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Investigation on reachable domain of contingency return trajectories in the circumlunar flight phase based on interval analysis

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Abstract: The investigation on reachable domain based on interval analysis is performed, which is aimed for the contingency return trajectories during the circumlunar flight stage for manned lunar landing engineering. First, the issue of contingency return trajectory which adopts a non-coplanar maneuver during the circumlunar flight phase is described. The mathematics model of the reachable domain of contingency return trajectories is established. Second, the contingency return trajectory is designed. An improved interval branch and bound algorithm is introduced and a reachable domain calculation method of the contingency return trajectories is proposed. Third, in numerical simulations, by comparing the reachable domain obtained by the traditional brute force method, the accuracy of the method presented in this paper is verified. It also indicates that this method has a higher calculation efficiency. Finally, abundant simulations are performed to discuss the parameter characteristic of reachable domain by adopting this method. The study conclusions can serve as an important reference for the selection and decision of contingency return trajectory plans in the future manned lunar landing engineering.

Key words: Manned lunar landing engineering; Contingency return trajectory; Interval analysis; Reachable domain; Parameter characteristic

1. Introduction

At the beginning of 21\textsuperscript{st} century, NASA presented the lunar exploration engineering goals of “global lunar access” and “anytime return” (Garn et al. 2008). In the last few years, the United States
positively conducts the “Artemis” program and plans to land on the lunar south pole in 2024 (Smith et al. 2020). The successful Chang’E-5 engineering means that the final stage of China’s unmanned lunar exploration missions has been successfully conducted (Qian et al. 2021; Zhou et al. 2021). China also states that the manned lunar landing missions is the next goal. Manned lunar landing is a huge systematic engineering and contains high hazard. During the entire course of conducting the engineering, if an urgent situation which endangers the astronauts’ safety happens, the contingency rescue plans which are made in advance can be used by the astronauts to ensure their safety. Because the trajectory design is one of the top-level design links of the engineering, the contingency return trajectory is a key component of contingency rescue plans. Accordingly, the study on the contingency trajectory issue has a great importance to the plan and construction of the missions.

With regard to the contingency trajectory in lunar exploration engineering, many scholars have performed some studies. Hyle et al. concluded the abort plan problems of the Apollo mission in the different phases (Hyle et al. 1970). Foggatt discussed an issue of contingency return by utilizing a service propulsion system in the Earth-Moon flight phase (Foggatt 1966). In terms of a non-free return flight, Babb investigated a return trajectory issue in emergency during the process of lunar orbit insertion (Babb 1969). Braud studied a contingency return issue about a non-coplanar circumlunar trajectory and analyzed the reentry terminal element characteristic (Braud 1962). Anselmo and Baker investigated the contingency issues during the two flight stages (Anselmo and Baker 1970). These researches promoted the manned lunar landing missions in the last century, but only several simple discussions and conclusions were given and concrete trajectory calculation methods were not demonstrated. In terms of the crew exploration vehicle (CEV), Beksinski studied the contingency return trajectories for manned lunar landing engineering and compared it to that for Apollo engineering (Beksinski 2007). Williams and Condon explored the return trajectories in emergency during each phase for asteroid redirection manned missions (Williams and Condon 2014). In terms of the lunar atmosphere & dust environment explorer (LADEE) mission, Genova planned a contingency trajectory when the failure happens in the lunar orbit insertion process (Genova 2014). Jesick solved the contingency trajectory returning from a halo orbit (Jesick 2013). Shen and Casalino adopted the indirect method to determine a contingency return trajectory during the lunar parking stage (Shen and Casalino 2014). Peng et al. proposed the calculation model of
lunar descending trajectory considering contingency constraints, and presented a design algorithm on the basis of Gauss pseudo-spectral method (Peng et al. 2010). Huang et al. used a two-body trajectory dynamic model to discuss the performances of contingency return trajectories during the Earth-Moon flight stage and the lunar orbit insertion stage (Huang et al. 2010a; Huang et al. 2010b). Lu et al. designed and analyzed the contingency trajectories in the Earth-Moon flight stage and Moon-Earth flight stage (Lu et al. 2022a; Lu et al. 2022b). Ocampo and Saudememont presented an initial value estimate method to determine a contingency trajectory (Ocampo and Saudememont 2010).

