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Research Article

Keywords: Discrete fractional, p-Laplacian, One weak solution, Infinitely many solutions Variational methods

Posted Date: November 30th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-2320135/v1

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Additional Declarations: No competing interests reported.
Existence results for a discrete fractional boundary value problem driven by $p$-Laplacian operator

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Abstract

Global optimization concentrates on finding the maximum or minimum over all input values. Optimization problems are ubiquitous in the mathematical modeling of real-world systems and cover a very broad range of applications arising in all branches of economics, finance, chemistry, materials science, astronomy, physics, structural and molecular biology, engineering, computer science, and medicine. In this paper, we shall discuss the existence of at least one weak solution and infinitely many weak solutions for a discrete fractional boundary value problems driven by $p$-Laplacian operator. In particular, in our work, we shall look for local minima for the Euler functional corresponding to discrete fractional boundary value problems involving $p$-Laplacian. On the other hand, equations involving the discrete $p$-Laplacian operator, subjected to classical or less classical boundary conditions, have been widely studied by many authors using various techniques. Our technical approach is based on variational methods. Some recent results are extended and improved. Moreover, some examples are presented to demonstrate the application of our main results.
Existence results for a discrete fractional

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1 Introduction

In this paper, we consider the following discrete fractional boundary value problems driven by $p$-Laplacian operator for any \( k \in [1, T]_{\mathbb{N}_0} \)

\[
\begin{aligned}
& \left\{ \begin{array}{l}
T+1 \nabla_0^\alpha (k \nabla_0^\alpha (u(k))) + k \nabla_0^\alpha (T+1 \nabla_k^\alpha (u(k))) + \varphi_p(u(k)) = \lambda f(k,u(k)), \\
\quad u(0) = u(T+1) = 0
\end{array} \right. \quad (P_\lambda^f)
\end{aligned}
\]

where \( 0 < \alpha < 1, \lambda > 0, k \nabla_0^\alpha \) is the left nabla discrete fractional difference and \( T+1 \nabla_k^\alpha \) is the right, \( f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R} \) is a continuous function, \( \varphi_p \) is the so called $p$-Laplacian operator defined as \( \varphi_p(s) = s^{p-2}s \). Fractional differential equations (FDEs) have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc, for instance by (Diethelm 2010; Hilfer 2000; Kilbas et al. 2006) and the references therein. Some classical tools such as the method of upper and lower solutions, fixed point theorems, the coincidence degree theory and the monotone iterative methods are used to study nonlinear FDEs, and we refer the reader to the papers of (Galewski and Molica Bisci 2016; Heidarkhani 2014; Rabinowitz 1986) and the references therein.

Due to the wide application in many fields such as science, economics, neural network, ecology, cybernetics, etc., the theory of nonlinear difference equations has been widely studied since the 1970s. At the same time, boundary value problems (BVPs) of difference equations have received much attention from many authors: (Cabada et al. 2009; Galewski and Gilkab 2012; Galewski and Wieteska 2013, Henderson and Thompson 2002) and the references therein.

Recently, many results have been established by applying variational methods. In this direction we mention the papers of (Faraci 2005; Li and He 1995; Wei and Agarwal 2008). In the last years, several researchers have studied nonlinear problems of this type through different approaches. Also, $p$-Laplacian boundary value problems for ordinary differential equations, finite difference equations and dynamic equations have been studied extensively, but there are few papers dealing with the fractional $p$-Laplacian boundary value problems, by (Cai 2012; Wang and Xiang 2010; Wang and Xiang 2009), especially for discrete fractional $p$-Laplacian boundary value problems involving Caputo fractional differences. For example (Lv 2012) by using Schaefer’s fixed point theorem, under certain nonlinear growth conditions of the nonlinearity obtained existence of solutions to a discrete fractional boundary value problem with a $p$-Laplacian operator.
Recently, fractional integro-differential equations are actively used to
describe a wide class of economical processes with power-law memory and spa-
tial nonlocality. Generalizations of basic economic concepts and notions of
the economic processes with memory were proposed. New mathematical models
with continuous time are proposed to describe the economic dynamics with a
long memory. Chu et al. (2021), through dynamical analysis, have introduce
for the first time a fractional-calculus based artificial macroeconomic model,
actually implemented in the Laboratory via a new hardware set up. Ming
et al. (2019) have applied Caputo-type fractional order calculus to simulate
China’s gross domestic product (GDP) growth based on R software, which is
a free software environment for statistical computing and graphics. Moreover,
they have compared the results for the fractional model with the integer order
model. In addition, they have shown the importance of variables according to
the BIC criterion. The study has shown that Caputo fractional order calcul-
cus can produce a better model and perform more accurately in predicting the
GDP values from 2012-2016. Johansyah et al. (2021) reviewed the applications
of fractional differential equation in economic growth models, and they com-
pared related literatures and evaluate them comprehensively. The results of
this study are the chronological order of the applications of the Fractional Dif-
ferential Equation (FDE) in economic growth models and the development on
theories of the FDE solutions, including the FRDE forms of economic growth
models. They provided a comparative analysis on solutions of linear and non-
linear FDE, and approximate solution of economic growth models involving
memory effects using various methods. The main contribution of thier research
is the cronological development of the theory to find necessary and sufficient
conditions to guarantee the existence and uniqueness of the FDE in economic
growth and the methods to obtain the solution.

Motivated by the above facts, in the present paper, using a smooth version
of Theorem 2.1 of Bonanno and Molica Bisci (2009) which is a more precise
version of Ricceri’s Variational Principle from (Ricceri 2000) we investigate
the existence of at least one weak solution and infinitely many weak solutions
for the problem \((P^f_\lambda)\). In fact, we shall study the existence of at least one non-
trivial weak solution for the problem \((P^f_\lambda)\) under an asymptotical behaviour
of the nonlinear datum at zero, see Theorem (3). We present Example (1)
in which the hy potheses of Theorem (3) are fulfilled. We give some remarks
on our results. Furthermore, under suitable oscillating behaviour at infinity
of the nonlinear datum we shall discuss the existence of infinitely many weak
solutions for the problem \((P^f_\lambda)\). We prove the existence of definite interval
about \(\lambda\) in which the problem \((P^f_\lambda)\) admits a sequence of solutions which is
unbounded in the space \(W\) which will be introduced later (Theorem 5). We
present an example to illustrate Theorem 6 (see Examples 2). Furthermore,
some consequences of Theorem 5 are listed. Replacing the conditions at infinity
on the nonlinear terms, by a similar one at zero, we obtain a sequence of
pairwise distinct solutions strongly converging at zero; see Theorem 11.
The remainder of the paper is organized as follows. In Section 2, we recall some basic definitions and our main tool. In Sections 3 and 4 we shall state and prove the main results of the paper.

2 Preliminaries

The key argument in our results is the following version of Ricceri’s variational principle (Ricceri 2000, Theorem 2.1) as given by Bonanno and Molica Bisci (2009).

**Theorem 1** Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gateaux differentiable functionals such that $\Phi$ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and $\Psi$ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \sup_{u \in \Phi^{-1}(-\infty, r)} \frac{\Psi(u) - \Psi(u) - \Phi(u)}{r - \Phi(u)}$$

and

$$\theta := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^{+}} \varphi(r).$$

Then, one has

(a) for every $r > \inf_X \Phi$ and every $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, the restriction of the functional $I_\lambda = \Phi - \lambda \Psi$ to $\Phi^{-1}(-\infty, r]$ admits a global minimum, which is a critical point (local minimum) of $I_\lambda$ in $X$.

