td>176661</td>
<td>215616</td>
</tr>
<tr>
<td>H6</td>
<td>5376</td>
<td>37562</td>
<td>182576</td>
<td>217615</td>
</tr>
<tr>
<td>H7</td>
<td>4982</td>
<td>33088</td>
<td>98880</td>
<td>167278</td>
</tr>
<tr>
<td>H8</td>
<td>4775</td>
<td>39122</td>
<td>136701</td>
<td>193393</td>
</tr>
<tr>
<td>H9</td>
<td>8046</td>
<td>42958</td>
<td>225138</td>
<td>256575</td>
</tr>
<tr>
<td>H10</td>
<td>8554</td>
<td>48955</td>
<td>257370</td>
<td>312877</td>
</tr>
<tr>
<td>H11</td>
<td>6147</td>
<td>45514</td>
<td>165274</td>
<td>227099</td>
</tr>
<tr>
<td>H12</td>
<td>8366</td>
<td>55140</td>
<td>203989</td>
<td>321623</td>
</tr>
<tr>
<td>H13</td>
<td>13479</td>
<td>68037</td>
<td>174270</td>
<td>341743</td>
</tr>
<tr>
<td>H14</td>
<td>21808</td>
<td>78302</td>
<td>322990</td>
<td>487539</td>
</tr>
</tbody>
</table>

Table 2  Efficiency score of model 3 and the corresponding weight values

<table>
<thead>
<tr>
<th>Hospitals</th>
<th>weight of I1</th>
<th>weight of I2</th>
<th>weigh of O1</th>
<th>weight of O2</th>
<th>w values</th>
<th>$E^C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>1.00e-06</td>
<td>6.42e-05</td>
<td>7.12e-06</td>
<td>1.00e-06</td>
<td>0.4053</td>
<td>1.0000</td>
</tr>
<tr>
<td>H2</td>
<td>2.60e-04</td>
<td>1.00e-06</td>
<td>6.14e-06</td>
<td>1.00e-06</td>
<td>0.07142</td>
<td>1.0000</td>
</tr>
<tr>
<td>H3</td>
<td>1.00e-06</td>
<td>3.83e-05</td>
<td>1.00e-06</td>
<td>5.08e-06</td>
<td>0.0449</td>
<td>1.0000</td>
</tr>
<tr>
<td>H4</td>
<td>3.31e-04</td>
<td>1.00e-06</td>
<td>1.00e-06</td>
<td>6.03e-06</td>
<td>0.2256</td>
<td>0.7649</td>
</tr>
<tr>
<td>H5</td>
<td>1.00e-06</td>
<td>2.53e-05</td>
<td>3.26e-06</td>
<td>1.09e-06</td>
<td>0.0499</td>
<td>0.8370</td>
</tr>
<tr>
<td>H6</td>
<td>1.46e-04</td>
<td>1.00e-06</td>
<td>1.08e-06</td>
<td>4.15e-06</td>
<td>-0.2026</td>
<td>1.0000</td>
</tr>
<tr>
<td>H7</td>
<td>2.22e-04</td>
<td>4.27e-07</td>
<td>1.00e-06</td>
<td>3.85e-06</td>
<td>0.1386</td>
<td>0.8120</td>
</tr>
<tr>
<td>H8</td>
<td>2.26e-04</td>
<td>1.00e-06</td>
<td>1.00e-06</td>
<td>4.05e-06</td>
<td>0.1368</td>
<td>0.9901</td>
</tr>
<tr>
<td>H9</td>
<td>1.00e-06</td>
<td>2.40e-05</td>
<td>3.14e-06</td>
<td>1.00e-06</td>
<td>0.0453</td>
<td>0.9868</td>
</tr>
<tr>
<td>H10</td>
<td>9.60e-05</td>
<td>1.00e-06</td>
<td>2.96e-06</td>
<td>1.00e-06</td>
<td>-0.1494</td>
<td>1.0000</td>
</tr>
<tr>
<td>H11</td>
<td>1.30e-04</td>
<td>1.00e-06</td>
<td>1.06e-06</td>
<td>3.67e-06</td>
<td>-0.1794</td>
<td>0.9994</td>
</tr>
<tr>
<td>H12</td>
<td>9.88e-05</td>
<td>1.00e-06</td>
<td>1.00e-06</td>
<td>2.63e-06</td>
<td>-0.1364</td>
<td>0.9820</td>
</tr>
<tr>
<td>H13</td>
<td>3.63e-05</td>
<td>1.00e-06</td>
<td>1.00e-06</td>
<td>2.55e-06</td>
<td>-0.5096</td>
<td>0.7923</td>
</tr>
<tr>
<td>H14</td>
<td>2.26e-05</td>
<td>1.00e-06</td>
<td>2.35e-06</td>
<td>1.00e-06</td>
<td>-0.4949</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Source of data [1]

