Experimental observation of fixed lines in a particle accelerator

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Experimental observation of fixed lines in a particle accelerator

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The motion of systems with linear restoring forces and recurring nonlinear perturbations is of central importance in physics: when a system’s natural oscillation frequencies and the frequency of the nonlinear restoring forces satisfy certain algebraic relations, the dynamics becomes resonant. Such a resonant dynamics is of interest in a broad range of applications and happens at any scale, from electron cyclotron resonances observed in plasma physics\(^1\) to condensed-matter physics\(^2\) and to the planetary motion of the solar system\(^3\). In accelerator physics\(^4\), the understanding of resonances and nonlinear dynamics is crucial to avoid particle loss. Here, we report on an experiment performed at the CERN Super Proton Synchrotron (SPS) to investigate a second order coupled resonance by measuring the position of a particle beam at discrete locations around the accelerator. Despite instrumental noise, we have constructed the “Poincaré surface of section”\(^5\), which is described by the Arnold resonant tori\(^6\), and found convincing evidence of a “fixed line structure” generated by the resonance.

The complexity of resonant dynamics depends on the number of degrees of freedom of the problem. A pendulum has one degree of freedom (1D)\(^8\), while a chain of \(N\) masses bounded by springs forming a Fermi-Pasta-Ulam system\(^9\), has \(N\) degrees of freedom. The main features of the dynamics in a periodic system are captured by the Poincaré surface of section, an approach invented by Henri Poincaré to study the dynamics of nonlinear systems\(^9\). The resonant dynamics for a 1D periodic system is characterized by special orbits in the Poincaré surface of section: these are fixed points, islands and separatrices, as shown in Fig. 1a top-left. The next level of complexity is a periodic system with 2 degrees of freedom (2D). In this case, the orbits in the Poincaré surface of section expand in four-dimensional phase space, the topology of which may elude our geometric intuition. In the simplest case, the two degrees of freedom (Fig. 1a) are decoupled, and the “mixed” coordinates \((q_1, q_2)\) and \((p_1, p_2)\) exhibit the characteristic rectangular shape\(^10\) as shown in Fig. 1a bottom (for experimental evidence see Ref\(^11\)). Instead, for resonant dynamics created by a nonlinear coupling force, the mixed coordinates may exhibit a specific correlation, for example, as shown in Fig. 1b. This feature can be quite baffling as each \((q, p)\) plane per degree of freedom seems unaffected, leaving the information of the resonant dynamics only in the mixed planes. Note that the four pictures in Fig. 1b show that the collection of all the red points lies on a *four-dimensional closed curve*, which we call “fixed line” since any resonant particle at any passage through the Poincaré surface of section is located somewhere on this curve.

Charged particles in circular accelerators have two degrees of freedom in the transverse plane. Nonlinear forces due to magnet imperfections may drive resonance structures in phase space, and this has always been a subject of practical concern to 1) avoid resonances\(^12–15\), and 2) keep the dynamic aperture (DA)\(^16\) large enough to ensure sufficient beam lifetime\(^17–19\). A prominent example of the impact of nonlinearities in accelerators is the CERN Large Hadron Collider (LHC)\(^20\) commissioned in 2008. The LHC is constructed with superconducting magnets that inherently generate unwanted nonlinear field components. The concerns of a significant reduction of DA and the lack of a clear experimental benchmark of computer modeling, led to a design with increased safety margin through a

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**Fig. 1:** Effect of the coupled resonance for a system of two degrees of freedom. **a** Poincaré surface of section for a one dimensional resonance acting in a 2 degrees of freedom system. The top-left shows the main features of the orbits for a 1D resonance; The top-right picture shows the non-resonant orbits for the second degree of freedom, which are now circles; The pictures at the bottom show the resonant orbits in the “mixed” planes \((q_1, q_2)\) and \((p_1, p_2)\). The characteristic rectangular shape of the areas covered by the orbits is the signature of the decoupling (green markers). **b** Poincaré surface of section in a system with two degrees of freedom subject to a 2D nonlinear resonance. The effect of the resonance is visible only in the mixed coordinate diagrams\(^7\).
comprehensive magnet optimization process. In 2012, a beam experiment demonstrated that the LHC 2D DA agreed within 10% with the predictions of the simulations. This successful benchmarking, followed by further confirmations for the correlation of real beam lifetime to DA, as for example shown in Ref. 22, will allow the use of smaller safety margins leading to cost-savings when designing future accelerators.