(b) If $\theta < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\theta}\right)$, the following alternative holds:

either

(b1) $I_\lambda$ possesses a global minimum,

or

(b2) there is a sequence $\{u_n\}$ of critical points (local minima) of $I_\lambda$ such that

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\delta}\right)$, the following alternative holds:

either

(c1) there is a global minimum of $\Phi$ which is a local minimum of $I_\lambda$,

or

(c2) there is a sequence of pairwise distinct critical points (local minima) of $I_\lambda$ which weakly converges to a global minimum of $\Phi$. 
We refer the interested reader to the papers of (Barilla et al. 2021; Lalewski and Molica Bisci 2016; Graef et al. 2021; Heidarkhani et al. 2016), in which Theorem 1 has been successfully employed to prove the existence of at least one non-trivial solution for boundary value problems and the papers of (Bonanno and Candito 2009; Graef et al. 2014; Heidarkhani 2013; Heidarkhani et al. 2017; Ricceri 2001), in which Theorem 1 has been successfully employed for the existence of infinitely many solutions for boundary value problems in the papers.

In this section, we will introduce several basic definitions, notations, lemmas, and propositions used all over this paper.

**Definition 1** (Atici 2009)

(i) Let \( m \) be a natural number, then the \( m \) rising factorial of \( t \) is written as

\[
t^{m} = \prod_{k=0}^{m-1} (t + k), \quad t^0 = 1.
\]

(ii) For any real number, the \( \alpha \) rising function is increasing on \( \mathbb{N}_0 \) and

\[
t^{\alpha} = \frac{\Gamma(t + \alpha)}{\Gamma(t)}
\]

such that \( t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}, 0^{\alpha} = 0 \).

**Definition 2** Let \( f \) be defined in \( \mathbb{N}_{a-1} \cap b+1 \mathbb{N} \), \( a < b \), \( \alpha \in (0,1) \) then the nabla discrete new (left Caputo) fractional difference is defined by

\[
\left( ^{C}_{k} \nabla_{a-1}^\alpha f \right)(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=a}^{k} \nabla s f(s)(k - \rho(s))^{-\alpha}, \quad k \in \mathbb{N}_{a}
\]

(1)

and the right Caputo One by

\[
\left( ^{C}_{b+1} \nabla_{k}^\alpha f \right)(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=k}^{b} (-\Delta s f(s)(s - \rho(k))^{-\alpha}, \quad k \in \mathbb{bN}
\]

(2)

and in the left Riemann sense by

\[
\left( ^{R}_{k} \nabla_{a-1}^\alpha f \right)(k) = \frac{1}{\Gamma(1-\alpha)} \nabla k \sum_{s=a}^{k} f(s)(k - \rho(s))^{-\alpha} = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{k} f(s)(k - \rho(s))^{-\alpha-1}, \quad k \in \mathbb{N}_{a},
\]

and the left Riemann sense by

\[
\left( ^{R}_{b+1} \nabla_{k}^\alpha f \right)(k) = \frac{1}{\Gamma(1-\alpha)} (-\Delta k) \sum_{s=k}^{b} f(s)(s - \rho(k))^{-\alpha} = \]
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\[ \frac{1}{\Gamma(-\alpha)} \sum_{s=k}^{b} (f(s))(s - \rho(k))^{-\alpha-1}, \quad k \in bN \]

where \( \rho(k) = k - 1 \) be the backward jump operator.

For example, let \( f(k) = 1 \) defined on \( \mathbb{N}_{a-1} \cap b+1 \mathbb{N} \), therefor from (1) and (2), we have

\[ C_{b+1}^{\alpha} 1 = C_{k}^{\alpha} 1 = 0, \quad k \in \mathbb{N}_{a} \cap bN. \quad (3) \]

The relation between the nabla left and right Caputo and Riemann fractional differences are as follow:

\[ (C_{k}^{\alpha} a_{-1} f)(k) = (R_{k}^{\alpha} a_{-1} f)(k) - \frac{(k - a + 1)^{-\alpha}}{\Gamma(1 - \alpha)} f(a - 1), \quad (4) \]

\[ (C_{b+1}^{\alpha} k f)(k) = (R_{b+1}^{\alpha} k f)(k) - \frac{(b + 1 - k)^{-\alpha}}{\Gamma(1 - \alpha)} f(b + 1). \quad (5) \]

Thus by (3), (4) and (5), we have for any \( k \in \mathbb{N}_{a} \cap bN \)

\[ R_{b+1}^{\alpha} k f(k) = \frac{(b + 1 - k)^{-\alpha}}{\Gamma(1 - \alpha)} f(b + 1), \quad R_{k}^{\alpha} a_{-1} f(k) = \frac{(k - a + 1)^{-\alpha}}{\Gamma(1 - \alpha)}. \]

Regarding the domains of the fractional type differences we observe:

(i) The nabla left fractional \( a_{-1} \nabla_{k}^{\alpha} \) maps functions defined on \( a_{-1} \mathbb{N} \) to functions defined on \( a\mathbb{N} \).

(ii) The nabla right fractional \( k \nabla_{b+1}^{\alpha} \) maps functions defined on \( b+1 \mathbb{N} \) to functions defined on \( b\mathbb{N} \).

One can show that for \( \alpha \to 0 \), one has \( a \nabla_{k}^{\alpha} (f(k)) \to f(t) \) and for \( \alpha \to 1 \), one has \( a \nabla_{k}^{\alpha} (f(k)) \to \nabla f(t) \). We note that the nabla Riemann and Caputo fractional difference for \( 0 < \alpha < 1 \) consider when \( f \) vanishes at the end points that is \( f(a-1) = 0 = f(b+1) \) (Abdeljawad 2013). Indeed when \( 0 < \alpha < 1 \) those conclude from (4) and (5). So for convenience from no one we will use the symbol \( \nabla^{\alpha} \) instead of \( R_{\nabla}^{\alpha} \) or \( C_{\nabla}^{\alpha} \).

Now we present summation by parts formula in new discrete fractional calculus.

**Theorem 2** (Abdeljawa 2012, Theorem 4.4, Integration by parts for fractional difference). For functions \( f \) and \( g \) defined on \( \mathbb{N}_{a} \cap b\mathbb{N} \), \( a = b \) and \( 0 < \alpha < 1 \), one has

\[ \sum_{k=a}^{b} f(k)(k \nabla_{a-1}^{\alpha} g)(k) = \sum_{k=a}^{b} g(k)(b+1 \nabla_{k}^{\alpha} f)(k). \]

Similarly

\[ \sum_{k=a}^{b} f(k)(b+1 \nabla_{k}^{\alpha} g)(k) = \sum_{k=a}^{b} g(k)(k \nabla_{a-1}^{\alpha} f)(k). \]
In order to give the variational formulation of the problem \( (P^f_\lambda) \). Let us define the finite \( T \)-dimensional Banach space

\[
W = \{ u : [0, T + 1]_{\mathbb{N}_0} \to \mathbb{R} : u(0) = u(T + 1) = 0 \},
\]

which is equipped with the norm

\[
\|u\| = \left( \sum_{k=1}^{T} |u(k)|^2 \right)^{\frac{1}{2}}.
\]

**Lemma 1** For every \( 0 < \alpha < 1 \) and \( u \in W \), we have

\[
\|u\|_{\infty} = \max_{k \in [1, T]_{\mathbb{N}_0}} |u(k)| \leq \|u\|. \tag{6}
\]

We define functionals \( \Phi, \Psi \) for every \( u \in W \), as follows

\[
\Phi(u) = \frac{1}{2} \sum_{k=1}^{T} |(k \nabla_0^\alpha u)(k)\|^2 + |(T+1 \nabla_k^\alpha u)(k)|^2 + \frac{1}{p} \sum_{k=1}^{T} |u(k)|^p \tag{7}
\]

and

\[
\Psi(u) = \sum_{k=1}^{T} F(k, u(k)), \tag{8}
\]

and we put

\[
I_\lambda(u) = \Phi(u) - \lambda \Psi(u)
\]

for every \( u \in W \).