It can be seen from these investigations related to the contingency trajectory that they are devoted to design and computation of a single contingency trajectory. It belongs to the field of point design issue, while analyzing the parameter space of contingency trajectories belongs to the field of interval design issue. By studying the interval design issue of the contingency trajectory, it is beneficial to reveal some general characteristics and rules, which can provide a comprehensive reference basis for mission decision. In recent years, reachable domain which is extended to the astrodynamics field is a classical interval design issue. In terms of the spacecraft near the Earth, reachable domain is defined as the space position set which the spacecraft can reach when the constraints are satisfied (Wen et al. 2014). As an intuitive concept to describe the reachable position of the spacecraft, some research progress has been made in the reachable domain with a single coplanar impulse considering the target-visit constraint (Xia et al. 2022), reachable domain after gravity assists of planets in elliptical orbits (Chen et al. 2020) and space debris removal on the basis of reachable domain (Liu et al. 2021). For lunar exploration missions, unlike the near-Earth trajectory, reachable domain of an Earth-Moon transfer trajectory is usually featured as the lunar surface reachable region, that is, the lunar surface which can be reached by the spacecraft under certain trajectory constraints, such as the longitude and latitude distribution of lunar landing points (Orloff 2000). However, the more generalized concept of the lunar surface reachable domain also refers to the distribution of feature state parameters of an Earth-Moon transfer trajectory in the high-dimensional space (He 2017). Similarly, the reachable domain of a Moon-Earth return trajectory can be defined as the distribution of the terminal feature state parameters of the Moon-Earth return trajectories in the high-dimensional space. In terms of the reachable domain issue of the trajectory in the Earth-Moon space,
some research progress has been made. Peng studied the parameter characteristic of the hybrid trajectories and the extensive ability of the lunar reachable domain boundary (Peng 2012). He et al. adopts the brute force method to calculate the reachable domain of different types of free return trajectories (He et al. 2017). Lu et al. revealed the reachable domain feature of the direct transfer trajectories from the near rectilinear halo orbits to the low lunar orbits (Lu et al. 2021).

In the light of the aforementioned research, it is drawn that some progress has been made on the reachable domain of the trajectories near Earth and the trajectories in Earth-Moon space. However, these studies which belong to interval design issue are performed for nominal trajectories, while there are few researches devoted to the interval design issue of contingency return trajectories. Thus, intended for the above deficiencies, this paper performs the investigation on the reachable domain issue of the contingency return trajectories during the circumlunar flight stage based on interval analysis. The relevant methods and theories of interval analysis have been applied widely in the fields of structural reliability analysis (Song and Zhang 2021), workspace determination of parallel robots (Jin et al. 2022) and stress assessment (Kumar et al. 2021), et al. In addition, interval analysis also starts to be applied in the aerospace field, such as determination of the nonlinear aircraft trim points (Kampen et al. 2007), integer ambiguity resolution for aircraft attitude determination (Kampen et al. 2009; Weerdt et al. 2008), optimization of Lambert rendezvous (Chen et al, 2013), robust optimal attitude determination of the satellites (Hossein et al. 2022) and fuel optimization for constrained spacecraft formation rotations (Weerdt et al. 2009), et al.

The main contributions of this investigation are as follows: Firstly, a mathematics model of reachable domain of contingency return trajectories is presented. Secondly, a non-coplanar contingency return trajectory during the circumlunar flight stage for manned lunar landing engineering is designed. On the basis of it, a reachable domain calculation method of the contingency return trajectories is proposed by introducing the improved interval branch and bound algorithm. Thirdly, the parameter characteristic of reachable domain is analyzed by adopting this method proposed in this paper, which reveals the general rule of contingency return trajectories. This paper includes five sections, containing the introduction. The contingency return trajectory is demonstrated and the mathematics model of reachable domain is established in Section 2. Section 3 is devoted to the reachable domain calculation method of the contingency return trajectories based
on interval analysis. By performing the numerical simulation, the accuracy and advantage of this method are verified and the parameter characteristic is analyzed in Section 4. Section 5 draws the brief conclusions of this investigation.

2. Problem statement

2.1. Contingency return trajectory statement

In the manned lunar landing engineering, circumlunar flight phase is one of vital phases in the entire mission. Before landing on the Moon, the spacecraft usually needs to park in the low lunar orbit and wait for the suitable moment to perform the lunar descending. After ascending from the Moon surface, the spacecraft also needs to park in the low lunar orbit and wait for the suitable moment to return to the Earth. Thus, the spacecraft once fails or other contingencies happen in the circumlunar flight phase, in order to ensure the astronauts’ safety, it is necessary to abort the nominal mission to return to the Earth. Because the distance between the spacecraft and the Earth is long, it is not beneficial to apply multi maneuvers in the urgent condition and a single maneuver can be applied to insect a contingency return trajectory. The single maneuver can adopt the coplanar or the non-coplanar way. Compared to the coplanar maneuver, the non-coplanar maneuver can result in more return window and satisfy the contingency requirement of anytime return. The study in this paper is focused on a contingency return trajectory which adopts the non-coplanar maneuver.

