Spin-1/2 one- and two- particle systems in physical space without eigen-algebra or tensor product

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Spin-1/2 one- and two- particle systems in physical space without eigen-algebra or tensor product.

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Abstract. A novel representation of spin 1/2 combines in a single geometric object the roles of the standard Pauli spin vector operator and spin state. Under the spin-position decoupling approximation it consists of three orthonormal vectors comprising a gauge phase. In the one-particle case the representation: (1) is Hermitian; (2) shows handedness; (3) reproduces all standard expectation values, including the total one-particle spin modulus $S_{\text{tot}} = \sqrt{3/2}$; (4) constrains basis opposite spins to have same handedness; (5) allows to formalize irreversibility in spin measurement. In the two-particle case: (1) entangled pairs have precisely related gauge phases; (2) the dimensionality of the spin space doubles due to variation of handedness; (3) the four maximally entangled states are naturally defined by the four improper rotations in 3D: reflections onto the three orthogonal frame planes (triplets) and inversion (singlet). Cross-product terms in the expression for the squared total spin of two particles relate to experiment and they yield all standard expectation values after measurement. Here spin is directly defined and transformed in 3D orientation space, without use of eigen algebra and tensor product as done in the standard formulation. The formalism allows working with whole geometric objects instead of only components, thereby helping keep a clear geometric picture of ‘on paper’ (controlled gauge phase) and ‘on lab’ (uncontrolled gauge phase) spin transformations.

1. Introduction

Key developments in Quantum Mechanics (QM), such as the first phenomenological description of spin 1/2 by Pauli [1] and the first quantum relativistic description of the electron by Dirac [2], ‘re-invented’ Clifford algebras (in the matrix representation), seemingly unaware of Grassmann’s [3] and Clifford’s [4] works more than half a century earlier. The promotion of vector-based Clifford algebras in Physics and in particular the development of the spacetime algebra (STA) was undertaken by Hestenes [5,6], with more scientists joining in during the last 2 – 3 decades [7-10]. Spin formalism is arguably the main application of STA in QM [7]. Here we address spin in the non-relativistic regime under the spin–position decoupling approximation [6, 7, 11]. The 3D physical space in this case reduces to orientation space at a point (the origin) and the relevant symmetry operations of interest are proper rotations and reflections. The three frame vectors $\sigma_j$ and the
scalar 1 generate a real 8D vector space matching the equivalent real dimensions of the 2 x 2 complex Pauli matrices. The space is endowed with the even sub-algebra of STA [5,7], which is isomorphic to the (Clifford) algebra of Pauli matrices [9], therefore the notation. A basis of the real vector space consists of one unit scalar, three orthonormal vectors, three bivectors and one trivector, or pseudoscalar (below \( \delta_{jk}, \epsilon_{jkl} \) are the Kronecker and the totally antisymmetric Levi-Civita symbols):

\[
\{ 1, \sigma_j (3), \sigma_j \sigma_k \equiv \sigma_{jk} (= \delta_{jk} + \epsilon_{jkl} l \sigma_l; j, k, l = 1,2,3) \} (3), \sigma_{123} \equiv l (1); l \sigma_j = \sigma_j l
\] (1)

Notice that \( \sigma_j \) are vectors, not matrices. Geometrically, the bivectors \( l \sigma_l \), e.g. \( l \sigma_2 = \sigma_{31} \), represent oriented surface elements, while the pseudoscalar \( l = \sigma_{123} \) is an oriented volume element. The geometric or Clifford product combines Hamilton’s scalar (symmetric) and Grassmann’s wedge (antisymmetric) products; if not zero it is invertible and as will become clear in definition (3) it is proportional to a rotor:

\[
\begin{align*}
uv = u \cdot v + u \wedge v &= v \cdot u - v \wedge u; \\
(uv)^{-1} &= vu/u^2v^2 .
\end{align*}
\] (2)

The geometric product generalizes to any dimension, whereas the cross product, valid only in 3D is related to the wedge product by: \( l(u \times v) = u \wedge v \). In the STA literature [6,7] it has become customary to represent spin either by a vector or by a bivector normal to it. Notably, such representations of spin \( s = 1/2 \) do not reproduce the standard squared total spin modulus \( S^2 \), which according to the rules of QM on angular momentum should equal \( S^2/\hbar^2 = s(s + 1) = 3/4 \). In the following we present a model in the STA formalism that reproduces all standard expectation values for spin 1/2 one and two particle systems.

The rest of the report comprises three parts. The new definition of spin opens Section 2. Transformation properties for the one and two particle cases will appear in this section in the vector form, i.e. as two-sided rotor (unitary) transformations. In Section 3 the transformation properties for the same two cases will be demonstrated in the spinor form, i.e. as one-sided rotor transformations. Orthogonality relations, e.g. between spin up and spin down hold in the spinor representation. The connection between the vector and the spinor forms will be also discussed in Section 3. Conclusions are presented in Section 4.
2. Definition and vector (two-sided rotor) transformations of spin 1/2 for one and two particle systems.