**Definition 3** We mean by a (weak) solution of the BVP \( (P^f_\lambda) \), any function \( u \in W \) such that

\[
\sum_{k=1}^{T} (k \nabla_0^\alpha u(k))(k \nabla_0^\alpha v(k)) + (T+1 \nabla_k^\alpha u(k))(T+1 \nabla_k^\alpha v(k)) + \sum_{k=1}^{T} u(k)^{p-2} u(k) v(k) - \lambda \sum_{k=1}^{T} f(k, u(k)) v(k) = 0
\]

for every \( v \in W \).

**Lemma 2** If \( u \in W \) be a critical point of \( I_\lambda \) in \( W \), iff \( u \in W \) be a solution of \( (P^f_\lambda) \).

**Proof** If \( u \in W \) be a critical point of \( I_\lambda \), for every \( v \in W \), we have

\[
\sum_{k=1}^{T} (k \nabla_0^\alpha u(k))(k \nabla_0^\alpha v(k)) + (T+1 \nabla_k^\alpha u(k))(T+1 \nabla_k^\alpha v(k))
\]
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\[ + \sum_{k=1}^{T} u(k)^{p-2} u(k) v(k) - \lambda \sum_{k=1}^{T} f(k, u(k)) v(k) = 0. \]

Bearing to mind \( v \in W \) is arbitrary, one get that

\[ T+1 \nabla_{k}^{\alpha} (k \nabla_{0}^{\alpha} (\bar{u}(k))) + k \nabla_{0}^{\alpha} (T+1 \nabla_{k}^{\alpha} (\bar{u}(k))) + |\bar{u}(k)|^{p-2} \bar{u}(k) - \lambda f(k, \bar{u}(k)) = 0 \]

for every \( k \in [1, T]_{\mathbb{N}} \). Therefore \( \bar{u} \) is a solution of the problem \((P_\lambda^f)\). Since \( \bar{u} \) is arbitrary, we conclude that every critical point of the functional \( I_\lambda \) in \( W \) is a solution of the problem \((P_\lambda^f)\). One the other hand, if \( \bar{u} \) be a solution of the problem \((P_\lambda^f)\), arguing backward, the proof is completed. \( \square \)

Put

\[ F(t, \xi) = \int_{0}^{\xi} f(t, x) dx \text{ for all } (t, \xi) \in [1, T]_{\mathbb{N}} \times \mathbb{R}. \]

3 One Solution

**Theorem 3** Assume that

\[ \sup_{\theta > 0} \frac{T}{\sum_{k=1}^{T} \max_{|x| \leq \theta} F(k, x)} > \frac{p}{(T+1)^{\frac{\nu(p-2)}{4}}} \quad (9) \]

and there are discrete intervals \([1, T_1]_{\mathbb{N}} \subseteq [1, T]_{\mathbb{N}} \) and \([1, T_2]_{\mathbb{N}} \subset [1, T_1]_{\mathbb{N}} \) where \( T_1, T_2 \geq 2 \), such that

\[ \limsup_{\xi \to 0^+} \frac{\text{ess inf}_{t \in B} F(t, \xi)}{\xi^p} = +\infty \]

and

\[ \liminf_{\xi \to 0^+} \frac{\text{ess inf}_{t \in D} F(t, \xi)}{\xi^p} > -\infty. \]

Then, for each

\[ \lambda \in \Lambda = \left( 0, \left( T+1 \right)^{\frac{\nu(p-2)}{4}} \sup_{\theta > 0} \frac{T}{\sum_{k=1}^{T} \max_{|x| \leq \theta} F(k, x)} \right), \]

the problem \((P_\lambda^f)\) admits at least one non-trivial weak solution \( u_\lambda \in W \).

**Proof** Our aim is to apply Theorem 1 to the problem \((P_\lambda^f)\). We introduce the functionals \( \Phi, \Psi \) as given in (7) and (8), respectively. Let us prove that the functionals \( \Phi \) and \( \Psi \) satisfy the required conditions in Theorem 1. Since \( W \) is compactly embedded in \((C^0([1, T]_{\mathbb{N}}), \mathbb{R})\), it is well known that \( \Psi \) is a Gâteaux differentiable functional whose Gâteaux derivative at the point \( u \in W \) is the functional \( \Psi'(u) \in W^* \), given by

\[ \Psi'(u)(v) = \sum_{k=1}^{T} f(k, u(k)) v(k) \]
for every \( v \in W \), and \( \Psi \) is sequentially weakly upper semicontinuous. Moreover, \( \Phi \) is a Gâteaux differentiable functional whose Gâteaux derivative at the point \( u \in W \) is the functional \( \Phi'(u) \in W^* \), given by

\[
\Phi'(u)(v) = \sum_{k=1}^{T} (k \nabla_0^\alpha u(k))(k \nabla_0^\alpha v(k)) + (T+1 \nabla_k^\alpha u(k))(T+1 \nabla_k^\alpha v(k)) + \sum_{k=1}^{T} |u(k)|^{p-2} u(k)v(k)
\]

for every \( v \in W \). Furthermore, by the definition of \( \Phi \) we observe that it is sequentially weakly lower semicontinuous and strongly continuous. Now, taking (7) into account, for every \( u \in W \) we have

\[
\frac{1}{p} \left( T+1 \right)^{\frac{p(p-2)}{4}} \|u\|^p \leq \Phi(u) \leq 2T(T+1)\|u\|^2 + \frac{1}{p} \left( T+1 \right)^{\frac{2-p}{2}} \|u\|^p,
\]

(10)

by using the first inequality in (10), it follows

\[
\lim_{\|u\| \to +\infty} \Phi(u) = +\infty,
\]

namely \( \Phi \) is coercive. From the condition (9), there exists \( \bar{\theta} > 0 \) such that

\[
\sum_{k=1}^{T} \max_{|x| \leq \bar{\theta}} F(k, x) > \frac{p}{(T+1)^{\frac{p(p-2)}{4}}} \bar{\theta}^p.
\]

(11)

Put

\[
r = \frac{(T+1)^{\frac{p(p-2)}{4}}}{p} \bar{\theta}^p.
\]

From the definition of \( \Phi \) and in view of (6), (7) and (10) for every \( r > 0 \), one has

\[
\Phi^{-1}(-\infty, r) = \left\{ u \in W; \Phi(u) < r \right\}
\]

\[
\subseteq \left\{ u \in W; \|u\|^p \leq \frac{pr}{(T+1)^{\frac{p(p-2)}{4}}} \right\} \subseteq \left\{ u \in W; \|u\|_\infty \leq \frac{pr}{(T+1)^{\frac{p(p-2)}{4}}} \right\}
\]

\[
= \left\{ u \in W; \|u\|_\infty \leq \bar{\theta}^p \right\}
\]

which follows

\[
\sup_{\Phi(u) < r} \Psi(u) = \sup_{\Phi(u) < r} \sum_{k=1}^{T} F(k, u(k)) \leq \sum_{k=1}^{T} \max_{|x| \leq \bar{\theta}} F(k, x).
\]