For lower energy accelerators, nonlinear resonances are of concern for high-intensity and high brightness beams, as for the SIS100 in the Facility for Antiproton and Ion Research (FAIR)26 project at GSI, and for the operation of the accelerator chain at CERN after the LHC injectors upgrade (LIU). Studies performed over the last 15 years on 1D resonances have shown that space charge induced resonance crossing is the mechanism behind halo formation and associated particle loss for high-intensity bunches. Recent studies have suggested that fixed lines cause asymmetric halo formation. However, the existence of fixed lines has never been proved experimentally, confining any discussion merely to computer simulations or analytical methods. This situation is unsatisfactory, especially when fixed lines are invoked as part of the complex mechanism of the periodic resonance crossing, motivating the experimental verification of their existence.

The dynamical coordinates of a particle in the transverse plane are $x$, $y$ and the conjugate momenta are $p_x$, $p_y$. The oscillation frequencies are expressed in terms of the number of oscillations per turn $Q_x$, $Q_y$. We consider the case of one or more normal sextupole magnets of integrated strength $K_2$ inserted in an otherwise linear accelerator structure. The resonance condition $Q_x + 2Q_y = N$ combines the phase advances of the particle transverse coordinates $(x, y)$ into a new phase advance per turn of the form $\Delta \phi_x + 2\phi_y$, which underlines the possible presence of a nonlinear coupling between the transverse particle coordinates $(x, y)$. When the accelerator tunes are set near this third-order resonance, i.e. when the distance to this resonance $\Delta_r = Q_x + 2Q_y - N \approx 0$, the particle dynamics acquire new features due to the presence of nonlinear fields. Computing the invariants from the particle coordinates, $\epsilon_x$, $\epsilon_y$, as prescribed by Courant and Snyder, we find that they are no longer invariant and the phase advances per turn $\Delta \phi_x$, $\Delta \phi_y$ are no longer constant. We call the values of the Courant-Snyder form resulting from the effect of the resonance $a_x$, $a_y$, to emphasize that these quantities vary turn after turn. A perturbative approach to the dynamics shows that $a_x$, $a_y$ must satisfy the relation $2a_x = a_y + C$, where $C$ is a constant determined by the initial conditions. Using only $a_y$, together with the phase advance $\Omega = \phi_x + 2\phi_y$ is thus sufficient to discuss the properties of the resonance. The key feature of the resonant dynamics for a fixed $C$ is the fact that a pair of values of $a_x$ and $\Omega$ exist such that these two dynamical variables become stationary, and the theory of fixed lines predicts the existence of an infinite set of these pairs. Back in the four-dimensional phase space $(x, p_x, y, p_y)$, this special solution, at a specific accelerator location, i.e. at the Poincaré surface of section, acquires the topology of a one-dimensional closed curve: the fixed line. Expressed in Courant-Snyder coordinates, a third-order fixed line reads as

$$\begin{align*}
\dot{x}(t) &= \sqrt{a_x} \cos(-2t - \alpha + \pi M), \\
\dot{y}(t) &= \sqrt{a_y} \cos(t), \\
\dot{p_x}(t) &= -\sqrt{a_x} \sin(-2t - \alpha + \pi M), \\
\dot{p_y}(t) &= -\sqrt{a_y} \sin(t),
\end{align*}$$

where $t$ is a parameterization variable $0 < t < 2\pi$, $a_x$, $a_y$ are now stationary, $\alpha$ is the resonance driving term angle with respect to the Poincaré surface of section, and the integer $M$ is either 0 or 1 according to the sign of $\Delta_r$ and of $\alpha$. From Eq. 1 we derive the stationary phase advance for a fixed line $\Omega_0 = -\alpha + \pi M$, which characterizes its geometric "orientation" in the phase space.