The entire flight process of the non-coplanar contingency return trajectory during the circumlunar flight phase is described as follows: When the spacecraft fails or other urgent cases happen, a non-coplanar maneuver is applied on the low lunar orbit and the spacecraft enters a contingency return trajectory to return to the boundary of the atmosphere. It finally lands on the Earth after the friction deceleration after the long-range reentry, as shown in Fig. 1.

2.2. Problem statement of reachable domain
In a continuous dynamic system as described in Eq. (1), \( t \in [t_0, t_f] \). When the initial state \( \Gamma(t_0) \in \Pi^n \) at the initial time \( t_0 \) is provided, if there is a control function \( u(t) \in U^n \) to acquire the final state \( \Gamma(t_f) \in \Theta^n \) at the end time \( t_f \), the set \( \Theta^n \) will be defined as reachable domain corresponding to \( \Pi^n \) (Grantham 1981).

\[
\begin{align*}
\dot{\Gamma}(t) &= f(t, \Gamma(t), u(t)) \\
\Gamma(t_0) &\in \Pi^n, u(t) \in U^n, \Gamma(t_f) \in \Theta^n \\
f(t) : \Pi^n \times U^n &\rightarrow \Pi^n
\end{align*}
\]  

(1)

where \( \dot{\ } \) denotes differentiation with respect to time.

The reachable domain is yet not straightforward and not each parameter is focused on at times. Only some parameters require attention and then the parameter function \( \delta(t) = c(t, \Gamma(t)) \) needs to be established. The corresponding reachable domain \( \Theta^k \) needs to be obtained.

\[
\begin{align*}
\delta(t) &= c(t, \Gamma(t)) \\
\delta(t_0) &\in \Theta^k \subseteq \Theta^n \\
c(t) : \Pi^n &\rightarrow \Theta^k
\end{align*}
\]  

(2)

As described in Eq. (3), if several inequality constraints exist during the control course and several equality constraints exist in the boundaries, \( \Theta^k \) will get smaller, undergoing a reduced-dimension, or even difficult to meet with the constraint conditions and grow equal to a null set.

\[
\begin{align*}
g(t, \Gamma(t), u(t)) &\leq 0 \\
x(t, \Gamma(t)) &= 0
\end{align*}
\]  

(3)

Thus, reachable domain of the contingency return trajectories is a multi-dimensional set consisted of the reentry point parameters. Fig. 2 shows the schematic diagram.
Fig. 2. Schematic diagram of the reachable domain of the contingency return trajectories

The initial state $\Pi_0^*$ of a contingency return trajectory can be expressed as

$$\Pi_0^* = \left\{ \Gamma(t_0) : \text{s.t. } h_0 = h_{\text{LLO}}, e_0 = e_{\text{LLO}}, i_0 = i_{\text{LLO}}, \right. \\
\left. \Omega_0 = \Omega_{\text{LLO}}, \omega_0 = \omega_{\text{LLO}}, f_0 = f_{\text{LLO}} \right\}$$

(4)

where s.t. denotes being constrained to several conditions, $h_{\text{LLO}}$ represents the altitude of low lunar orbit, $e_{\text{LLO}}$ represents the eccentricity of low lunar orbit, $i_{\text{LLO}}$ represents the orbital inclination of low lunar orbit, $\Omega_{\text{LLO}}$ represents the right ascension of the ascending node (RAAN) of low lunar orbit, $\omega_{\text{LLO}}$ represents the argument of perilune of low lunar orbit, $f_{\text{LLO}}$ represents the true anomaly of departure point.

The reachable domain of the contingency return trajectories can be expressed as

$$\Theta^* = \left\{ \Gamma(t_0) : \exists t_0, \text{s.t. } h_0 = h_{\text{RP}}, \right. \\
\left. \gamma_t = \gamma_{\text{RP}}, (t_1 - t_0) \leq T_{\text{max}} \right\}$$

(5)

where $h_{\text{RP}}$ represents the reentry point altitude, $\gamma_{\text{RP}}$ represents the reentry angle, $T_{\text{max}}$ represents the allowable maximum return flight time.

In actual missions, some parameters are generally paid attention to, such as the orbital inclination, RAAN, and the longitude and the latitude of the reentry point. Their reachable domain can be expressed as
$$\Theta_k = \{(i_{RP}, \Omega_{RP}, \lambda_{RP}, \phi_{RP}) : \forall \Gamma(t_r) \in I_{n}^u \} \quad (6)$$

where $i_{RP}$ denotes the orbital inclination of reentry point, $\Omega_{RP}$ denotes the RAAN of reentry point, $\lambda_{RP}$ represents the longitude of reentry point, and $\phi_{RP}$ denotes the latitude of reentry point.

3. Reachable domain calculation method of the contingency return trajectories based on interval analysis

The basic concept of interval analysis is presented firstly. Then a contingency return trajectory is designed. The improved interval branch and bound algorithm is used to further propose a reachable domain calculation method of the contingency return trajectories.