By considering the above computations, since \( 0 \in \Phi^{-1}(-\infty, r) \) and \( \Phi(0) = \Psi(0) = 0 \), one has

\[
\varphi(r) = \inf_{u \in \Phi^{-1}(-\infty, r)} \left( \sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) - \Psi(u) \right) \leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)}{r}
\]

\[
\leq \frac{p}{(T+1)^{\frac{p(p-2)}{4}}} \sum_{k=1}^{T} \max_{|x| \leq \bar{\theta}} F(k, x) \bar{\theta}^p.
\]
Hence, putting
\[ \lambda^* = \left( T + 1 \right) \frac{p(p-2)}{4} \sup_{\theta > 0} T \max_{k=1}^{T} F(k, x) \theta^p. \]

At this point, thanks to Theorem 1, for every \( \lambda \in (0, \lambda^*) \subseteq \left( 0, \frac{1}{\varphi(r)} \right) \), Theorem 1 ensures that the functional \( I_\lambda \) admits at least one critical point (local minima) \( u_\lambda \in \Phi^{-1}(-\infty, r) \). We will show that the function \( u_\lambda \) cannot be trivial. Let us show that
\[ \limsup_{|u| \to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \] (12)

Due to our assumptions at zero, we can fix a sequence \( \{\xi_k\} \subset \mathbb{R}^+ \) converging to zero and two constants \( \zeta, \kappa \) (with \( \zeta > 0 \)) such that
\[ \lim_{k \to +\infty} \text{ess inf}_{t \in B} F(t, \xi_k) \geq \kappa |\xi_k|^p, \]
\( \xi \in [0, \zeta] \). Now, fix a set \([1, T_3]_{N_0} \subset [1, T_2]_{N_0} \) where \( T_3 \geq 2 \) and a function \( v \in W \) such that:

(i) \( v(k) \in [0, 1] \) for every \( k \in [1, T]_{N_0} \),

(ii) \( v(k) = 1 \in \mathbb{R} \), for every \( k \in [1, T_3]_{N_0} \),

(iii) \( v(k) = 0 \), for every \( k \in [T_1 + 1, T]_{N_0} \).

Hence, fix \( Y > 0 \) and consider a real positive number \( \eta \) with
\[ \eta T_3 + \kappa \sum_{k=T_3+1}^{T} |v(k)|^p \]
\[ < \frac{\frac{1}{p} \left( T + 1 \right)}{(T + 1)^{\frac{2}{p} - 1}} \|v\|^p. \]

Then, there is \( k_0 \in \mathbb{N} \) such that \( \varepsilon^k < \zeta \) and
\[ \text{ess inf}_{t \in B} F(t, \xi_k) \geq \eta |\xi_k|^p \]
for every \( k > k_0 \). Now, for every \( k > k_0 \), by using the properties of the function \( v \) (that is \( 0 \leq \xi_k v(k) < \zeta \) for \( k \) large enough), by (10), one has
\[ \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = \sum_{k=1}^{T_3} F(k, \xi_n) + \sum_{k=T_3+1}^{T} F(t, \xi_n v(k)) \]
\[ > \frac{\eta T_3 + \kappa \sum_{k=T_3+1}^{T} |v(k)|^p}{\frac{1}{p} \left( T + 1 \right)} \geq Y. \]

Since \( Y \) could be consider arbitrarily large, it is concluded that
\[ \lim_{k \to \infty} \frac{\Psi(\xi_k v)}{\Phi(\xi_k v)} = +\infty, \]
from which (12) clearly follows. 
Hence, there exists a sequence \( \{w_k\} \subset W \) strongly converging to zero, \( w_k \in \Phi^{-1}(-\infty, r) \) and 
\[
I_\lambda(w_k) = \Phi(w_k) - \lambda \Psi(w_k) < 0.
\]
Since \( u_\lambda \) is a global minimum of the restriction of \( I_\lambda \) to \( \Phi^{-1}(-\infty, r) \), we conclude that 
\[
I_\lambda(u_\lambda) < 0,
\]
so that \( u_\lambda \) is not trivial. The proof is complete. \( \square \)

We give some remarks of our results.

**Remark 1** In Theorem 3 we looked for the critical points of the functional \( I_\lambda \) naturally associated with the problem \((P^f_\lambda)\). We note that, in general, \( I_\lambda \) can be unbounded from the following in \( W \). Indeed, for example, in the case when \( f(\xi) = 1 + |\xi|^\gamma - p |\xi|^{p-1} \) for every \( \xi \in \mathbb{R} \) with \( \gamma > p \), for any fixed \( u \in W \setminus \{0\} \) and \( \iota \in \mathbb{R} \), we obtain
\[
I_\lambda(\iota u) = \Phi(\iota u) - \lambda \sum_{k=1}^{T} F(\iota u(k))
\leq 2\iota^2 T(T + 1)\|u\|^2 + \frac{\iota^p}{p} (T + 1)^{\frac{2-p}{2}} \|u\|^p - \lambda \iota^p T \|u\|^p - \lambda \iota^\gamma \|u\|^\gamma \to -\infty
\]
as \( \iota \to +\infty \). Therefore, the condition by (Rabinowitz 1986, Theorem 2.2) is not satisfied. Hence, we can not use direct minimization to find critical points of the functional \( I_\lambda \).

Now we want to point out that the energy functional \( I_\lambda \) associated with the problem \((P^f_\lambda)\) is not coercive. Indeed, when \( F(\xi) = |\xi|^s \) with \( s \in (p, +\infty) \) for every \( \xi \in \mathbb{R} \), for any fixed \( u \in W \setminus \{0\} \) and \( \iota \in \mathbb{R} \) we have
\[
I_\lambda(\iota u) = \Phi(\iota u) - \lambda \sum_{k=1}^{T} F(\iota u(k))
\leq 2\iota^2 T(T + 1)\|u\|^2 + \frac{\iota^p}{p} (T + 1)^{\frac{2-p}{2}} \|u\|^p - \lambda \iota^p T \|u\|^p - \lambda \iota^s T \|u\|^s \to -\infty
\]
as \( \iota \to -\infty \).