We report here on the measurement of fixed lines performed at the CERN SPS. With a set of kicker magnets, we induce transverse oscillations of a proton beam and study how they are affected by the third-order resonance excited by a few strongly powered sextupoles. The beam positions are measured at each turn using the available Beam Position Monitors (BPMs). With four consecutive BPMs (two per plane) we can reconstruct the Courant-Snyder coordinates $(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y)$ at one location of the machine. Taking advantage of the action-angle representation $\hat{x} = \sqrt{a_x} \cos(\hat{x}_x)$, $\hat{p}_x = -\sqrt{a_x} \sin(\hat{x}_x)$, $\hat{y} = \sqrt{a_y} \cos(\hat{x}_y)$, $\hat{p}_y = -\sqrt{a_y} \sin(\hat{x}_y)$ we can retrieve $a_x$, $a_y$, $\phi_x$, $\phi_y$ and $\Omega$ (see Methods). This procedure allows visualizing the Poincaré surface of section in Courant-Snyder coordinates and inspecting the resonant dynamics in the $(\Omega, a_y)$ space for assessing whether the beam is locked to a stable fixed line or not as we expect the measured $\Omega$ and $a_y$ to be constant when the beam is on a fixed line.

This experiment faces three major difficulties: a) the inherent fragility of the effect being searched for; Tune modulation due to power converter ripple and therefore fluctuations of magnetic fields perturb the experimental conditions for detecting fixed lines. To mitigate these effects, we accelerated the beam to 100 GeV/c before exciting transverse beam oscillations, and machine settings were adjusted carefully (see Methods); b) intrinsic manufacturing tolerances of accelerator quadrupoles create a well-known effect called \textit{beta-beating} that may easily reach a level of the order of $\sim 5\%$. This unwanted optics perturbation must be considered in the analysis of the measurement results; c) we have to be able to kick the beam onto a fixed line: The SPS has only one vertical and one horizontal
Fig. 2: Measured Poincaré surface of section. These pictures show four projections of a Poincaré surface of section. The blue markers are the normalized beam coordinates obtained experimentally from 3000 passages through the selected longitudinal observation point. The red line is the prediction from Eq. 1. a) Projection of horizontal position and momentum expressed in scaled Courant-Snyder coordinates. The circular shape shows that the amplitude $p_x$ is constant. b) Projection in the $y, p_y$ plane. The orbit is circular showing that $\bar{a}_y$ is constant. c) In the $x, y$ coordinates the motion of the beam follows a Lissajous curve. d) Projection in $x, p_y$, the C shape is due to the nonlinear coupling created by the resonance.

kicker suitable for this experiment. This setup restricts the fixed line orientations we can explore to the unique value of $\Omega = \Omega_u$ (see Methods). In addition, the synchronization of the kickers needs to be taken into account (see Methods). The third-order resonance was therefore excited using two sextupoles located at locations enabling us to vary $\alpha$ and thus the orientation of the fixed line. To find the proper sextupole strengths a sequence of measurements was performed systematically varying the strength of the two sextupoles $K_{2,1}, K_{2,2}$, the distance from the resonance $\Delta_r$, and the strength of the horizontal and vertical kicks $\theta_x$ and $\theta_y$, respectively (see Methods).

For analyzing the experimental results we scale the Courant-Snyder coordinates and invariants to $\bar{x}, \bar{p}_x, \bar{y}, \bar{p}_y$, and $\bar{a}_x, \bar{a}_y$ (see Methods). In Fig. 2 we plot the projections of the measured Poincaré surface of section for 3000 turns of one selected data set (all six projections are shown in Extended Data Fig. 1). The circular orbits in the horizontal and vertical phase space projections (cf. Fig. 2a,b) show the usual Courant-Snyder invariance, from which we obtain the values of $\bar{a}_x, \bar{a}_y$. The other projections of Fig. 2 exhibit Lissajous patterns. Using these values of $\bar{a}_x, \bar{a}_y$ and applying a least square minimization, we find the best fit of Eq. (1) to the experimental data. This curve is shown by the red line in all the projections of Fig. 2 indicating a dynamics consistent with the topology of a fixed line.