Operating expenses: It is the sum of twelve entities given as (a) Payments to and provisions for employees, (b) Rent, taxes and lighting, (c) Printing and stationery, (d) Advertisement and publicity, (e) Depreciation on bank’s property, (f) Directors’ fees, allowances, and expenses, (g) Auditors’ fees and expenses, (h) Law charges, (i) Postage, telegrams, telephones, etc., (j) Repairs and maintenance, (k) Insurance, and (l) Other expenditure.

Interest earned: It is the sum of four entities, namely (a) Interest/Discount earned on advances/bills, (b) Income on investments, (c) Interest on balances with RBI and other inter-bank funds, and (d) Others.

Other income: It is the sum of six entities, (a) Commission, exchange, and brokerage, (b) Net profit (loss) on sale of investments, (c) Net profit (loss) on revaluation of investments, (d) Net profit (loss) on sale of land and other assets, (e) Net profit (loss) on exchange transactions, and (f) Miscellaneous income. The dataset has been extracted from the RBI website (https://www.rbi.org.in).
Table 3  Sine shaped fuzzy inputs/outputs

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Input 1</th>
<th>Input 2</th>
<th>Output 1</th>
<th>Output 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(17,20,23)</td>
<td>(148,151,153)</td>
<td>(97,100,103)</td>
<td>(86,90,95)</td>
</tr>
<tr>
<td>2</td>
<td>(15,19,23)</td>
<td>(129,131,134)</td>
<td>(148,150,151)</td>
<td>(48,50,53)</td>
</tr>
<tr>
<td>3</td>
<td>(22,25,28)</td>
<td>(158,160,163)</td>
<td>(157,160,163)</td>
<td>(53,55,57)</td>
</tr>
<tr>
<td>4</td>
<td>(23,27,29)</td>
<td>(165,168,170)</td>
<td>(178,180,183)</td>
<td>(70,72,74)</td>
</tr>
<tr>
<td>5</td>
<td>(20,22,25)</td>
<td>(154,158,162)</td>
<td>(90,94,99)</td>
<td>(63,66,69)</td>
</tr>
<tr>
<td>6</td>
<td>(52,55,59)</td>
<td>(250,255,259)</td>
<td>(228,230,231)</td>
<td>(87,90,93)</td>
</tr>
<tr>
<td>8</td>
<td>(27,31,33)</td>
<td>(202,206,208)</td>
<td>(151,152,225)</td>
<td>(81,88,92)</td>
</tr>
<tr>
<td>9</td>
<td>(25,30,35)</td>
<td>(241,244,246)</td>
<td>(187,190,194)</td>
<td>(97,100,102)</td>
</tr>
<tr>
<td>10</td>
<td>(47,50,54)</td>
<td>(262,268,271)</td>
<td>(246,250,253)</td>
<td>(97,100,104)</td>
</tr>
<tr>
<td>12</td>
<td>(34,38,40)</td>
<td>(281,284,286)</td>
<td>(248,252,256)</td>
<td>(119,120,123)</td>
</tr>
</tbody>
</table>

Source of dataset [12]