3.1. Basic concept of interval analysis

Interval analysis is the theory involved to interval numbers and interval arithmetic operations (Hickey et al. 2011; Moore 1966). An interval number $[I]$ is defined as the set of all real numbers $I$ which are between a lower bound $L_I$ and an upper bound $U_I$:

$$[I] = [I^L, I^U] = \{ I \in \mathbb{R} | I^L \leq I \leq I^U \} \quad (7)$$

If $I^L = I^U$, then $[I]$ is called a point interval. If $I^L > 0$, then $[I]$ is positive. If $I^U < 0$, then $[I]$ is negative. If $I^L = -I^U$, $[I]$ will be symmetric. The width or diameter of $[I]$ is denoted as

$$w([I]) = I^U - I^L \quad (8)$$

The midpoint of $[I]$ can be computed by

$$\text{mid}([I]) = \frac{I^U - I^L}{2} \quad (9)$$

Operations in the ordinary number system also can be expanded to cover the interval numbers, such as the basic calculation operations of addition, subtraction, multiplication and division:

$$[I] + [J] = [I^L + J^L, I^U + J^U]$$
$$[I] - [J] = [I^L - J^U, I^U - J^L]$$
$$\left[\frac{I}{J}\right] = \left[\frac{I^L}{J^U}, \frac{I^U}{J^L}\right], 0 \notin [J]$$

The more generic form of these equations is described as
\[ [I] \oplus [J] = \{I' \oplus J' | I' \in [I], J' \in [J]\} \quad (11) \]

where \( \oplus \) denotes +, −, × and /. When \( \oplus = /, 0 \notin [J] \).

In addition, the interval arithmetic operations include basic index function, trigonometric function, interval number intersection, function integration and differentiation, et al. (Moore et al. 2009). Regarding these basic arithmetic operations, an interval function which expresses a real function with interval numbers can be computed by

\[ F([I_1], \ldots, [I_n]) = \Xi([I_1], \ldots, [I_n]) \quad (12) \]

where \( \Xi(\cdot) \) represents the interval arithmetic operations for an interval function with interval numbers \([I_1, \ldots, I_n]\).

Just as ordinary numbers can be grouped in vectors, interval numbers also can be stacked in an interval vector, which is generally called a box. Two or more subintervals can be obtained by splitting an interval. For interval vectors, these subdivisions are defined as sub-boxes.

3.2. Contingency return trajectory design

Because the trajectory computation in the high-fidelity dynamics model consumes longer time, it is not suitable for extensive calculations. The simplified trajectory dynamics model has higher calculation efficiency and can indicate the general characteristic of high-fidelity trajectories, so the contingency return trajectory design is performed in the simplified trajectory dynamics model in this investigation.

The spacecraft is assumed to apply the contingency maneuver at the time \( t_0 \) to enter the contingency return trajectory. The initial position and the initial velocity of the spacecraft are denoted by \( r_0 \) and \( v_0 \) in the lunar J2000 coordinate system. Once the longitude \( \lambda_{ex} \) of the exit point and the latitude \( \phi_{ex} \) of the exit point are known, the position of the exit point in lunar fixed coordinate system can be obtained by

\[ r_{ex}^{LF} = \begin{bmatrix} \rho_L \cos \phi_{ex} \cos \lambda_{ex} \\ \rho_L \cos \phi_{ex} \sin \lambda_{ex} \\ \rho_L \sin \phi_{ex} \end{bmatrix} \quad (13) \]

where \( LF \) represents lunar fixed coordinate system and \( \rho_L \) represents the radius of lunar sphere.
of influence (LSI).

Given the transfer time $\Delta t_i$ from the contingency departure point to the exit point of LSI, the position of the exit point in the lunar J2000 coordinate system is expressed as

$$r_{\text{ex}}^{\text{LJ}} = M(t_{\text{ex}})r_{\text{ex}}^{\text{LF}}$$  \hspace{1cm} (14)

where $\text{LJ}$ represents lunar J2000 coordinate system, $t_{\text{ex}}$ represents the moment at the exit point, $M(t_{\text{ex}})$ represents the coordinate transformation matrix from the lunar fixed coordinate system to the lunar J2000 coordinate system at the moment $t_{\text{ex}}$.