From the same data set, in Fig. 3 (top) we show that the distance from the resonance as obtained from the beam tunes remains small throughout the storage time (cf. Fig. 3a), that $\bar{a}_x, \bar{a}_y$ exhibit only small correlated oscillations around their corresponding average values (cf. Fig. 3b), and that the beam is revolving around a fixed point in the $(\Omega, \bar{a}_y)$ diagram (cf. Fig. 3c) whose orientation we determine as $\Omega^{\text{exp}} = 0.275$ (in units of $2\pi$). The line $\Omega^{\text{fit}} = 0.30$ shows the expected orientation of the fixed line from the sextupole settings. We attribute the difference between $\Omega^{\text{exp}}$ and $\Omega^{\text{fit}}$ to the unavoidable presence of beta-beating. In fact, a 5% beta-beating, a value that is pretty normal in hadron accelerators, is sufficient to create an rms spread in $\Omega^{\text{fit}}$ of 0.016 (see Methods), consistent with the experimental findings. The presence of the residual beam oscillations around the fixed line in the $(\Omega, \bar{a}_y)$ diagram stems from the experimental inability of moving the beam exactly onto the fixed line. For a given accelerator optics, there is a unique fixed line orientation allowed by the location of the kicker magnets (see Methods), which for the ideal SPS optics is $\Omega_u = 0.34$. We interpret the difference between the orientation from the ideal lattice $\Omega_u$, the orientation expected from the sextupole driving term $\Omega^{\text{fit}}$ and the orientation determined experimentally $\Omega^{\text{exp}}$ as a result of the combined effect of beta-beating and the granularity of the sextupole scan.

In Fig. 3 (bottom), we show a data set affected by an uncontrolled drift of the SPS machine parameters. In this
Fig. 3: Finding a fixed line. Two distinct measurement results: on the top a case with the stable features of Fig. 2, and on the bottom a case where the machine conditions (in particular the machine tunes) are subject to unwanted drift. a shows the evolution of the distance from the resonance calculated using the measured beam tunes, $q_x$ and $q_y$. b in dark blue we show the evolution of $\bar{a}_x$, $\bar{a}_y$ and in light blue their corresponding moving average. c this picture demonstrates that this beam revolves over 3000 turns around a “fixed point” in the $\bar{a}_y$ versus $\Omega$ diagram. d here we find that the distance from the resonance calculated using the measured beam tunes exhibits a larger oscillation over time. e the variation of $\bar{a}_x$, $\bar{a}_y$ is larger, but $\bar{a}_x$ and $\bar{a}_y$ are still correlated. f in the $\bar{a}_y$ versus $\Omega$ diagram, we see that the quantity $\bar{a}_y$ spans over a large range, while $\Omega$ keeps oscillating around one specific $\Omega_{\text{exp}}$ (the location of the fixed line). This finding suggests that the beam is trapped on the resonance.