Table 4  The parametric efficiency values of model 5

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Interval Parametric Efficiency Score = [max $E_{o_k}^L$, max $E_{o_k}^U$]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 0$</td>
</tr>
<tr>
<td>1</td>
<td>[0.9189,1.0000]</td>
</tr>
<tr>
<td>2</td>
<td>[0.9600,1.0000]</td>
</tr>
<tr>
<td>3</td>
<td>[0.8691,0.9188]</td>
</tr>
<tr>
<td>4</td>
<td>[0.9597,1.0000]</td>
</tr>
<tr>
<td>5</td>
<td>[0.7209,0.7944]</td>
</tr>
<tr>
<td>6</td>
<td>[0.8893,0.9540]</td>
</tr>
<tr>
<td>7</td>
<td>[0.9326,1.0000]</td>
</tr>
<tr>
<td>8</td>
<td>[0.7541,0.8253]</td>
</tr>
<tr>
<td>9</td>
<td>[0.8421,1.0000]</td>
</tr>
<tr>
<td>10</td>
<td>[0.9627,1.0000]</td>
</tr>
<tr>
<td>11</td>
<td>[0.9862,1.0000]</td>
</tr>
<tr>
<td>12</td>
<td>[0.9611,1.0000]</td>
</tr>
</tbody>
</table>

Fuzzification of dataset
Real-world phenomena always appear with some degree of fluctuation that implies imprecision and vagueness. Consequently, in the banking industry, the inputs and outputs are millions. The imprecision occurs due to differences between actual data and the given data, for instance, the actual data of operating expenses in a bank is 16343.68 crores, and the given data of operating expenses for the different area might be 16343.6 or 16343.7 or 16344, crores. Then, the absolute difference between the actual data with the given data, i.e., 0.08, 0.02, and 0.32 crores, results in low accuracy of the performance measurement of a bank. Therefore, we fuzzify the data as sine-shaped fuzzy numbers. The mathematical representation of a sine-shaped fuzzy number is a triplet $(x^L, x^M, x^U)$, where $x^L$ is the lower bound, $x^M$ shows the middle bound, and $x^U$ displays the upper bound of a fuzzy number. Therefore, in this study, we represent the collected crisp data as the middle bound of sine-shaped fuzzy numbers, i.e., $x^M$. The lower and upper bound have been selected concerning inputs and outputs. According to the information from RBI, there is a variation of 2% for the lower bound of the inputs and a variation of 1% for the upper bound of the inputs. Therefore, the 2% variation of $x_{ik}$ is considered as lower bound $(x^L)$, 1% variation of $x_{ik}$ is considered as upper bound $(x^U)$ and the middle bound $x^M$ is given by $x_{ik}$ itself. Similarly, there is a variation
of 1.5\% for the lower bound of outputs and a variation of 1\% for the upper bound of the outputs. 
Hence, the 1.5\% variation of \( y_{rk} \) is considered as lower bound (\( y^L \)), 1\% variation of \( y_{rk} \) as upper bound (\( y^U \)) and the middle bound (\( y^M \)) is given as \( y_{rk} \) itself.

### Result and discussion

The performance of Indian public sector banks for 2019 and 2020 has been evaluated with model 5 and model 6. Model 5 estimates the upper bound parametric efficiency score for different values of \( \alpha \), whereas the lower bound parametric efficiency score of DMUs is considered by model 6. Table 8 represents the efficiency intervals for the bank dataset for 2019, and Table 9 represents the efficiency intervals for 2020. The graphical representation of Table 8 is shown in Fig. 4 while Fig. 5 indicates the graphic illustration of Table 9. Fig. 4 and Fig. 5 depict the scatter plot of efficiency values for the Indian public sector banks concerning
Table 7  Dataset with two Inputs (Interest expended and operating expenses) and two Outputs (Interest earned and other income)