Because $r_0$, $r_{\text{ex}}^{\text{LJ}}$ and $\Delta t_i$ are given, the selenocentric segment trajectory is determined by resolving the Lambert problem. Now many calculation algorithms about the Lambert problem can be adopted, and the concrete calculation process isn’t duplicated here. This study uses the Lambert method proposed in the reference (Izzo 2015) to obtain the initial velocity $v_{\text{ex}0}$ and the final velocity $v_{\text{ex}}$. The contingency maneuver is obtained by

$$\Delta V = v_{\text{ex}0} - v_0$$  \hspace{1cm} (15)

According to $t_{\text{ex}}$, $r_{\text{ex}}^{\text{LJ}}$ and $v_{\text{ex}}$, the position and the velocity of the exit point in Earth J2000 coordinate system are expressed as

$$r_{\text{ex}}^{\text{EJ}} = r_{\text{ex}}^{\text{LJ}} + r_{\text{M}}^{\text{EJ}}$$  \hspace{1cm} (16)

$$v_{\text{ex}}^{\text{EJ}} = v_{\text{ex}} + v_{\text{M}}^{\text{EJ}}$$  \hspace{1cm} (17)

where $r_{\text{M}}^{\text{EJ}}$ and $v_{\text{M}}^{\text{EJ}}$ denote the position and the velocity of the Moon in Earth J2000 coordinate system, which are calculated from JPL DE430 ephemeris.

Accordingly, the geocentric segment trajectory elements can be acquired by transformation. The transfer time between the exit point and the reentry point is calculated by

$$\Delta t_2 = \frac{a_{\text{gs}}}{\mu_{\text{E}}} \left( E - e_{\text{gs}} \sinh E \right)$$  \hspace{1cm} (18)

where $\mu_{\text{E}}$ denotes the Earth gravitational parameter, $a_{\text{gs}}$ denotes the semi-major axis of geocentric segment trajectory, $e_{\text{gs}}$ denotes the orbital eccentricity of geocentric segment trajectory, $E$ denotes the eccentrical anomaly of reentry point which is generated by its true anomaly.
This true anomaly can be expressed as

\[ f_{RP} = \begin{cases} 
2\pi - \arccos \left( \frac{1}{e_{RP}} \left( \frac{r_{up}}{r_{RP}} (1 + e_{RP}) - 1 \right) \right), & r_{up} \leq r_{RP} \\
2\pi, & r_{up} > r_{RP}
\end{cases} \]  \hspace{1cm} (19)

where \( r_{up} \) denotes the radius of vacuum perigee, and \( r_{RP} \) denotes the radius of reentry point. The orbital elements of reentry point are acquired accordingly.

In terms of the design of a contingency return trajectory, design variables are the latitude and longitude of LSI exit point and the transfer time of the selenocentric segment. It can be seen from the above discussion that if a group of design variables is given, an entire trajectory will be generated.

\[ X_i = (\lambda_{ex}, \phi_{ex}, \Delta t_i) \]  \hspace{1cm} (20)

In the urgent case, the fuel may be insufficient, so it is often desired that less fuel is consumed in the contingency maneuver process. Thus, the objective function of the contingency return trajectory design is generally desired to minimum the magnitude of maneuver, as described in Eq. (21).

\[ J = (\| \Delta V \|)_{\text{min}} \]  \hspace{1cm} (21)

This paper adopts the SQP_Snopt algorithm to optimize and solve a contingency return trajectory design problem. The SQP_Snopt algorithm is an improved SQP algorithm (Philip et al. 2002), which is usually used to resolve the smooth nonlinear programming problems.

### 3.3. Reachable domain calculation method of the contingency return trajectories

On the basis of the trajectory design model in the above subsection, this subsection proposes a reachable domain calculation method by introducing an improved interval branch and bound algorithm.

The interval branch and bound algorithm generally contains box splitting, box eliminating and box contracting (Ma and Xu 2015). Firstly, the box is split into several sub-boxes according to the search space of all interval variables. Then, the sub-boxes which satisfy the eliminating condition are eliminated. Finally, the sub-boxes which are not eliminated are further contracted. By sufficiently splitting, eliminating and contracting, an optimal solution can be obtained. This paper improves the interval branch and bound algorithm for the reachable domain calculation of contingency return trajectories, which is intended to obtain all the solutions satisfying the constraints.
rather than to obtain an optimal solution. The specific process of an improved interval branch and
bound algorithm is as follows:

(a) First, perform box splitting. In terms of the interval function \( F \) with the interval variables
\([I_1], \ldots, [I_k], \ldots, [I_n]\), the search space of the interval variable \([I_k]\) is divided into \( l \) sub-intervals.

The interval function \( F \) is expressed firstly as

\[
\begin{align*}
F([I_1], \ldots, [I_k], \ldots, [I_n]) &= \Xi([I_1], \ldots, [I_k], \ldots, [I_n]) \\
& \quad \ldots \\
F([I_1], \ldots, [I_n], \ldots, [I_n]) &= \Xi([I_1], \ldots, [I_n], \ldots, [I_n]) \\
& \quad \ldots \\
F([I_1], \ldots, [I_n], \ldots, [I_n]) &= \Xi([I_1], \ldots, [I_n], \ldots, [I_n])
\end{align*}
\]  

(22)

The search spaces of other interval variables are also divided into the corresponding sub-intervals.