2.1. One-particle systems. Spin is represented here by the sum of three orthonormal vectors together with a (non-observable) ‘gauge’ rotation. Specifically, spin up $\uparrow$ and spin down $\downarrow$ along the $\sigma_3$ axis is the sum of the three orthonormal vectors $\sigma_3, \sigma_1, \sigma_2$ with a two-sided rotor $R_\varphi$ in the plane $\sigma_{12} = I \sigma_3$ (see figure 1):

$$S_{1\sigma} = \hbar R_\varphi (\sigma_3 + \sigma_1 + \sigma_2) R_\varphi^\dagger = \frac{\hbar}{2} (\sigma_3 + R_\varphi (\sigma_1 + \sigma_2) R_\varphi^\dagger) \equiv \frac{\hbar}{2} (\sigma_3 + (\sigma_1 + \sigma_2)\varphi) = \frac{\hbar}{2} (\sigma_3 + (\sigma_1 + \sigma_2)\varphi)$$

$$(u_\perp + u_2)_{\varphi-\varphi_0}; \ R_\varphi = e^{-i\sigma_3 \varphi \frac{\hbar}{2}} = \cos \varphi \frac{\hbar}{2} - i \sin \varphi \frac{\hbar}{2}; \ R_\varphi^\dagger = e^{-i\sigma_3 \varphi \frac{\hbar}{2}} = R_\varphi^{-1}$$

$$S_{1\sigma} = u_2 S_{1\sigma} u_2 = \frac{\hbar}{2} (-\sigma_3 + (-u_\perp + u_2)_{\varphi-\varphi}) = \frac{\hbar}{2} (-\sigma_3 + (-\sigma_1 + \sigma_2)_{\varphi}) = S_{1\sigma}$$

$$\frac{3\hbar^2}{4}; \ S_0 = S_{\sigma}; S_{-\sigma} = -S_\sigma; \langle S_{\pm\sigma} \rangle \equiv \frac{\hbar}{2} (\pm \sigma_3 + (R_{\pm\varphi}^\dagger)(\pm \sigma_1 + \sigma_2)) = \pm \frac{\hbar}{2} \sigma_3 \quad \text{(spin vector)} \quad (3)$$

![Figure 1: 3D and 2D rendering of (a) the coordinate and (b) spinor transformation.](image)

$u_\perp, u_2$ are any two orthonormal vectors in the $\sigma_{12}$ plane, so that $u_\perp + u_2 = R_{\varphi_u}(\sigma_1 + \sigma_2) R_{\varphi_u}^\dagger$ and $u_\perp u_2 = \sigma_1 \sigma_2$ (see fig. 1a). Handedness in (3) is defined by the duplet $(\sigma_3, \varphi)$, where $\varphi, -\varphi$ are anticlockwise and clockwise rotations, respectively. By this definition $(\sigma_3, \varphi), (-\sigma_3, -\varphi)$ are right-handed ($R$), while $(-\sigma_3, -\varphi), (-\sigma_3, \varphi)$ are left-handed ($L$).

For ‘book keeping’ purposes we will also characterize the spin in (3) by the order of vectors with respective signs as they appear in the expression of spin, so that independent of $\varphi$, $(\sigma_3, \sigma_1, \sigma_2)$ or $(-\sigma_3, -u_\perp, u_2)$ form right-hand triplets ($\uparrow$), while $(-\sigma_3, u_\perp, u_2)$, or $(-\sigma_3, -\sigma_1, -\sigma_2)$ form left-hand triplets ($\downarrow$). $S_{1\sigma}, S_{1\sigma}$ in (3) are both ($R$, $r$). For one-particle systems one can use exchangeably handedness or order: $R$ or $r$ (and $L$ or $\ell$).

A proper rotation of a vector is a two-sided operator, as in (3), which is a unitary transformation preserving both handedness and order. The axial reflection $u_2 S_\sigma u_2$ is equivalent to a $\pi$-rotation of $S_\sigma$ around $u_2$. The phase factor $\varphi$ in (3) being non-observable is expressed by taking the expected values $\langle \sin \varphi \rangle = \langle \cos \varphi \rangle = 0$. Angled brackets denote either the expected value (a scalar) or the average (a scalar, vector, etc.). I will also use the notation $\langle \rangle_0$ for the expected value, if needed for clarity. We model measurement of spin by extracting the phase-sensitive part of a spin or spin combination (see (6-7), (12-13)). The full spin modulus $\sqrt{S_\sigma^2}$, alike the overall phase, is not measurable; we call it an ‘on-paper’ quantity relating to QM angular momentum selection rules. It includes equal contributions from all the three components and exceeds by a factor of $\sqrt{3}$ the measured (‘on lab’) spin vector modulus $\sqrt{\langle S_\sigma^2 \rangle} = \hbar/2$. 