The previous analysis shows that the behavior of $\Omega$ reveals the properties of the resonant dynamics. In particular, the average $\langle \Omega \rangle$ and the standard deviation $\sigma_{\Omega}$ over the observation period are key quantities for characterizing each beam. Figure 4a shows the measured beam tunes for the complete set of around 400 different shots, of which around 150 were on the resonance. Figure 4b shows the behavior of $\langle \Omega \rangle$ for each beam in the ($\langle \Omega \rangle$, $\sigma_{\Omega}$) diagram. The shots cluster into two distinct groups: a) a cluster of off-resonance shots, for which $\Omega/(2\pi)$ spans all possible angles averaging to $\langle \Omega \rangle/(2\pi) = 1/2$ with a standard deviation $\sigma_{\Omega}/(2\pi) = 1/\sqrt{12}$; b) a second cluster close to $\langle \Omega \rangle = \Omega_{\text{ref}}$ with much lower $\sigma_{\Omega}$, i.e. shots where the beam is trapped onto the resonance structure. Note that the orientation of all the experimentally found fixed lines is slightly offset compared to the unique orientation, as for the fixed line discussed in the example of Fig. 3. In order to show more general properties of the fixed lines, we select the data from Fig. 4b that are closer to $\Omega_{\text{exp}}^{\text{ref}}$ by requiring $\sigma_{\Omega} < 0.1$ and plot them in an ($\bar{a}_x$, $\bar{a}_y$) chart in Fig. 4c. The colors of the markers show $D$, the normalized rms distance to the fixed line (see Methods), that $\bar{a}_x$, $\bar{a}_y$ have due to drifts of the SPS parameters or due to an oscillation around the fixed line. The distribution of the fixed lines in this chart is very similar to what has been found in Ref.7.

We report the first experimental observation of fixed lines created by a controlled, third-order, normal, 2D resonance in a particle accelerator. In the range of experimentally tested kick amplitudes, we observe the existence of many fixed lines, all having the same phase space orientation for the same accelerator settings. This confirms the theoretical prediction of the dynamics for single 2D coupled resonances. It has been demonstrated that we can predict the topology and orientation of the fixed lines, which will allow the development of mitigation strategies to combat beam degradation mechanisms such as the periodic resonance crossing induced by any modulation of the particle tune or by amplitude-dependent detuning. These findings are relevant for achieving high intensity and high brightness.
Fig. 4: All shots measured. a Tune diagram showing the measured average beam tunes along the interval of 3000 turns. From the total of about 400 shots measured, around 150 shots are on the resonance. b For each “shot” we compute the turn-by-turn evolution of $\Omega$ along the storage period, and the corresponding mean value $\langle \Omega \rangle$ and standard deviation $\sigma_\Omega$. The black solid line represents $\Omega_{3\text{rd}}^{\text{4}}$. The blue area shows the selected shots close to the excited fixed line. c The close-to-the-resonance shots are shown here in a $(\bar{a}_x, \bar{a}_y)$ chart indicating the mean values $(\bar{a}_x)$ and $(\bar{a}_y)$ of the measured shots. The color indicates the degree of shifts in the $\bar{a}_x, \bar{a}_y$ values caused by drifts of SPS machine parameters (see Methods, Eq. (9)).

beams as required for both current and future accelerator projects.


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**METHODS**

**Scaled Courant-Snyder coordinates.** For an on-momentum particle at a fixed energy, the theory of Courant-Snyder defines the normalized dynamical variables as

\[
\dot{x} = \frac{1}{\sqrt{\beta_x}}x, \\
\dot{p}_x = \frac{\alpha_x}{\sqrt{\beta_x}}x + \sqrt{\beta_x}p_x, \\
\dot{y} = \frac{1}{\sqrt{\beta_y}}y, \\
\dot{p}_y = \frac{\alpha_y}{\sqrt{\beta_y}}y + \sqrt{\beta_y}p_y,
\]

(2)

where \(x, p_x, y, p_y\) are the particle coordinates and \(\beta_x, \alpha_x, \beta_y, \alpha_y\) the Twiss parameters at the location of the particle\(^{31}\). The units of the normalized Courant-Snyder coordinates are \(\sqrt{\text{m} \cdot \text{rad}}\), hence the Courant-Snyder invariants \(\alpha_x, \alpha_y\) have units \(\text{m} \cdot \text{rad}\). Using these new coordinates allows for a significant simplification of the topology of the linear dynamics by making the uncoupled planes of the phase space highly symmetric (linear normal form). For our data analysis we make use of the scaled Courant-Snyder coordinates defined as \(x = \sqrt{\beta_x}x = x, \; \bar{p}_x = \sqrt{\beta_x}p_x, \; y = \sqrt{\beta_y}y = y, \; \bar{p}_y = \sqrt{\beta_y}p_y\), which have units m, and automatically imply \(\alpha_x, \alpha_y\) becoming \(\alpha_x = \beta_x \alpha_x, \; \alpha_y = \beta_y \alpha_y\) with units \(\text{m}^2\). These coordinates are especially convenient as they are directly retrieved from the measurement data from two consecutive BPMs if the phase advance between them is \(\pi/2\), which holds in the SPS within 2%. In this case