Assuming that each interval variable is divided into \( l \) sub-intervals, all the interval variables
\([I_1], \ldots, [I_k], \ldots, [I_n]\) are split as

\[
[I_k] = [I_{m_1}] \times \cdots \times [I_{m_m}] \times \cdots \times [I_{m_n}]
\]  

(23)

where the subscript \( m_i \) denotes any value of \( 1 \to l \) expressed in Eq. (23). \( l^n \) sub-boxes can be
acquired and then \( l^n \) interval functions can be acquired after performing box splitting.

(b) Second, performing box eliminating. The interval equation of an interval function with
interval variables is expressed as

\[
F([I_1], \ldots, [I_k], \ldots, [I_n]) = \Xi([I_1], \ldots, [I_k], \ldots, [I_n]) = 0
\]  

(24)

Regarding each split sub-box \([I_k]\) in Eq. (23), if the interval output of an interval function
\(F([I_k])\) does not include the element 0, which indicates the interval equation has no solution, the
split sub-box \([I_k]\) will be deleted safely. Otherwise, the split sub-box \([I_k]\) will be retained to
perform box contracting.

(c) Finally, performing box contracting. If the interval equation with the split sub-box \([I_k]\) has
a solution, the split sub-box will be contracted further to determine all the solutions in this split sub-
box. The classical midpoint contracting method is used as the contracting method. Two sub interval
variables are further obtained by dividing each interval variable of this split sub-box with its
midpoint as the boundary, and several sub-boxes are acquired after performing box contracting. Therefore, on the basis of the contingency return trajectory design, the reachable domain can be calculated by introducing the improved interval branch and bound algorithm. The specific steps of the calculation method are described as follows:

**Step (1):** among the design variables, $\lambda_{\alpha}$ and $\phi_{\alpha}$ are selected as the interval variables $[\lambda_{\alpha}]$ and $[\phi_{\alpha}]$, and the transfer time $\Delta t_i$ from the contingency departure point to the LSI exit point is selected as the optimization variable. According to the search space of the interval variables, the interval variables $[\lambda_{\alpha}]$ and $[\phi_{\alpha}]$ are split into $p$ sub-intervals and $q$ sub-intervals. Then several sub-boxes can be obtained.

**Step (2):** The contingency return trajectory is generated with the lower bound and the upper bound of interval variables in each split sub-box. If the lower bound and the upper bound of interval variables in a split sub-box both have no solution, this split sub-box will be deleted, or the next step is performed.

**Step (3):** The split sub-box which exists the solution is performed box contracting to obtain several sub-boxes.

**Step (4):** Whether the bound of interval variables in a contracted sub-box existing a solution is judged. If the solution does not exist, **Step (2)** will be performed and the box eliminating will be conducted. If the solution exists, **Step (3)** will be performed and the box contracting will be conducted further until the minimum interval width of a sub-box is satisfied.

**Step (5):** After all the sub-boxes in search space are dealt, all the solutions of the contingency return trajectories can be acquired and then the reachable domain is obtained.

In conclusion, the whole flow diagram of the reachable domain calculation method of contingency return trajectories is depicted in Fig. 3.
4. Simulations and analysis

In this section, by adopting numerical simulation method, the reachable domain calculation method of the contingency return trajectories is verified and the parameter characteristic of reachable domain is analyzed.

4.1. Example

Firstly, an example is adopted to verify the effectiveness and accuracy of reachable domain
calculation method. The parameters of this simulation example are set as follows: the contingency return time is 2028-02-17 00:00:00. In the lunar J2000 coordinate system, the altitude of low lunar orbit is 100 km, the eccentricity is 0, the orbital inclination is 156°, RAAN is 194°, the argument of perilune and the true anomaly is 0°. The allowable maximum contingency return flight time is 3 days. The reentry point altitude is 122 km and the reentry angle is -6°. In addition, the search space of the longitude of exit point of LSI is -180° ~ 180°, and the search space of the latitude of LSI exit point is -90° ~ 90°. When adopting the reachable domain calculation method proposed in this investigation, the interval variables $[\lambda_{ex}]$ and $[\phi_{ex}]$ are split into several sub-intervals, and the minimum interval width of sub-interval contracting is set to be different values. In order to verify the accuracy of the method proposed in this investigation, the brute force method is used to calculate the reachable domain of the contingency return trajectories. When using the brute force method, the longitude and latitude of exit point of LSI are traversal searched in a stepsize of 1°. Table 1 presents the computation time of the brute force method and the calculation times of different configurations of the proposed method. The simulation calculations are conducted on an Intel Core i5-7500 CPU 3.40GHz computer running a code in C++ language.