**Remark 2** If in Theorem 3 the function \( f(k, x) \geq 0 \) for a.e. \( (k, x) \in [1, T]N_0 \times \mathbb{R} \), the condition (9) becomes to the more simple form
\[
\sup_{\theta > 0} \frac{\theta^p}{\sum_{k=1}^{T} F(k, \theta)} > \frac{p}{(T + 1) \frac{p(p-2)}{4}}.
\]
(\( S_\lambda \))

Moreover, if the following assumption is verified
\[
\limsup_{\theta \to +\infty} \frac{\theta^p}{\sum_{k=1}^{T} F(k, \theta)} > \frac{p}{(T + 1) \frac{p(p-2)}{4}},
\]
then the condition (\( S_\lambda \)) automatically holds.
Indeed, by considering that $\Phi$ is coercive and that for $\lambda \in (0, \lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant $L$ such that $\|u_\lambda\| \leq L$ for every $\lambda \in (0, \lambda^*)$. After that, it is easy to see that there exists a positive constant $M$ such that

$$\left| \sum_{k=1}^{T} f(k, u_\lambda(k))u_\lambda(k) \right| \leq M\|u_\lambda\| \leq ML$$

for every $\lambda \in (0, \lambda^*)$. Since $u_\lambda$ is a critical point of $I_\lambda$, we have $I'_\lambda(u_\lambda)(v) = 0$, for any $v \in W$ and every $\lambda \in (0, \lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \sum_{k=1}^{T} f(k, u_\lambda(k))u_\lambda(k)$$

for every $\lambda \in (0, \lambda^*)$. Then, since

$$0 \leq (T + 1) \frac{p(p-2)}{4} \|u\|^p \leq \Phi'(u_\lambda)(u_\lambda)$$

by using (16) it is concluded that

$$0 \leq (T + 1) \frac{p(p-2)}{4} \|u\|^p \leq \Phi'(u_\lambda)(u_\lambda) \leq \lambda \sum_{k=1}^{T} f(k, u_\lambda(k))u_\lambda(k)$$

for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^+$, by (15) we have $\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$. One has

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\infty} = 0.$$  

Finally, we show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. For our goal we observe that for any $u \in W$, one has

$$I_\lambda(u) = \lambda \left( \frac{\Phi(u)}{\lambda} - \Psi(u) \right).$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let $u_{\lambda_1}$, $u_{\lambda_2}$ are the global minimums of the functional $I_\lambda$, restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} = \left( \frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left( \frac{\Phi(v)}{\lambda_i} - \Psi(v) \right),$$

for $i = 1, 2$.

Clearly, (14) together with (18) and since $\lambda > 0$ implies that

$$m_{\lambda_i} < 0, \text{ for } i = 1, 2.$$  

Moreover

$$m_{\lambda_2} \leq m_{\lambda_1},$$

due to that $0 < \lambda_1 < \lambda_2$. Then by considering equations (18)-(20) and again since $0 < \lambda_1 < \lambda_2$, we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. Since $\lambda < \lambda^*$ is arbitrary, we show $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$.  

Remark 4 We observe that Theorem 3 above is a bifurcation result in the sense that the pair \((0,0)\) belongs to the closure of the set
\[
\left\{(u_\lambda, \lambda) \in W \times (0, +\infty) : u_\lambda \text{ is a non-trivial weak solution of } (P_\lambda^f)\right\}
\]
in \(W \times \mathbb{R}\). Indeed, by Remark 3 we have that \(\|u_\lambda\| \to 0\) as \(\lambda \to 0\).

Hence, there exist two sequences \(\{u_j\}\) in \(W\) and \(\lambda\) in \(\mathbb{R}^+\) (here \(u_j = u_\lambda\)) such that \(\lambda_i \to 0^+\) and \(\|u_j\| \to 0\), as \(j \to +\infty\). Moreover, we want to emphasize that due to the fact that the map
\[(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)\]
is strictly decreasing, for every \(\lambda_1, \lambda_2 \in (0, \lambda^*)\), with \(\lambda_1 \neq \lambda_2\), the solutions \(\bar{u}_{\lambda_1}\) and \(\bar{u}_{\lambda_2}\) ensured by Remark 3 are different.

Remark 5 If \(f\) is non-negative then the weak solution ensured in Theorem 3 is non-negative. Indeed, let \(u_0\) be a non-trivial weak solution of the problem \((P_\lambda^f)\). Arguing by a contradiction, assume that the set \(A = \{k \in [1, T]_{N_0} : u_0(k) < 0\}\) is non-empty and of positive measure. Put \(\bar{v}(k) = \min\{0, u_0(k)\}\) for all \(k \in [1, T]_{N_0}\). Clearly, \(\bar{v} \in W\) and one has
\[
\begin{align*}
\sum_{k=1}^T (k \nabla_0^\alpha u_0(k))(k \nabla_0^\alpha \bar{v}(k)) &+ (T+1) \nabla_k^\alpha u_0(k)(T+1) \nabla_k^\alpha \bar{v}(k) \\
&+ \sum_{k=1}^T |u_0(k)|^{p-2}u_0(k)\bar{v}(k) - \lambda \sum_{k=1}^T f(k, u_0(k))\bar{v}(k) = 0.
\end{align*}
\]
Thus, from our sign assumptions on the data we have
\[
0 \leq (T+1) \frac{p(p-2)}{4} |u_0|^p \leq \sum_{A} (k \nabla_0^\alpha u_0(k))^2 + (T+1) \nabla_k^\alpha u_0(k)^2 + \sum_{A} |u_0(k)|^p
\]
\[
= \lambda \sum_{A} f(k, u_0(k))u_0(k) \leq 0
\]
. Hence, \(u_0 = 0\) in \(A\) and this is absurd.

The next theorem is concerned with a particular case of our results.

Theorem 4 Let \(f : \mathbb{R} \to \mathbb{R}\) be a non-negative continuous function. Assume that
\[
\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} = +\infty.
\]
Then, for each
\[
\lambda \in \left(0, \frac{(T+1)^{p(p-2)}}{Tp} \sup_{\theta > 0} \frac{\theta^p}{F(\theta)}\right),
\]
the problem
\[
\begin{align*}
&T+1 \nabla_k^\alpha (k \nabla_0^\alpha (u(k))) \nabla_k^\alpha \nabla_0^\alpha (T+1 \nabla_k^\alpha (u(k))) + \varphi_p(u(k)) + \psi_p(u(k)) = \lambda f(u(k)), \\
u(0) = u(T+1) = 0
\end{align*}
\]
admits at least one nontrivial weak solution \( u_\lambda \in W \) such that
\[
\lim_{\lambda \to 0^+} \| u_\lambda \| = 0
\]
and the real function
\[
\lambda \to \frac{1}{2} \sum_{k=1}^{T} |(k^{\alpha}u(k))|^2 + \sum_{k=1}^{T} |u(k)|^p - \lambda \sum_{k=1}^{T} F(u(k))
\]
is negative and strictly decreasing in \( \left( 0, \frac{(T+1)^{\frac{p(p-2)}{4}}} {T^p} \text{sup}_{\theta > 0} \frac{\theta^p}{F(\theta)} \right) \).

Finally, we present the following example to illustrate Theorem 4.

**Example 1** Let \( p = 4 \) and \( T = 2 \). We consider the problem
\[
\begin{cases}
g \left( k^{\frac{3}{2}} \frac{1}{k^{\alpha}}(u(k)) \right) + k^{\frac{1}{2}} \frac{1}{k^{\alpha}}(3^{\frac{3}{2}} \frac{1}{k^{\alpha}}(u(k))) + \varphi_4(u(k)) = \lambda f(u(k)), \\
u(0) = u(3) = 0
\end{cases}
\tag{21}
\]
where
\[
f(\xi) = 34\xi^3 + 2 \tan(\xi) \sec^2(\xi) + e^\xi
\]
for every \( \xi \in \mathbb{R} \). By simple computations, we have
\[
F(\xi) = \xi^4 + \sec^2(\xi) + e^\xi - 2
\]
for every \( \xi \in \mathbb{R} \). We see that, all the assumptions of Theorem 4 are satisfied, and it implies that the problem (21) for each \( \lambda \in \left( 0, \frac{9}{8} \right) \), admits at least one nontrivial weak solution \( u_\lambda \in W \) such that
\[
\lim_{\lambda \to 0^+} \| u_\lambda \| = 0
\]
and the real function
\[
\lambda \to \frac{1}{2} \sum_{k=1}^{T} |(k^{\alpha}u(k))|^2 + \sum_{k=1}^{T} |u(k)|^4 - \lambda \sum_{k=1}^{T} F(u(k))
\]
is negative and strictly decreasing in \( \left( 0, \frac{9}{8} \right) \).