3
By convention $S_\sigma$ depicts either $S_{1\sigma}$ or $S_{2\sigma}$, while $S_{\sigma}$ and $S_{-\sigma}$ stand for opposite spins of same handedness, as in (3). Any left-handed ($L, \ell$) spin-up (down) is the inverse of a right-handed ($R, r$) spin-down (up):

$$S_{\sigma L} = IS_{-\sigma R} = -S_{-\sigma R}; \quad L = L, \ell; \quad R = R, r$$

The two spaces $L = (L \text{ or } \ell)$ and $R = (R \text{ or } r)$ are disjoint under proper rotations, while plane reflections and inversion (improper rotations), convert a spin from one space to the other. Focusing on $R$, the combined rotation $R_{u} \equiv R_{\theta_u} R_{\phi_u}$ transforms ‘on paper’ (i.e. with conserved gauge phase $\varphi$) $S_{\sigma}$ to $S_{u}$ as follows:

$$\langle S_{\sigma} \rangle \rightarrow R_{u} S_{\sigma} R_{u}^\dagger = \frac{h}{2} R_{\theta_u} \left( \sigma_3 + (u_1 + u_2) \right) R_{\theta_u}^\dagger = \frac{h}{2} \left( u^+ \cos \theta_u + u^- \sin \theta_u + R_{\theta_u}(u_1 + u_2) \phi R_{\theta_u}^\dagger \right) = \frac{h}{2} \left[ u + (u_1 + u_2) \phi \right] = \langle S_{u} \rangle \phi'; \quad \phi'(u_{12 \text{plane}}) = \varphi(\sigma_{12 \text{plane}}); \quad R_{\theta_u} = e^{B \theta_u/2}; \quad B = u_1 \sigma_3 = u_{13} = u^- u^+ = -j u_2$$

(5)

A spin measurement of $S_{\sigma}$ along $u$ by a Stern-Gerlach (SG) magnet, i.e. an ‘on lab’ transformation, differs from (5) by the two gauge angles $\varphi'$ and $\varphi$ becoming uncorrelated:

$$\langle S_{\sigma} \rangle \rightarrow \langle S_{u} \rangle \phi' = \frac{h}{2} \left[ u + (u_1 + u_2) \phi \right]; \quad \langle \sin \varphi' \sin \varphi \rangle_{S-G} = \langle \sin \varphi' \rangle \langle \sin \varphi \rangle = 0$$

(6)

In many instances, it is sufficient to apply the transformation to only the phase-insensitive spin vectors:

$$\frac{h}{2} \sigma_3 \rightarrow \frac{h}{2} R_{\theta_u} \sigma_3 R_{\theta_u}^\dagger = \frac{h}{2} \left( u^+ \cos \theta_u + u^- \sin \theta_u \right) = \frac{h}{2} u, \quad \text{where: } R_{\theta_u} \sigma_3 = u^+; \quad R_{\pi-\theta_u}(-\sigma_3) = u^-$$

(6')

The halfway vectors $u^+, u^-$ (see figure 1b) depict one form of reduced spinor representation and as shown by the last two equalities in (6') they arise by the action of one-sided rotor (spinor) onto the spin vectors up and down, respectively. Of course, one can obtain both $u^+$ and $u^-$ by action of one-sided rotors on spin-up alone; even then, two terms are needed with distinct probability amplitudes for coincidence and anti-coincidence. In order to render the discussion systematic, we insist, as in the standard formalism that the spin basis consist of two opposite spins; then they must have same handedness in order for the spinor representation illustrated in fig. 1b to close into a proper rotation, as recounted in Section 3, eq. (16).

The expectation value for the ‘on lab’ transformation (6) is:

$$\langle S_{u} \rangle_{0(SG)} = \langle \sigma_3 u \rangle_0 = \sigma_3 \cdot u = (-\sigma_3) \cdot (-u) = \cos \theta_u = \cos^2 \frac{\theta_u}{2} - \sin^2 \frac{\theta_u}{2}$$

(7)

From (7) the probability for coincidence and anticoincidence outcomes is $\cos^2 \theta_u/2$ and $\sin^2 \theta_u/2$. If the
relative probabilities and probability amplitudes show up in relations involving only spin vectors, like in (6'), (7) and (19), why do we then need the full spin (3) and its transformations (5) or (16)? Well, the full spin allows to account for the total modulus, compatible with quantum selection rules for angular momentum of spin 1/2. In this sense, it is analogue to the standard Pauli spin vector. The full spin also constrains the form and handedness of the basis spins to comply with the ‘on paper’ transformations (5), (16). In addition, the explicit gauge phase in (3) formalizes the entrance of irreversibility in measurement as due to the loss of phase correlation, thus resulting into an expression equivalent to the restricted transformations (6'), (19').

The full spin expression (3) becomes more relevant in the two-spin case discussed in the next section; gauge phase and handedness – order are key concepts for understanding entanglements in the present framing.