<table>
<thead>
<tr>
<th>Parameter configuration of the proposed method</th>
<th>Calculation time of the proposed method/(s)</th>
<th>Calculation time of the brute force method/(s)</th>
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<td>20 sub-intervals and the minimum interval width of sub-interval contracting of 1°</td>
<td>788.279</td>
<td></td>
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<tr>
<td>40 sub-intervals and the minimum interval width of sub-interval contracting of 1°</td>
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<td>40 sub-intervals and the minimum interval width of sub-interval contracting of 0.5°</td>
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</tbody>
</table>
It can be drawn from the table that the calculation time of the brute force method is 20324.349 s. When adopting the proposed method, the calculation time is 788.279 s by splitting the interval variables into 20 sub-intervals. The calculation time is about 2400 s by splitting the interval variables into 40 sub-intervals. The calculation time is 8908.101 s by splitting the interval variables into 80 sub-intervals. It can be concluded that, under different parameter configurations, compared to the brute force method, the proposed method can reduce the calculation time by at least 56% and at most 96%. The comparison reflects the proposed method can greatly improve the calculation efficiency of the reachable domain of contingency return trajectories.

Fig. 4(a) depicts the reachable domain of the contingency return trajectories obtained by the brute force method, and Fig. 4(b)~(f) show the reachable domain obtained by the proposed method with different parameter configurations. It can be seen from the figure that the topological feature of the reachable domain obtained by the proposed method is basically consistent with that obtained by the brute force method. Both of them are characterized by “arch” with the opening to the left. Thus, it verifies the accuracy of the method proposed in this investigation. Combined with the calculation time under different parameter configurations and the similarity of the topological feature of the reachable domain, when analyzing the characteristic of reachable domain subsequently, the parameter configuration is that the interval variables are split into 40 sub-intervals and the minimum interval width of sub-interval contracting is $1^\circ$.
Second, analysis of reachable domain characteristic is performed. Considering contingency return from the low lunar orbits with different RAANs, RAAN is set to be $0^\circ$, $90^\circ$, $180^\circ$ and $270^\circ$ , and other parameters are set to be the same as that in the above subsection. By adopting the method in this paper, reachable domains of the contingency return trajectories departing from different right ascensions of the ascending node are shown in Fig. 5. It is concluded from the figure that reachable domains basically coincide. The RAAN of the low lunar orbit has no effect on the reachable domain of the contingency return trajectories.
Fig. 5. Reachable domain of the contingency return trajectories departing from the different RAANs

Fig. 6 depicts the contingency maneuver magnitude distribution within the reachable domain of departing from the different RAANs. It can be drawn from the figure that the distribution features of the contingency maneuver magnitude within the reachable domain of departing from the different RAANs are different. The contingency maneuver magnitude is small as a whole when the RAAN is $180^\circ$, while the contingency maneuver magnitude is large as a whole when the RAAN is $90^\circ$. 

(a) RAAN of $0^\circ$  
(b) RAAN of $90^\circ$
The contingency return flight time distribution within the reachable domain of departing from the different RAANs is described in Fig. 7. It can be drawn from the figure that the distribution characteristics are basically similar within the reachable domain, ranging from 50 hours to 72 hours. Therefore, the RAAN has little influence on the distribution characteristic of the contingency return flight time.
Fig. 7. Contingency return flight time distribution within the reachable domain of departing from different RAANs.

Considering contingency return from the low lunar orbits with different orbital inclinations, the orbital inclination is set to be 0°, 90°, 180° and 270°, and other parameters are set to be the same as that in the above subsection. By adopting the method in this paper, reachable domains of the contingency return trajectories departing from different orbital inclinations are shown in Fig. 8. It is concluded from the figure that reachable domains basically coincide. The orbital inclination of the low lunar orbit has no effect on the reachable domain of the contingency return trajectories.

Fig. 8. Reachable domain of the contingency return trajectories departing from the different orbital inclinations.

Fig. 9 depicts the contingency maneuver magnitude distribution within the reachable domain of departing from different orbital inclinations. It can be drawn from the figure that the distribution features of the contingency maneuver magnitude within the reachable domain of departing from the different orbital inclinations are different. The contingency maneuver magnitude is small as a whole when the orbital inclination is 180°, while the contingency maneuver magnitude is large as a whole when the orbital inclination is 270°.
The contingency return flight time distribution within the reachable domain of departing from the different orbital inclinations is described in Fig. 10. It is drawn from the figure that distribution characteristics within the reachable domain are basically similar, ranging from 50 hours to 72 hours. Therefore, the orbital inclination has little influence on the distribution characteristics of the contingency return flight time.
Considering contingency return from the low lunar orbits with different orbital altitudes, the orbital altitude is set to be 100 km, 200 km, 300 km and 400 km, and other parameters are set to be the same as that in the above subsection. By adopting the method proposed in this paper, reachable domains of contingency return trajectories departing from different orbital altitudes are shown in Fig. 11. It is drawn from the figure that reachable domains basically coincide. The orbital altitude of the low lunar orbit has no effect on the reachable domain of the contingency return trajectories.