\[
\bar{x}_m = x_m, \\
\bar{p}_{x,m} = \frac{\beta_x m}{\beta_{x,m+1}} x_{m+1}, \\
\bar{y}_m = y_m, \\
\bar{p}_{y,m} = \frac{\beta_y m}{\beta_{y,m+1}} y_{m+1},
\]

(3)

where \(x_m, y_m, x_{m+1}, y_{m+1}\) are the physical beam positions measured by the \(m\)th BPM and in the direction of the beam by the \((m+1)\)th BPM. In the SPS, the situation is even further simplified as the Twiss parameters are almost equal at the BPM locations and thus

\[
\bar{x}_m = x_m, \\
\bar{p}_{x,m} = x_{m+1}, \\
\bar{y}_m = y_m, \\
\bar{p}_{y,m} = y_{m+1}.
\]

(4)

The sequence of BPMs in the SPS alternates vertical with horizontal position measurements, therefore Eq. 4 needs to be applied using the data of a group of four consecutive BPM which we call \(V_m, H_m, V_{m+1}, H_{m+1}\). For computing the Poincaré surface of section we take the location of \(H_m\) as our reference position. While the horizontal phase space is automatically retrieved at the location of \(H_m\), a further rotation of the coordinates \(\bar{g}_m, \bar{p}_{y,m}\) by \(\pi/4\) is necessary to transport the vertical phase space from \(V_m\) to the location of \(H_m\). This is the only treatment of the BPM data necessary to visualize the Poincaré surface of section.

**Constraint on the fixed line orientation.** The possible kick sequence of the two kicker magnets used to deflect the beam from the central orbit has an inherent limitation. In fact, while these two accelerator elements provide two degrees of freedom in displacing the beam in phase space (the kick angles \(\theta_x\) and \(\theta_y\)), they are not sufficient to deflect the beam to any point in phase space. This is because with two degrees of freedom we can access only a two-dimensional surface in the four-dimensional phase space. The accessible phase space points in the Poincaré surface of section where the measurement is done, are determined by the optics at the location of the kickers and the phase advance in the accelerator.

We consider first the case in which the sequence consists of the vertical kick followed by the horizontal kick. Using Eq. 1 and solving for the \(t, \theta_x, \theta_y\) which bring the beam onto the fixed line, we find that the only allowed kicks are

\[
\theta_x = (-1)^{1+N_k} \sqrt{\frac{\alpha_x}{\beta_{h,x}}},
\]

(5)

\[
\theta_y = \pm \sqrt{\frac{\alpha_y}{\beta_{v,y}}},
\]

(6)
where $\beta_{h,x}$ is the horizontal beta function at the location of the horizontal kicker, and $\beta_{v,y}$ is the vertical beta function at the location of the vertical kicker, with the integer $N_k$ given by the sign of $\theta_x$. The consequent “unique orientation” of the fixed line is

$$\Omega_u = -\alpha + \pi M = 2\Delta \phi_{h,x} - \frac{\pi}{2} + \pi N_k - \Delta \Omega,$$

(7)

where $\Delta \phi_{h,y}$ is the difference of vertical phase advance between the two kickers.