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Interest expended (I1)</th>
<th>Operating expenses (I2)</th>
<th>Interest earned (O1)</th>
<th>Other income (O2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALB</td>
<td>11474.62</td>
<td>4818.58</td>
<td>16925.02</td>
<td>2415.69</td>
</tr>
<tr>
<td>ANB</td>
<td>12795.49</td>
<td>4639.04</td>
<td>19784.18</td>
<td>2742.94</td>
</tr>
<tr>
<td>BOB</td>
<td>48532.37</td>
<td>18872.39</td>
<td>7593.66</td>
<td>10317.32</td>
</tr>
<tr>
<td>BOI</td>
<td>27096.29</td>
<td>10451.40</td>
<td>4235.27</td>
<td>6713.07</td>
</tr>
<tr>
<td>BOMH</td>
<td>7216.65</td>
<td>3080.96</td>
<td>11495.45</td>
<td>1649.23</td>
</tr>
<tr>
<td>CB</td>
<td>35811.08</td>
<td>11577.23</td>
<td>48934.99</td>
<td>7813.15</td>
</tr>
<tr>
<td>CBH</td>
<td>15933.62</td>
<td>6921.52</td>
<td>23562.47</td>
<td>3636.82</td>
</tr>
<tr>
<td>COPB</td>
<td>10942.1</td>
<td>5175.19</td>
<td>16170.58</td>
<td>3749.20</td>
</tr>
<tr>
<td>IB</td>
<td>13798.55</td>
<td>4420.84</td>
<td>21404.97</td>
<td>3312.46</td>
</tr>
<tr>
<td>IOVB</td>
<td>12103.28</td>
<td>5128.83</td>
<td>17406.11</td>
<td>3359.68</td>
</tr>
<tr>
<td>OBC</td>
<td>1357.35</td>
<td>5836.02</td>
<td>19193.07</td>
<td>3204.93</td>
</tr>
<tr>
<td>PSB</td>
<td>5871.98</td>
<td>1858.03</td>
<td>79295.53</td>
<td>897.40</td>
</tr>
<tr>
<td>FND</td>
<td>36362.24</td>
<td>11973.37</td>
<td>35800.03</td>
<td>9274.13</td>
</tr>
<tr>
<td>SBI</td>
<td>15938.77</td>
<td>75374.69</td>
<td>257333.59</td>
<td>4522.48</td>
</tr>
<tr>
<td>SB</td>
<td>14744.58</td>
<td>6744.78</td>
<td>21915.21</td>
<td>3047.12</td>
</tr>
<tr>
<td>UB</td>
<td>10042.06</td>
<td>3127.89</td>
<td>15134.33</td>
<td>2871.21</td>
</tr>
<tr>
<td>UBI</td>
<td>25794.37</td>
<td>7516.41</td>
<td>37231.12</td>
<td>5260.79</td>
</tr>
<tr>
<td>UNBI</td>
<td>6669.88</td>
<td>5404.00</td>
<td>9655.54</td>
<td>2589.04</td>
</tr>
</tbody>
</table>

Table 8  The parametric efficiency values for Indian public sector banks for the year 2019

<table>
<thead>
<tr>
<th>DMUs</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.3$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.7$</th>
<th>$\alpha = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALB</td>
<td>[0.4458,0.9523]</td>
<td>[0.5176,0.9512]</td>
<td>[0.5750,0.9504]</td>
<td>[0.6481,0.9494]</td>
<td>[0.9462,0.9462]</td>
</tr>
<tr>
<td>ANB</td>
<td>[0.4976,1.0000]</td>
<td>[0.5689,1.0000]</td>
<td>[0.6267,1.0000]</td>
<td>[0.7003,1.0000]</td>
<td>[1.0000,1.0000]</td>
</tr>
<tr>
<td>BOB</td>
<td>[0.7692,1.0000]</td>
<td>[0.8900,1.0000]</td>
<td>[0.8389,1.0000]</td>
<td>[0.8748,1.0000]</td>
<td>[1.0000,1.0000]</td>
</tr>
<tr>
<td>BOI</td>
<td>[0.6934,0.9491]</td>
<td>[0.7362,0.9486]</td>
<td>[0.7687,0.9482]</td>
<td>[0.8078,0.9478]</td>
<td>[0.9463,0.9463]</td>
</tr>
<tr>
<td>BOMH</td>
<td>[0.3063,1.0000]</td>
<td>[0.3896,1.0000]</td>
<td>[0.4596,1.0000]</td>
<td>[0.5533,1.0000]</td>
<td>[1.0000,1.0000]</td>
</tr>
<tr>
<td>CB</td>
<td>[0.7383,0.9946]</td>
<td>[0.7812,0.9946]</td>
<td>[0.8139,0.9945]</td>
<td>[0.8537,0.9945]</td>
<td>[0.9944,0.9944]</td>
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<td>[0.4408,1.0000]</td>
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<td>[0.6227,0.9570]</td>
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</table>