For reference in the following discussion of the two-particle case, let spend a few lines to formalize improper rotations (reflections and inversion). As already noted, an axial reflection is equivalent to a proper rotation: 

\[ R_{\pi,s \bar{\pi}} S_{\bar{\pi},s \pi} R_{\pi,s \bar{\pi}} = e^{-\frac{i \pi}{2} S_u e^{-\frac{i \pi}{2}} = -i \sigma_k S_u I \sigma_k = \sigma_k S_u \sigma_k } \]  

(8)

In comparison, a plane reflection like \( I \sigma_k S_u I \sigma_k = -\sigma_k S_u \sigma_k \) reverses the sign. In 3D plane reflections and inversion (‘reflection’ onto the directed volume \( I \)), can be shown by the unified expression (taking \( \sigma_0 = 1 \)):

\[ S_u \rightarrow I \sigma_\mu S_u I \sigma_\mu = -\sigma_\mu S_u \sigma_\mu = \begin{cases} \text{Inversion} & \text{for } \mu = 0 \\ \text{Three base plane reflections} & \text{for } \mu = j = 1, 2, 3 \end{cases} \]  

(9)

In the following paragraph we will obtain a unified expression for maximally entangled two particle states by applying improper rotations only to the vector part of spin, leaving the gauge phase unaffected. Therefore, if a certain reflection reverses the spin vector then that transformation will reverse handedness; if it leaves the spin vector unchanged then the handedness will not change either. Improper rotations always reverse order.

2.2. Two-particle systems. The above discussion for the one-particle case generalizes in the following way to two-particle systems, again preserving a clear geometric picture of spin. First, the total spin and its square are now (see (2) for the definition of scalar product in STA):

\[ S_{\text{tot}} = S_{(1)} + S_{(2)}; S_{\text{tot}}^2 = S_{(1)}^2 + S_{(2)}^2 + S_{(1)} S_{(2)} + S_{(2)} S_{(1)} = \frac{3 \hbar^2}{2} + 2 S_{(1)} \cdot S_{(2)} \]  

(10)

\( S_{\text{tot}} \) is Hermitian and symmetric with respect to the spin swap \( S_{(1)} \leftrightarrow S_{(2)} \). Therefore, in the following discussion, which will revolve around equation (10), the shorthand \( S_{(1)} \) and \( S_{(2)} \) will stand for \( S_{(1 \text{or} 2)} \) and...
\[ S_{\text{tot}}(\mu) = S_\sigma + l_\sigma S_\sigma, \sigma_\mu = (S_\sigma)_\phi - (\sigma_\sigma S_\sigma \sigma_\mu)_{\phi} = \begin{cases} 0 & \text{for } \mu = 0 \quad (\Psi^-) \\
2[S_\sigma - (S_\sigma \cdot \sigma_j)], (S_{\text{tot}(j)})^2 = 2\hbar^2 & \text{for } \mu = j = 1, 2, 3 \quad (\Phi^+, \Phi^-, \Psi^+) \end{cases} \] (11a)

\[ 2(S_{(1)} \cdot S_{(2)})_{(\mu)} = -2(S_\sigma)_{\phi} \cdot (\sigma_\mu S_\sigma \sigma_\mu)_{\phi} = 2S_\sigma \cdot (S_\sigma - 2(S_\sigma \cdot \sigma_\mu)\sigma_\mu) = 2 \left( S_\sigma^2 - 2(S_\sigma \cdot \sigma_\mu)^2 \right) = \begin{cases} -2S_\sigma^2 = -3\hbar^2/2 & \text{for } \mu = 0 \quad (\Psi^-) \\
2[3\hbar^2/4 - \hbar^2/2] = \hbar^2/2 & \text{for } \mu = j = 1, 2, 3 \quad (\Phi^+, \Phi^-, \Psi^+) \end{cases} \] (11b)

\( \Psi^- \), where the spin pair relate by inversion is a singlet state, while the spin pairs in the three triplet states \( \Psi^+, \Phi^-, \Phi^+ \) relate by reflections onto the three frame planes defined by \( \sigma_{12}, \sigma_{23}, \sigma_{31} \), respectively. The sense of the gauge phase can be chosen (consistently) either anti-clockwise (as shown in (11)), or clockwise.

With this, relations (11) define the two spins and their relative phases in 3D; together with the spin swap symmetry for the cross-product in (10) they determine the two-particle states within gauge freedom. From (11) the two spins have \textit{opposite order} for all entangled pairs. They have opposite (same) handedness for \( \Psi^-, \Psi^+ \) (resp. \( \Phi^+, \Phi^- \)). Taking the handedness and order of one particle as \( (\mathcal{R}, r) \) the other entangled particle from (11) is \( (\mathcal{L}, \ell) \) for \( \Psi^- \), \( \Psi^+ \) and \( (\mathcal{R}, \ell) \) for \( \Phi^+, \Phi^- \), see equations (11'). A subtlety in the definitions of the last two triplet states is that the first comprises two spin up, while the second two spin down, therefore the superscripts; this is rendered explicit further down in (11'), (14).