The quantity $\Delta \Omega$ is $\Delta \Omega = \Omega_h - \Omega_p$, with $\Omega_h$ the resonance phase at the location of the horizontal kicker, and $\Omega_p$ the resonance phase at the location of the Poincaré surface of section. $\Delta \Omega$ is computed counting the phases from the Poincaré surface of section to the kickers. Since the horizontal kicker of the SPS can only generate positive deflections, the unique orientation of the fixed line which is consistent with the SPS kicker system is $\Omega_u = 2\Delta \phi_{h,y} + \pi/2 - \Delta \Omega$ (mod. $2\pi$). Only if the resonance is excited with phase $\Omega_p$ and the correct deflecting angles $\theta_h, \theta_p$ are used can the SPS kicker system shift the beam exactly onto the excited fixed line.

For completeness, we consider also the case in which the kicker sequence is inverted, i.e. horizontal kick followed by vertical kick. Using the same approach as above we find the unique fixed line orientation given by

$$\Omega_u = -\alpha + \pi M = 2\Delta \phi_{h,x} - \frac{\pi}{2} + \pi N_k - \Delta \Omega,$$

(8)

where we have used the corresponding notation and meaning of quantities as above.

**Mitigation of power converter ripple.** Increasing the beam energy helps to mitigate the impact of power converter ripple, since the relative amplitude of ripple decreases with the higher current required for higher beam energy.

**Kicker synchronization.** As the resonance phase at the location of the kickers and the phase advance between the kickers determines the orientation of the fixed line, the synchronization of the kickers is critical. During the experimental campaign, the horizontal kicker was fired one turn after the vertical kicker, and therefore the sequence of beam deflections is vertical followed by horizontal (see paragraph “Constraint on the fixed line orientation”).

**Driving term created by two sextupoles.** By knowing the accelerator optics at the location of the BPMs, we compute the driving term, which has a strength $\Lambda$ and the angle $\alpha$. By acting on two independent sextupole magnets with strengths $K_{2,1}$ and $K_{2,2}$, respectively, we can easily reach any value of $\alpha$ as long as $\alpha_{K_{2,2}} - \alpha_{K_{2,1}} \simeq \pi/2$, where $\alpha_{K_{2,1}}$ and $\alpha_{K_{2,1}}$ denote the angle of the driving term generated by the two sextupoles. During the experimental scan of the sextupole settings, the strength of the driving term was kept constant.

**Programming the SPS.** The kickers will shift the beam onto a fixed line only if the angle $\alpha$ of the driving term is consistent with the combined effect of both kickers and the accelerator lattice between them. As the SPS has no single sextupole at the proper distance to fulfill this condition, we have searched for the fixed lines by scanning the angle $\alpha$ looking for suitable experimental conditions. Once we had an indication of a fixed line, we scanned the distance from the resonance $\Delta \epsilon_k$ and the strength of the resonance, i.e. the strength of the two sextupoles and the kicker strengths $\theta_x, \theta_y$. For each measurement the beam position was stored turn by turn from all available BPMs in the SPS.

**Beam and machine parameters.** Optimal experimental conditions were found by setting the SPS to the following parameters: betatron tunes $Q_x = 26.104, Q_y = 26.448$, compensating the natural chromaticity of the SPS with dedicated sextupole magnets without exciting or significantly influencing the third-order resonance (i.e. chromaticities $Q'_x \approx 0$ and $Q'_y \approx 0$). The optimal values of sextupole strengths for exciting the third-order resonance (normalized to the beam rigidity) and kicking the beam onto the fixed line were found to be $K_{2,1} = -0.12$ m$^{-3}$ and $K_{2,2} = -0.21$ m$^{-3}$. In addition, a family of weak octupoles was powered to create some small amplitude-dependent detuning required to stabilize the beam.

The measurements were performed at a plateau of constant beam energy corresponding to a momentum of 100 GeV/c, such that electro-magnetic interactions between the particles (collective effects) are negligible compared to the external magnetic guiding fields of the machine. In this case, the beam behaves almost like a single charged particle (“pencil beam”), which makes it an ideal probing tool to investigate the nonlinear dynamics of the third-order resonance. A beam with the following characteristics was used: single bunch of $4 \times 10^{10}$ protons with rms normalized transverse emittances of $\varepsilon_x \approx 0.5$ $\mu$m and $\varepsilon_y \approx 0.5$ $\mu$m and a bunch length of about 2.5 ns ($4 \sigma$).