### 4 Infinitely Many Solutions

We formulate our main result as follows: we discuss the existence of infinitely many weak solutions for the problem \((P_\lambda^F)\).

Put
\[
B^\infty = \limsup_{\xi \to \infty} \frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k^{\alpha}u(k))|^2 + \frac{T \xi^p}{p}
\]
and
\[
\frac{\varphi_4}{\text{sup}_{\theta > 0} \frac{\theta^p}{F(\theta)}}.
\]
Theorem 5 Assume that there exist two sequences \( \{a_n\} \) and \( \{b_n\} \) with \( \lim_{n \to \infty} b_n = \infty \) such that \\

\[
\begin{align*}
(A_1) & \quad \frac{a_n^2}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k)\alpha|^2 + \frac{T a_n^p}{p} < \frac{(T + 1)^{\frac{p-2}{4}}}{p} b_n^p \quad \text{for every } n \in \mathbb{N}, \\
(A_2) & \quad A^\infty = \lim_{n \to \infty} \frac{\sum_{k=1}^{T} \max_{t \leq b_n} F(k, t) - \sum_{k=1}^{T} F(k, a_n)}{(T + 1)^{\frac{p-2}{4}} \frac{a_n^2}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k)\alpha|^2 + \frac{T a_n^p}{p}} < B^\infty.
\end{align*}
\]

Then, for each \( \lambda \in \left( \frac{1}{B^\infty}, \frac{1}{A^\infty} \right) \), the problem \((P^\lambda f)\) admits an unbounded sequence of solutions.

Proof Our goal is to apply Theorem 1. Consider the functionals \( \Phi \) and \( \Psi \) as given in (7) and (8), respectively. Therefore, we observe that the regularity assumptions on \( \Phi \) and \( \Psi \), as requested in Theorem 1 are verified. We verify that \( u \in W \) is a solution of the problem \((P^\lambda f)\) if and only if \( u \) is a critical point of the function \( I^\lambda \). Put \\

\[
r_n = \frac{(T + 1)^{\frac{p-2}{4}}}{p} b_n^p
\]

for all \( n \in \mathbb{N} \). We see that \( r_n > 0 \) for all \( n \in \mathbb{N} \). From the definition of \( \Phi \) and in view of (6), (8) and (10) for every \( r_n > 0 \), one has \\

\[
\Phi^{-1}(\infty, r_n) = \{ u \in W; \Phi(u) < r_n \}
\]

\[
\subseteq \left\{ u \in W; \|u\|^p \leq \frac{p r_n}{(T + 1)^{\frac{p-2}{4}}} \right\} \subseteq \left\{ u \in W; \|u\|^p \leq \frac{p r_n}{(T + 1)^{\frac{p-2}{4}}} \right\}
\]

\[
= \left\{ u \in W; \|u\|_\infty^p \leq b_n^p \right\},
\]

which follows \\

\[
\sup_{\Phi(u) < r_n} \Psi(u) = \sup_{\Phi(u) < r_n} \sum_{k=1}^{T} F(k, u(k)) \leq \sum_{k=1}^{T} \max_{|x| \leq b_n} F(k, x).
\]

Now, for each \( n \in \mathbb{N} \), we define \\

\[
w(k) = \begin{cases} a_n, & k \in [1, T]_{\mathbb{N}_0}, \\ 0, & k \in 0, T + 1. \end{cases} \tag{22}
\]

Clearly, \( w \in W \). Since \( w \) vanishes at the end points that is \( w(0) = 0 = w(T + 1) \), thus its Riemann and Caputo fractional differences coincide, hence for any \( k \in \mathbb{N}_1 \cap T_{\mathbb{N}} \) \\

\[
(T+1\nabla_k^\alpha w)(k) = \frac{R_{T+1}^\alpha}{T+1} \nabla_k^\alpha w)(k) = \frac{C_{T+1}^\alpha}{T+1} \nabla_k^\alpha w)(k) = \frac{a_n(T + 1 - k)^{-\alpha}}{\Gamma(1 - \alpha)},
\]
Thus, if we consider a sequence

\[(k \nabla_0^\alpha w)(k) = (R_k \nabla_0^\alpha w)(k) = (C_k \nabla_0^\alpha w)(k) = \frac{a_n(k)^{-\alpha}}{\Gamma(1 - \alpha)}.\]

So, we have

\[
\Phi(w) = \frac{1}{2} \sum_{k=1}^{T} |(k \nabla_0^\alpha w)(k)|^2 + |(T+1 \nabla_k^\alpha w)(k)|^2 + \frac{1}{p} \sum_{k=1}^{T} |w(k)|^p
\]
\[
= \frac{1}{2} \sum_{k=1}^{T} \left( \frac{a_n(k)^{-\alpha}}{\Gamma(1 - \alpha)} \right)^2 + \left( \frac{a_n(T + 1 - k)^{-\alpha}}{\Gamma(1 - \alpha)} \right)^2 + \frac{T a_n^p}{n}
\]
\[
= \frac{a_n^2}{2(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k)^{-\alpha}|^2 + |(T + 1 - k)^{-\alpha}|^2 + \frac{T a_n^p}{n}
\]

We have

\[
\Psi(w) = \sum_{k=1}^{T} F(k, w(k)) = \sum_{k=1}^{T} F(k, a_n).
\]

From the conditions Moreover, from \((A_1)\) one has \(\Phi(w_n) < r_n\). Therefore, for every \(n\) large enough, one has

\[
\varphi(r_n) = \inf_{u \in \Phi^{-1}(-\infty, r_n)} \frac{\sup_{u \in \Phi^{-1}(-\infty, r_n)} \Psi(u) - \Psi(u)}{r_n - \Phi(u)}
\]
\[
\leq \sup_{u \in \Phi^{-1}(-\infty, r_n)} \Psi(u) - \sum_{k=1}^{T} F(k, u(k))
\]
\[
\leq \frac{T \max_{|x| \leq b_n} F(k, x) - \sum_{k=1}^{T} F(k, a_n)}{(T + 1)^{p(p-2)/4} b_n^p - \left( \frac{a_n^2}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k)^{-\alpha}|^2 + \frac{T a_n^p}{n} \right)}.
\]

Hence, bearing in mind \((A_2)\), \(\gamma \leq \lim_{n \to \infty} \varphi(r_n) \leq A^\infty < \infty\) follows. Now, we verify that \(I_\lambda\) is unbounded from below. First, assume that \(B^\infty = \infty\). Accordingly, fix \(N\) such that \(N < \sum_{k=1}^{T} |(k)^{-\alpha}|^2 + \frac{T c_n^p}{n}\) and let \(\{c_n\}\) be a sequence of positive numbers, with \(\lim_{n \to \infty} c_n = \infty\), such that

\[
\sum_{k=1}^{T} F(k, c_n) > N, \quad \forall n \in \mathbb{N}.
\]