Spinor representation of the four states is presented in Section 3, see relations (20-21). \( S_{\text{tot}}^2 = 2\hbar^2 \) (or 0) for triplet (singlet) in (11a) correspond to angular momentum of \( \hbar \) (or 0) for each entangled pair. It stands clear from the expressions above that entangled states are not just combinations of spin vector pairs up and down; the spins’ relative handedness and the gauge phases have to satisfy precise relations as well. The vectors for \( S_{\text{tot}} \) of the triplet states in (11a) ‘live’ in the mutually perpendicular planes \( l_\sigma = \sigma_{23}, l_\sigma = \sigma_{31}, l_\sigma = \sigma_{12} \), respectively. Relations (11, 11a-b) demonstrate in a nutshell why and how the full spin (3) is necessary
to define two-spin entanglements. One particular choice of mutually orthogonal states, as shown in Section 3.2, is the following, with the handedness $(R, L)$ and order $(r, l)$ shown explicitly:

\[
S_{(1\text{or}2)} = \begin{cases}
\frac{\hbar}{2} \left( \sigma_3 + \sigma_1 + \sigma_2 \right)_\varphi; (R, r) \\
\left( \sigma_3 + \sigma_1 + \sigma_2 \right)_\varphi; (R, r) \\
\left( -\sigma_3 - \sigma_1 - \sigma_2 \right)_\varphi; (L, l) \\
\left( -\sigma_3 - \sigma_1 - \sigma_2 \right)_\varphi; (L, l)
\end{cases}
\]

\[
S_{(2\text{or}1)} = \begin{cases}
\frac{\hbar}{2} \left( -\sigma_3 - \sigma_1 - \sigma_2 \right)_\varphi; (L, \ell) \\
\left( -\sigma_3 - \sigma_1 - \sigma_2 \right)_\varphi; (L, r) \\
\left( -\sigma_3 - \sigma_1 - \sigma_2 \right)_\varphi; (R, \ell) \\
\left( -\sigma_3 - \sigma_1 - \sigma_2 \right)_\varphi; (R, r)
\end{cases}
\]

The reflection operations for the triplet states have to apply before rendering the phase explicit, because the reflection planes are fixed, bound to the reference frame. As noted above all entangled pair members show opposite order, while for handedness this true only for pair members from $\Psi^-, \Psi^+$. Any measurement of spin along a chosen axis $u$ will produce either $+\frac{\hbar}{2}u$ or $-\frac{\hbar}{2}u$. Therefore the measured square value of total spin for two particles cannot exceed $\hbar^2$, which as seen in (11a) falls short by a factor of 2 relative to the ‘on paper’ value of $S_{\text{tot}}^2$ for the triplet states. The same reason as for the one-particle case is valid here, the measurement projecting out two of each spin components. Applying the ‘on lab’ rule of transformation to the cross-product terms in (10), which is the part of the squared total spin affected by a measurement, one obtains the correlation expectation values (reflections apply to the transformed spins!):

\[
\langle \Upsilon_{(\mu)} \rangle_{S-G} \equiv \frac{1}{2} \left( S_{(1)} S_{(2)} + S_{(2)} S_{(1)} \right)_{(\mu)} = -\frac{4}{\hbar^2} \left( R_u S_R R_u^\dagger \right)_{S-G} \cdot \left( \sigma_\mu R_v S_R R_v^\dagger \sigma_\mu \right)_{S-G} = -u \cdot (\sigma_\mu v \sigma_\mu) = u \cdot v - 2(u \cdot \sigma_\mu) \cdot (v \cdot \sigma_\mu) =
\]

\[
\begin{cases}
- (u \cdot v) = - \cos \theta_{uv} = \sin^2 \frac{\theta_{uv}}{2} - \cos^2 \frac{\theta_{uv}}{2}; & \mu = 0 \\
u \cdot v - 2(u \cdot \sigma_k) (v \cdot \sigma_k) = \cos \theta_{uv} - 2u_k v_k = \cos \theta_{uv} - 2 \cos \theta_{uk} \cos \theta_{vk}; & \mu = k = 1, 2, 3
\end{cases}
\]

$u_k, v_k$ are the scalar components of $u$ and $v$ along $\sigma_k$, i.e. the expected values for triplet states depend on the initial reference frame. The experimental bounds $-\hbar^2/2 \leq (2S_{(1)} \cdot S_{(2)})_{S-G} \leq \hbar^2/2$ become explicit in (12) for all two-particle states. In addition, the expected values do not change under swapped reflection $S_u \cdot (I \sigma_\mu S_v I \sigma_\mu) = (I \sigma_\mu S_u I \sigma_\mu) \cdot S_v$, as they should in respect of the spin swap symmetry $S_{(1)} \simeq S_{(2)}$. Both ‘on paper’ and ‘on lab’ expectations satisfy the frame-independent ‘closure’ relation: $\sum_{\mu} (S_{(1)} \cdot S_{(2)})_{(\mu)} = 0$.