$\alpha$ values for the years 2019 and 2020, respectively. The parameter $\alpha = 0$ provides the smallest lower and greatest upper bound for each sine-shaped fuzzy input and output. Then, the lower bound of each sine-shaped fuzzy input and the output value equal their respective upper bounds when $\alpha = 1$. Therefore, both Table 8 and Table 9 and their corresponding Fig. 4 and Fig. 5 portray the variation in upper parametric efficiency score and lower parametric efficiency score for different values of $\alpha$ except $\alpha = 1$. With decreasing
Table 9 The parametric efficiency values for Indian public sector banks for the year 2020

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Interval parametric efficiency score = $[\max E_{\alpha k}, \max E_{\alpha k}]$</th>
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<tbody>
<tr>
<td>ALB</td>
<td>$\alpha = 0$</td>
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<tr>
<td>ANB</td>
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<td>BOB</td>
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<td>BOI</td>
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<td>BOMH</td>
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<tr>
<td>CB</td>
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<tr>
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<td>$0.5225,0.9320$</td>
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<tr>
<td>COPB</td>
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<td>IB</td>
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</tr>
<tr>
<td>OBC</td>
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<tr>
<td>PSB</td>
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<tr>
<td>PNB</td>
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<tr>
<td>SBI</td>
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<td>SB</td>
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<td>UNBI</td>
<td>$0.3859,1.0000$</td>
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</table>

values of $\alpha$ from 1 to 0, the upper parametric efficiency score is increasing, and the lower parametric efficiency score is decreasing such that $\alpha = 0$ provides the greatest upper parametric efficiency score and smallest lower parametric efficiency score. This observation from Table 8 and Table 9 and their graphical illustration Fig. 4 and Fig. 5 give the best and the worst performances evaluation of Indian public sector banks for 2019 and 2020, respectively. Based on the lower parametric efficiency score, the best performances are given by SBI (state bank of India) and PNB (Punjab national bank), while the worst performances are given by BOMH (bank of Maharashtra) and PSB (Punjab and Sind banks).

6 Conclusion

In the paper, a novel fuzzy data envelopment analysis (FDEA) model has been introduced to assess the performance of banks, known as a novel Pythagorean fuzzy BCC model. This proposed model has been further divided into two efficiency models: upper and lower parametric. The upper parametric efficiency model identifies the DMU giving the best performance among others to measure growth. In contrast, the lower parametric efficiency model identifies the DMU with the worst performance among others during a recession. Here, the Indian public sector banks are considered an empirical part of the paper to evaluate the best and the worst performance of various banks under the uncertain environment. We have also fuzzified the crisp dataset to sine-shaped fuzzy numbers to represent the uncertainty in the given dataset in the non-linear form. The traditional DEA and FDEA models either maximize the outputs or minimize the inputs, but the corresponding DMUs remain inefficient. Therefore, in the paper, we come across a different approach called the Pythagorean approach that provides a way to improve the performance of each DMU. It uses the shortest distance between the inefficient DMU and the efficient frontier, simultaneously considering the outputs maximization and the inputs minimization. As a result, each DMU can be made efficient to create an optimal workplace. As a part of future work, we will extend our proposed Pythagorean approach to the dynamical system two-stage data envelopment analysis in the presence of uncertainty.

Funding: The research work is not funded by any organization.

Author contribution: The authors confirm contribution to the paper as follows: Mohammad Aqil Sahil: study conception and design, data collection, analysis and interpretation of results;
Meenakshi Kaushal: draft manuscript preparation; Q.M. Danish Lohani reviewed the results and approved the final version of the manuscript. 

Conflict of interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability statement: The research work has used dataset from the website (https://www.rbi.org.in).

References


Model (1998)