Fig. 11. Reachable domain of the contingency return trajectories departing from the different orbital altitudes

Fig. 12 describes the contingency maneuver magnitude distribution within the reachable domain of departing from the different orbital altitudes. It is concluded from the figure that the distribution
characteristics of the contingency maneuver magnitude within the reachable domain are basically similar. The distribution ranges are also basically similar.

(a) Orbital altitude of 100 km

(b) Orbital altitude of 200 km

(c) Orbital altitude of 300 km

(d) Orbital altitude of 400 km

Fig. 12. Contingency maneuver magnitude distribution within the reachable domain of departing from different orbital altitudes

The contingency return flight time distribution within the reachable domain of departing from the different orbital inclinations is described in Fig. 13. It is drawn from the figure that distribution characteristics within the reachable domain are basically similar, ranging from 50 hours to 72 hours. Therefore, the orbital altitude has little influence on the distribution characteristics of the contingency return flight time.
Considering different contingency return dates, with 2028-02-17 as the starting date and other parameters unchanged, the daily reachable domain of the contingency return trajectories within 60 days can be calculated by adopting the proposed method. Fig. 14 shows the variations of $\lambda_{RP}$ and $\phi_{RP}$ within reachable domain with the contingency return date. It can be seen from the figure that the distribution range of $\lambda_{RP}$ is large, covering the range of $-180^\circ$ ~$180^\circ$, while the distribution range of $\phi_{RP}$ is small, ranging from $-60^\circ$ ~$60^\circ$. At the same time, it can be also found that the distributions of $\lambda_{RP}$ and $\phi_{RP}$ change periodically as a whole, and the change period is about 29 days.
Fig. 14. Relationship between the longitude and latitude distributions of the reentry point and the contingency return date

Fig. 15 depicts the relationship between the mapping distribution of RAAN within reachable domain and the contingency return date. It is concluded from the figure that distribution range of the reachable orbital inclination is $0^\circ \sim 180^\circ$, and the distribution range of the reachable orbital inclination changes periodically with a change period of about 14 days. The distribution range of the reachable RAAN is $0^\circ \sim 360^\circ$, and the mapping distribution characteristic changes periodically with a change period of about 29 days.

Fig. 16 describes the relationship between the mapping distribution of the contingency maneuver magnitude within the reachable domain and the contingency return date. The figure indicates that distribution range of contingency maneuver magnitude is 1000 m/s~3200 m/s, and the mapping...
distribution characteristic changes periodically with a change period of about 29 days.

Fig. 16. Relationship between the mapping distribution of the contingency maneuver magnitude and the contingency return date

Fig. 17 describes the relationship between the mapping distribution of the contingency return flight time within the reachable domain and the contingency return date. It is concluded from the figure that distribution range of contingency return flight time is 45 hours–72 hours, and the mapping distribution characteristic changes periodically with a change period of about 29 days.

Fig. 17. Relationship between the mapping distribution of the contingency return flight time and the contingency return date

Thus, according to the above reachable domain characteristic analysis, some conclusions can be summarized as follows: the reachable domain of the contingency return trajectories is not influenced
by the parameters of the low lunar orbit, including the RAAN, the orbital inclination and the orbital altitude. In addition, the contingency return date influences the reachable domain of the contingency return trajectories. The entire characteristic of the reachable domain takes on a periodic variation and the variation period is about 29 days.

5. Conclusion

Aimed at the contingency return trajectory which adopts a non-coplanar maneuver way in the circumlunar flight stage for manned lunar landing engineering, this paper proposes a reachable domain calculation method based on interval analysis. According to the problem statement of a contingency return trajectory, the mathematics model of its reachable domain is established. On the basis of the design process of a contingency return trajectory, a reachable domain calculation method is proposed by introducing the improved interval branch and bound algorithm. By comparing the simulation results, the accuracy of the proposed method is verified and it has higher calculation efficiency. Compared to the brute force method, the method proposed in this investigation can reduce 56%–96% of calculation time under different parameter configurations. By analyzing the reachable domain characteristic, it can be found that different low lunar orbit parameters cannot affect the reachable domain characteristic and the contingency return date can influence the reachable domain characteristic. In addition, the entire characteristic of the reachable domain has a periodic variation and the variation period is about 29 days.

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