The advantage of (12) is that the two terms render the spin swap symmetry explicit and directly yield the scalar product of the two spins; however, for calculation purposes, we could also have defined the expectation values by the scalar part of only one term:
\[ \langle Y(\mu) \rangle_{S-G} \equiv -\frac{4}{\hbar^2} \langle S_S \sigma_\mu S_V \sigma_\mu \rangle_{S-G} = \begin{cases} - (u \cdot v) & \text{for } \langle \Psi^\mu \rangle (\mu = 0) \\ (u \cdot v - 2(u \cdot \sigma_R)(v \cdot \sigma_R)) & (\mu = k = 1,2,3) \\ \sum_\mu \langle Y(\mu) \rangle = 0 \end{cases} \quad (13) \]

The form of correlation expectation value in (13) hints to a spinor representation of the four maximally entangled states that is discussed in Section 3.2 (see relations (20-21)).

As an additional detail one can schematically characterize the entanglement states by the four (not unique) sign combinations below in front of the basis vectors; the first column of signs stands for \( S_{(1)} \) (see (11')):

\[ \Psi^- : \frac{\sigma_3}{+ -}, \Phi^+ : \frac{+ +}{+ -}, \Psi^- : \frac{- -}{- +}, \Phi^+ : \frac{+ +}{- -} ; \text{ or } \sum_\mu s(\sigma_{j(1)}) s(\sigma_{j(2)}) = \begin{cases} -3 & \text{singlet} \\ +1 & \text{triplets} \end{cases} \quad (14) \]

The last equality is a succinct expression for the ‘on paper’ scalar \( S_{(1)} \cdot S_{(2)} \) in units of \( \hbar^2/4 \), see (11b). What about the orthogonality of the maximally entangled states? Alike the one-particle case (remember \( u^+, u^- \) in fig. 1b), orthogonality is experienced in the spinor (one-sided rotor) representation, as described by equations (22-24) and the relative discussions in Section 3.2.

If the spins are not entangled relations (12, 13) are invalid. In this case the gauge components of the spins \( S_{(1)}, S_{(2)} \) are not correlated before measurement and the two spins form a mixed state. For example, instead of the singlet state in (12), there will be a pair of antiparallel spins along the \( \sigma_3 \) axis, while the total spin will lie in the \( \sigma_{12} \) plane and vary randomly between the limiting values of \( \pm \hbar \). The expectation value in this case will be the product of the separate expectation values, instead of the expectation value of the product in (13):

\[ \langle S_{(1)} S_{(2)} \rangle = \frac{4}{\hbar^2} \langle S_S u \rangle \langle S_S v \rangle = -(\sigma_3 \cdot u)(\sigma_3 \cdot v) = - \cos \theta_u \cos \theta_v \quad (15) \]

One future scope could be to adapt the present STA formalism to describe spin pairs with degrees of entanglement between the maximally entangled states from relations (11)-(13) and the mixed states from (15), as depending on the uncertainty of the relative gauge phase.

3. Spinor (one-sided rotor) transformation of spin 1/2 one and two particle systems.

In spinor form, the transformation of a given spin appears as a sum of one-sided rotor transformations of spin-up and spin-down with the standard probability amplitudes for coincidence and anti-coincidence.
3.1. One-particle systems. The vector transformation (5) looks like a ‘deterministic’ spin transformation from a given orientation or handedness in the \( \mathbf{a} \) system into a definite handedness in the \( \mathbf{u} \) system. One can improve on this impression by applying the STA spinor transformation, which uses one-sided rotors to transform the spin, as illustrated in Fig. 1b:

\[
S_\sigma \rightarrow R_{\theta_u} S_{\sigma(\phi)} \cos \frac{\theta_u}{2} + R_{\pi-\theta_u} S_{-\sigma(-\phi)} \sin \frac{\theta_u}{2} = R_{\theta_u} S_{\sigma(\phi)} \cos \frac{\theta_u}{2} + R_{\pi-\theta_u} R_{\pi u} S_{\sigma(\phi)} R_{\pi u}^\dagger \sin \frac{\theta_u}{2} = \\
R_{\theta_u} S_{\sigma(\phi)} \left( \cos \frac{\theta_u}{2} + I u_2 \sin \frac{\theta_u}{2} \right) = R_{\theta_u} S_{\sigma(\phi)} R_{\theta_u}^\dagger = S_u(\varphi-\varphi_u),
\]

or:

\[
S_\sigma \rightarrow \frac{\hbar}{2} \left( (u^+ + R_{\theta_u} (\mathbf{u}_1 + \mathbf{u}_2)) \cos \frac{\theta_u}{2} + (u^- - R_{\pi-\theta_u} (\mathbf{u}_1 - \mathbf{u}_2)) - \varphi_u \sin \frac{\theta_u}{2} \right) = \frac{\hbar}{2} \left( u^+ \cos \frac{\theta_u}{2} + u^- \sin \frac{\theta_u}{2} + R_{\theta_u} \left( (u_1 + u_2) - \varphi_u \sin \frac{\theta_u}{2} \right) \right) = \frac{\hbar}{2} \left( u + R_{\theta_u} (u_1 + u_2) \right) = (S_u)_{\varphi-\varphi_u}
\]