Thus, if we consider a sequence \(\{y_n\}\) in \(W\) defined by setting

\[y_n(k) = c_n, \quad \text{for all} \; k \in [1, T].\]

Thus \(y_n \in W\) and

\[
\Phi(y_n) = \frac{c_n^2}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k)^{-\alpha}|^2 + \frac{T c_n^p}{n}.
\]
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So

$$I_\lambda(y_n) = \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T c_p}{p} - \lambda \sum_{k=1}^{T} F(k, c_n)$$

$$< \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T c_p}{p} - \lambda N$$

that is, \(\lim_{n \to \infty} I_\lambda(y_n) = -\infty\). Next, assume that \(B^\infty < \infty\). Since \(\lambda > 1\), we can fix \(\epsilon > 0\) such that \(\epsilon < B^\infty - \frac{1}{\lambda}\). Therefore, also calling \(\{c_n\}\) a sequence of positive numbers such that \(\lim_{n \to \infty} c_n = \infty\) and

$$\sum_{k=1}^{T} F(k, c_n) > B^\infty - \epsilon, \quad \forall n \in \mathbb{N},$$

arguing as before and by choosing \(y_n \in W\) as above, one has

$$I_\lambda(y_n) = \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T c_p}{p} - \lambda \sum_{k=1}^{T} F(k, c_n)$$

$$< \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T c_p}{p} - \lambda (B^\infty - \epsilon).$$

So, \(\lim_{n \to \infty} I_\lambda(y_n) = -\infty\). Hence, in both cases \(I_\lambda\) is unbounded from below and the proof is complete. \(\square\)

**Remark 6** If \(\{a_n\}\) and \(\{b_n\}\) are two real sequences with \(\lim_{n \to \infty} b_n = \infty\), such that the assumption \((A_1)\) in Theorem 5 is satisfied. Then, under the conditions \(A^\infty = 0\) and \(B^\infty = \infty\), Theorem 5 assures that for every \(\lambda > 0\) the problem \((P_\lambda^f)\) admits infinitely many weak solutions.

**Theorem 6** Assume that

\[(A_3)\]

$$\liminf_{\xi \to \infty} \sum_{k=1}^{T} \max_{|x| \leq \xi} F(k, x) \frac{(T+1)^{p(p-2)}}{p} \xi^p < \limsup_{\xi \to \infty} \sum_{k=1}^{T} F(k, \xi) \frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T \xi^p}{p}.$$

Then, for each

$$\lambda \in \left(\frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T \xi^p}{p}, \frac{(T+1)^{p(p-2)}}{p} \frac{\xi^p}{\sum_{k=1}^{T} \max_{x \leq \xi} F(k, x)}\right),$$

the problem \((P_\lambda^f)\) has an unbounded sequence of weak solutions.
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Proof We choose the sequence \( \{b_n\} \) of positive numbers which goes to infinity and

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{T} \max_{|x| \leq b_n} F(k, x)}{(T+1) \left( \frac{p(p-2)}{p} b_n \right)^{p \left( \frac{p}{p-2} \right)}} = \liminf_{\xi \to \infty} \frac{\sum_{k=1}^{T} \max_{|x| \leq \xi} F(k, x)}{(T+1) \left( \frac{p(p-2)}{p} \xi \right)^{p \left( \frac{p}{p-2} \right)}}.
\]

Now, since \( \Phi(0) = \Psi(0) = 0 \), we can taking \( a_n = 0 \) for every \( n \in \mathbb{N} \) in (23), from Theorem 1 the conclusion follows.

Now, we give an application of Theorem 6.

Example 2 Let \( p = 4 \) and \( T = 3 \). We consider the following problem

\[
\begin{cases}
4 \nabla_\alpha^0 (k \nabla_\alpha^0 (u(k))) + k \nabla_\alpha^0 (4 \nabla_\alpha^0 (u(k))) + \varphi_4 (u(k)) = \lambda f(u(k)), \\
u(0) = u(T + 1) = 0
\end{cases}
\]

(23)

where \( \alpha = \frac{1}{2} \) and

\[
f(x) = \begin{cases}
0, & \text{if } x \in (-\infty, 0], \\
4x^3 + 4x^3 \sin(\pi e^x) + \pi x^4 e^x \cos(\pi e^x), & \text{if } x \in (0, \infty).
\end{cases}
\]

By the expression of \( f \), we have

\[
F(x) = \begin{cases}
0, & \text{if } x \in (-\infty, 0], \\
x^4 (1 + \sin(\pi e^x)), & \text{if } x \in (0, \infty).
\end{cases}
\]

Since

\[
\liminf_{\xi \to \infty} \frac{\sum_{k=1}^{3} \max_{|x| \leq \xi} F(x)}{64 \xi^4} = 0
\]

and

\[
\limsup_{\xi \to \infty} \frac{\sum_{k=1}^{3} F(\xi)}{(\Gamma(\frac{1}{2}))^2 \sum_{k=1}^{T} \frac{1}{2} |(k) - \frac{1}{2}|^2 + \frac{3\xi^4}{4}} = 8,
\]

we clearly see that all assumptions of Theorem 6 are satisfied, and then the problem (23) for every \( \lambda \in \left( \frac{1}{8}, \infty \right) \) has an unbounded sequence of weak solutions in \( \{u : [0, 4] \cap \xi \to \mathbb{R} : u(0) = u(4) = 0\} \).

Here, we point out two simple consequences of main results.

Corollary 7 Assume that there exist two real sequences \( \{a_n\} \) and \( \{b_n\} \) with \( \lim_{n \to \infty} b_n = \infty \), such that the assumption \((A_1)\) in Theorem 5 holds, \( A_\infty < 1 \) and \( B_\infty > 1 \). Then, the problem

\[
\begin{cases}
T+1 \nabla_\alpha^0 (k \nabla_\alpha^0 (u(k))) + k \nabla_\alpha^0 (T+1 \nabla_\alpha^0 (u(k))) + \varphi_p (u(k)) = f(k, u(k)), \\
u(0) = u(T + 1) = 0
\end{cases}
\]

(\( Pf \))

has an unbounded sequence of weak solutions.
Corollary 8 Assume that $B_\infty > 1$ and
\[
\liminf_{\xi \to \infty} \frac{\sum_{k=1}^{T} \max_{|x| \leq \xi} F(k, x)}{(T+1)^{\frac{p(p-2)}{4}} \xi^p} < 1.
\]
Then, the problem $(P_f)$ has an unbounded sequence of weak solutions.

We here give the following two consequences of the main result.