**Removing the BPMs noise from $\bar{a}_x, \bar{a}_y$.** As the turn-by-turn data suffer from instrumental noise, a 100 turn moving average filter was applied to yield the light blue traces in Fig. 3.

**Computing $a_x, a_y$ and error bars.** The values of $a_x, a_y$ are averages over 3000 turns after the beam is kicked. The associated error bar is found from the unbiased standard deviation of these data.

**Selection of the experimental data and drift parameter.** To select the data sets to be analysed, we adopted the same procedure used to verify the data in Fig. 4, namely investigating the oscillatory properties in a ($\Omega, \bar{a}_y$) diagram. The location
of the kickers constrains the fixed line to a unique orientation $\Omega_u$, however, the available SPS sextupoles do not allow the same driving term orientation to be created. We therefore adopt the standard deviation $\sigma_\Omega$ over the storage time, i.e. on the oscillations amplitude of $\Omega$, as a measure of closeness to the resonance. We consider a beam to be locked onto a resonance if $\sigma_\Omega \leq 10\%$. In order to identify a fixed line, in addition to the locking property, the quantities $\bar{a}_x, \bar{a}_y$ should not suffer from large variations. We measure this effect by defining a parameter $D$ as

$$D = \sqrt{\frac{\sigma^2_{\bar{a}_x}}{\langle \bar{a}_x \rangle^2} + \frac{\sigma^2_{\bar{a}_y}}{\langle \bar{a}_y \rangle^2}},$$

(9)

where $\sigma^2_{\bar{a}_x}$ and $\sigma^2_{\bar{a}_y}$ are the variances of $\bar{a}_x$ and $\bar{a}_y$. Therefore, $D$ corresponds to the normalized rms distance to the fixed line. Small values of $D$ correspond to cases where the beam is very close to a fixed line with stationary machine parameters (e.g. $D = 0.11$ for Fig. 3c). On the other hand, a large $D$ means that either the beam has a large distance to the fixed line, or that there has been a drift of the machine parameters (e.g. $D = 0.36$ for Fig. 3f).

**Effect of beta-beating.** The effect of beta-beating is estimated by taking the ideal accelerator structure and imparting a tiny random error in the strength of each quadrupole. This perturbed structure is used to compute the driving term angle, hence the associated fixed line orientation $\Omega_{fl}^{\text{drt}}$, as well as the perturbed beta functions $\beta_x, \beta_y$. A repetition of this procedure allows for statistical analysis. Extensive simulations have confirmed that the method of averaging the instantaneous $\Omega$ allows $\Omega_{fl}^{\text{drt}}$ to be retrieved, and that $\sigma_{\Omega_{fl}^{\text{drt}}}/[\sigma_{\beta_y}/\langle \beta_y \rangle] = 3.23 \times 10^{-3}$, where $\sigma_{\Omega_{fl}^{\text{drt}}}$ is the standard deviation of the set of perturbed $\Omega_{fl}^{\text{drt}}$, and $\sigma_{\beta_y}/\langle \beta_y \rangle$ is the beta-beating at the location of the sextupoles expressed in percent. Therefore, for a beta-beating of $\sim 5\%$, we find $\sigma_{\Omega_{fl}^{\text{drt}}} \sim 0.0161$.

**Simulation with MAD-X.** The experimental conditions are very well defined and we have a sophisticated tracking model of the ideal SPS lattice in a MAD-X format. The results of the simulation are shown in Fig. 1(g-l). The simulation shows very similar orientations in the six different projections of the Poincaré surface of section as found in the experimental data (Fig. 1(a-f)). The fixed line orientation for the ideal SPS lattice is $\Omega_{fl}^{\text{sim}} = 0.290$. 
Extended Data Fig. 1: Poincaré surface of section as measured in the experiment (a-f) compared to the results obtained in simulation (g-l). In both cases, all six different projections of the Poincaré surface of section are shown.