By insisting that a basis consist of opposite spins then in order for transformation (16) to complete into a proper rotation as it does, the two spins must have same handedness. The plane for the phase angle in the subscripts in (16) follows the plane defined by the two vectors in brackets. The concise form of the transformation in the top two lines of (16) is typical for STA, where full geometric objects transform as a whole, with none or minimal need to work with components. Vector components appear explicitly in the second form of transformation. The full spinor representations for spin up and spin down are:

\[
R_{\theta_u} S_{\sigma(\phi)} = u^+ + R_{\theta_u} (u_1 + u_2) \varphi-\varphi_u \quad \text{and} \quad R_{\theta_u} S_{\sigma(\phi)} I u_2 = u^- + R_{\theta_u} (u_1 + u_2) \varphi-\varphi_u I u_2
\]

Notice that spinor representations are not spin according to definition (3). As already mentioned, the midway vectors \( u^+ \) and \( u^- \) are reduced spinor representations of the two spins; they are manifestly orthonormal.

Spinor representations are not unique. Another reduced spinor representation comprises the factors in front of the trigonometric functions of the rotor \( R_{\theta_u} \), a scalar 1 and a bivector \( I u_2 \), respectively, which also appear in (17). These are even-grade elements of the 3D algebra and a standard choice in the STA literature [7]. In order for the reduced representation to be an orthonormal spinor basis one must have a zero scalar (grade 0) for \( \langle I u_2 \rangle_0 = \langle u^+ u^- \rangle_0 = 0 \), which is indeed the case. The orthogonality relation for the full spinor representation is (remember that \( S_\sigma \) is Hermitian):

\[
\langle (R_{\theta_u} S_{\sigma(\phi)} I u_2) \dagger (R_{\theta_u} S_{\sigma(\phi)}) \rangle_0 = -\left( \frac{3\hbar^2}{4} I u_2 \right)_0 = 0, \quad \text{which is clearly satisfied}
\]
The orthogonality condition for the reduced representation is a normalized version of (18). There are two things to notice in (16). First, it is clear that the new spin has been expressed by the two basis spins in \( \sigma \), each contributing an amplitude given by the trigonometric factors. The basis spins, in addition to being opposite must comply with precise phase angle differences and have same handedness in order for the transformation to complete. Second, it is probably more significant to read transformation (16) backwards. Start with \( S_u \) and transform it with probability amplitudes of \( \cos \frac{\theta_u}{2} \) and \( \sin \frac{\theta_u}{2} \) into \( S_\sigma \) (coincidence) or \( S_{-\sigma} \) (anticoincidence), respectively:

\[
S_u = R_{\theta_u} S_\sigma \cos \frac{\theta_u}{2} + R_{\theta_u} S_\sigma l u_2 \sin \frac{\theta_u}{2} = R_{\theta_u} S_\sigma \cos \frac{\theta_u}{2} + R_{\theta_u} R_{\pi u} S_{-\sigma} \sin \frac{\theta_u}{2}
\]

(16’)

From (16’), if \( S_u \) is spin-up (down) then \( R_{\theta_u} S_\sigma \) represents spin-up (down) and \( l u_2 R_{\theta_u} S_{-\sigma} \) or \( l u_2 \) stand for coincidene, while \( R_{\theta_u - \pi} S_{-\sigma} \) or \( l u_2 \) stand for anti-coincidence. It is also relevant to point out that the controlled phase angle in (16) actually ensures the reversibility of the ‘on paper’ spin transformation, ultimately enabling to read it backwards as in (16’). On the other side, the action of a \( S-G \) magnet makes the transformation irreversible relative to the gauge plane; in that case (16’) cannot remain an equality, changing to:

\[
S_u(\varphi) \rightarrow R_{\theta_u} S_\sigma(\varphi') \cos \frac{\theta_u}{2} + R_{\pi - \theta_u} S_{-\sigma}(\varphi') \sin \frac{\theta_u}{2} = \frac{\hbar}{2} \left( u^+ \cos \frac{\theta_u}{2} + u^- \sin \frac{\theta_u}{2} \right)
\]

(19)

(\( \varphi, \varphi', \varphi'' \) uncorrelated; \( \langle \cdot \rangle \) means average with respect to gauge phases)

In the ‘on lab’ case it is straightforward to apply the spinor transformation to the spin vector alone:

\[
\frac{\hbar}{2} u = \frac{\hbar}{2} R_{\theta_u} \sigma_3 \left( \cos \frac{\theta_u}{2} + l u_2 \sin \frac{\theta_u}{2} \right) = \frac{\hbar}{2} \left( u^+ \cos \frac{\theta_u}{2} + u^- \sin \frac{\theta_u}{2} \right)
\]

(19’)

The final expression in (19’) shows explicitly the halfway vectors \( u^+, u^- \) as an orthonormal basis for the spinor representation. As already mentioned in Section 2.1, being insensitive to gauge phase, the spin vector is well suited to represent spin transformation by \( S-G \) measurement.