Corollary 9 Assume that there exist two real sequences $\{a_n\}$ and $\{b_n\}$ with $\lim_{n \to \infty} b_n = \infty$, such that the assumption $(A_1)$ in Theorem 5 holds, $f_1 \in C([1, T]_{N_0} \times \mathbb{R})$ and $F_1(t, x) = \int_{0}^{x} f_1(t, \xi) d\xi$ for all $(t, x) \in [0, T]_{N_0} \times \mathbb{R}$. Moreover, assume that
\[
(\sum_{k=1}^{T} \max_{|x| \leq b_n} F_1(k, x) - \sum_{k=1}^{T} F_1(k, a_n)) \frac{a_n^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T a_n^p}{p} < \infty;
\]

\[
(\sum_{k=1}^{T} F_1(k, \xi)) \frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k-\alpha)|^2 + \frac{T \xi^p}{p} = \infty.
\]

Then, for every function $f_i \in C([1, T]_{N_0} \times \mathbb{R})$, denoting $F_i(t, x) = \int_{0}^{x} f_i(t, \xi) d\xi$ for all $(t, x) \in [1, T]_{N_0} \times \mathbb{R}$ for $2 \leq i \leq n$, satisfying
\[
\max \left\{ \sup_{\xi \in \mathbb{R}} F_i(t, \xi); \; 2 \leq i \leq n \right\} \leq 0
\]
and
\[
\min \left\{ \liminf_{\xi \to \infty} \frac{F_i(t, \xi)}{\xi^2}; \; 2 \leq i \leq n \right\} > -\infty,
\]
for each
20  

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the problem

\[
\begin{aligned}
& T + 1 \nabla_k^\alpha (k \nabla_0^\alpha (u(k))) + k \nabla_0^\alpha (T + 1 \nabla_k^\alpha (u(k))) + \varphi_p(u(k)) = \lambda f_i(k, u(k)), \\
& u(0) = u(T + 1) = 0
\end{aligned}
\]  \tag{24}

has an unbounded sequence of weak solutions.

**Proof** Set \( F(t, \xi) = \sum_{i=1}^{n} F_i(t, \xi) \) for all \((t, \xi) \in [1, T]_{\mathbb{N}_0} \times \mathbb{R} \). Assumption \((A_4)\) along with the condition

\[
\min \left\{ \liminf_{\xi \to +\infty} \frac{F_i(t, \xi)}{\xi^2}; \quad 2 \leq i \leq n \right\} > -\infty
\]

ensures

\[
\limsup_{\xi \to \infty} \frac{T}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k - \alpha)|^2 + T \xi^p / p = \limsup_{\xi \to \infty} \frac{T}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k - \alpha)|^2 + T \xi^p / p = \infty.
\]

Moreover, Assumption \((A_4)\) together with the condition

\[
\max \left\{ \sup_{\xi \in \mathbb{R}} F_i(t, \xi); \quad 2 \leq i \leq n \right\} \leq 0,
\]

implies

\[
\lim_{n \to \infty} \frac{T}{(T + 1)^{\frac{p(p - 2)}{4}} b_n^p} - \left( \frac{\sigma_n^2}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k - \alpha)|^2 + T \sigma_n^p / p \right) \leq \lim_{n \to \infty} \frac{T}{(T + 1)^{\frac{p(p - 2)}{4}} b_n^p} - \left( \frac{\sigma_n^2}{(\Gamma(1 - \alpha))^2} \sum_{k=1}^{T} |(k - \alpha)|^2 + T \sigma_n^p / p \right) < \infty.
\]

Hence, the conclusion follows from Theorem 5. \( \square \)

**Corollary 10** Let \( f_1 \in C([1, T]_{\mathbb{N}_0} \times \mathbb{R}, \mathbb{R}) \) and let \( F_1(t, x) = \int_{0}^{x} f_1(t, \xi) d\xi \) for all \((t, x) \in [1, T]_{\mathbb{N}_0} \times \mathbb{R} \). Assume that

\[
\liminf_{\xi \to \infty} \frac{T}{(T + 1)^{\frac{p(p - 2)}{4}} \xi^p} \sum_{k=1}^{T} \max_{|x| \leq \sigma(\xi)} F_1(k, x) < \infty
\]
and
\[
\limsup_{\xi \to \infty} \sum_{k=1}^{T} F_1(k, \xi) = \infty.
\]

Then, for every function \( f_i \in C([1, T]_{\mathbb{N}_0} \times \mathbb{R}, \mathbb{R}) \), denoting \( F_i(t, x) = \int_0^x f_i(t, \xi) d\xi \) for all \((t, x) \in [1, T]_{\mathbb{N}_0} \times \mathbb{R}\) for \(2 \leq i \leq n\), satisfying
\[
\max \left\{ \sup_{\xi \in \mathbb{R}} F_i(t, \xi); \ 2 \leq i \leq n \right\} \leq 0
\]
and
\[
\min \left\{ \liminf_{\xi \to \infty} \frac{F_i(t, \xi)}{\xi^p}; \ 2 \leq i \leq n \right\} > -\infty,
\]
for each
\[
\lambda \in \left\{ \begin{array}{c}
1 \\
\lim_{n \to \infty} \sum_{k=1}^{T} \max_{|x| \leq \xi} F(k, x) \\
\lim \inf_{\xi \to \infty} \frac{\sum_{k=1}^{T} |(k - \alpha)|^2 + T\xi^p}{(T+1)^{\frac{p(p-2)}{4}}\xi^p}
\end{array} \right\},
\]
the problem (24) has an unbounded sequence of weak solutions.

Now put
\[
\mathcal{B}^0 = \limsup_{\xi \to 0} \sum_{k=1}^{T} F(k, \xi)
\]

Arguing as in the proof of Theorem 5, but using conclusion (c) of Theorem 1 instead of (b), one establishes the following result.

**Theorem 11** Assume that there exist two real sequences \( \{d_n\} \) and \( \{e_n\} \) with \( \lim_{n \to \infty} e_n = 0 \) such that
\[
(A_6) \quad \frac{d_n^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k - \alpha)|^2 + \frac{T d_n^p}{p} < \frac{(T+1)^{\frac{p(p-2)}{4}}}{p} e_n^p \text{ for every } n \in \mathbb{N}
\]
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\[(A_7)\]

\[A^0 = \lim_{n \to \infty} \frac{T}{(T+1)^{\frac{p(p-2)}{4}} \varepsilon_n^2} \sum_{k=1}^{T} \max_{|x| \leq e_n} F(k, x) - \sum_{k=1}^{T} F(k, d_n) < B^0.\]

Then, for each \(\lambda \in (\lambda_3, \lambda_4)\) with \(\lambda_3 := \frac{1}{B^0}\) and \(\lambda_4 := \frac{1}{A_0}\), the problem \((P_{\lambda}^f)\) has a sequence of pairwise distinct weak solutions which strongly converges to 0 in \(W\).

**Theorem 12** Assume that

\[(A_8)\]

\[\lim_{\xi \to 0^+} \frac{\sum_{k=1}^{T} \max_{|x| \leq \xi} F(k, x)}{(T+1)^{\frac{p(p-2)}{4}} \xi^p} < \lim_{\xi \to 0^+} \frac{\sum_{k=1}^{T} F(k, \xi)}{(\Gamma(1-\alpha))^2 \sum_{k=1}^{T} |(k)^{-\alpha}|^2 + T\xi^p}.\]

Then, for each

\[\lambda \in \left\{ \begin{array}{c} 1 \sum_{k=1}^{T} F(k, \xi) \frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k)^{-\alpha}|^2 + T\xi^p \frac{\xi^2}{(T+1)^{\frac{p(p-2)}{4}} \xi^p} \end{array} \right\},\]

the problem \((P_{\lambda}^f)\) has a sequence of pairwise distinct weak solutions which strongly converges to 0 in \(W\).

**Remark 7** Applying Theorem 11, results similar to Remark 6 and Corollaries 7, 8, 9 and 10 can be obtained.

** Declarations**

- **Funding**
  Not applicable.

- **Conflict of interest/Competing interests**
  The authors declare that they have no conflict of interest.
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- Consent to participate
  Not applicable.

- Consent for publication
  Not applicable.

- Availability of data and materials
  Not applicable.

- Code availability
  Not applicable.

- Authors’ contributions
  All authors contributed equally to this work.

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