3.2. Two-particle systems. By writing the non-normalized expression in (13) for \( u = v = \sigma_3 \) as:

\[
\langle S_\sigma l \sigma_3 S_\sigma l \sigma_3 \rangle = \langle -l \sigma_3 S_\sigma \rangle^\dagger \langle S_\sigma l \sigma_3 \rangle (\text{notice that } S_\sigma l \sigma_3 \text{ is not Hermitian})
\]

one realizes that a non-normalized spinor form for the entangled pairs consists of the conjugate pair:
The spinor representations (21) for the four states comprise only terms that are either even (bivectors) for the singlet state ($\mu = 0$) or odd (vectors and pseudoscalar) for the triplet states ($\mu = 1$), therefore:

$$\langle \Psi_{(\mu)} \mid \Psi_{(\nu)} \rangle = 0$$

for both ‘on paper’ and ‘on lab’ (22)

In words, (22) proves that the singlet state is orthogonal to the triplet states. By taking the spins defined at the left in (11’) as $S_\alpha, S'_\alpha$ one can prove mutual orthogonality for all pairs of distinguished states ($\mu \neq \nu$):

$$\langle \Psi_{(\mu)} \mid \Psi_{(\nu)} \rangle = \langle S_\alpha \mid S'_\alpha \rangle = 0; C = 1 \text{ for } \mu \nu = 0; C = I \epsilon_{jkl} \text{ for } \mu \nu = jk; j, k, l = 1, 2, 3$$

(23)

Finally, the orthogonality relations among the entangled states ‘on lab’ can be rendered by the equations:

$$\langle \Psi_{(\mu)} \mid \Psi_{(\nu)} \rangle = \pm \langle S_\alpha \mid S'_\alpha \rangle = 0 \text{ for } \mu \neq \nu; \mu, \nu = 0, 1, 2, 3$$

(24)

From (23, 24) the ‘on paper’ and ‘on lab’ spinor representations for the maximally entangled states are mutually orthogonal. The corresponding reduced spinor representations $\pm I S_\alpha$ are manifestly orthonormal for the four entanglement states $\Psi_{(\mu)}, \mu = 0, 1, 2, 3$, which is equivalent to equations (24). We do not need calculate the expected values for entangled pairs in spinor representation as by construction they are equal to (13).

A swift comparison of the two-particle spinor representation (21) with the vector representation (11) reveals that inversion and reflection operations apply to one of the spins in (11); the same operations are split between the two spinor representations in (21). Therefore, given that the two spins are experimentally monitored at different locations, one might get the impression that the spinor representation is ‘more nonlocal’ than the vector representation (11). However, as pointed out in relation to expression (10) there is full symmetry under the exchange $S_{(1)} \leftrightarrow S_{(2)}$. In addition, the cross-product term in the form (13) is the same for both representations and as a correlation expectation value it takes account of both entangled spins under the same angled brackets. This would represent a nonlocal operation, as the two measurements actually take place at different locations. The common angled brackets refer to the statistical dependence between the two measurements, directly connected to the correlation between gauge phases at the event of creation of entangled pairs. It is relevant to remind here that we are working in the spin-position decoupling.
approximation and further development of the STA formalism is necessary to describe nonlocality, which necessarily requires coupling of spin and position, and that most correctly has to be treated relativistically [7]. Due to decoupling, the operation is local.

4. Conclusions

All the above results were obtained by direct use and transformation (rotation, reflection) of spin in 3D physical orientation space, without invoking concepts like eigen algebra and tensor product, which seem so fundamental in the standard formulation of QM. This also provides a clear geometric meaning for the four Bell states: one singlet state of spin pairs related by inversion and three triplet states, each comprising spin pairs related by reflection onto one of the three coordinate planes.

In conclusion, the STA spin model in (3) – sum of base vectors with a gauge phase, displays many attractive features in the spin-position decoupling approximation. For one and two particle systems the same 3D geometric object (3) shows the correct representation of spin 1/2 relative to both ‘on paper’ and ‘on lab’ expectations. In a STA context (3) merges the roles played by the Pauli spin matrix-vector and spin states in the standard quantum formalism. The explicit gauge phase in (3) allows formalizing the irreversibility related to measurement (‘on lab’ transformation) as due to loss of phase correlation. Two entangled spins relate by improper rotations and show precise correlations of gauge phases. A clear geometric picture can be maintained throughout the process of ‘on paper’ and ‘on lab’ transformations, not the least due to STA’s distinguished ability to work with whole geometric objects instead of components.

References


