

Modelling based in the stochastic dynamics for the time evolution of the COVID-19

Leonardo dos Santos Lima (✉ lslima@cefetmg.br)

Federal Education Center for Technological of Minas Gerais <https://orcid.org/0000-0003-0410-3640>

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Modelling based in the stochastic dynamics for the time evolution of the COVID-19

L. S. Lima

Department of Physics, Federal Education Center for Technological of Minas Gerais, 30510-000, Belo Horizonte, MG, Brazil.

The stochastic differential equation (SDE) corresponding to nonlinear Fokker-Planck equation where the nonlinearity appearing in this evolution equation can be interpreted as providing an effective description of a system of particles interacting is obtained. Additionally, we propose a stochastic model for time dynamics of the COVID-19 based in the set of data supported by the Brazilian health agencies.

Nonlinear equations (NL) have become an important subject in all branches of the physics due to their ability to explain several complex behaviors in the nature [1, 2]. Other areas of the physics as plasma physics and non-equilibrium physics have benefited from the study of nonlinear equations [3]. However, it is well known that nonlinear problems are impossible to solve analytically. The essential difference is that linear systems can be broken down into parts. Where each part can be solved separately and finally recombined to get the answer. This idea allows a fantastic simplification of complex systems, and underlies such methods as normal modes, Laplace transforms, superposition arguments, and Fourier analysis. But many things in nature do not act this way. Whether parts of a system interfere, there are nonlinear interactions going on. Therefore, the principle of superposition fails spectacularly. Within this realm of physics, nonlinearity is vital for many areas of physics, as for instance, for the superconductivity and Josephson junctions arrays. [4]

In this work, we derive the corresponding stochastic differential equation in Itô calculus for the nonlinear Fokker-Planck equation derived in the framework of the non-additive statistical mechanics and which may obey to stylized features of the financial market such as the inverse of cubic law [5–13]. Furthermore, we propose a stochastic model for propagation of COVID-19 based on nonlinear stochastic equation.

Stochastic nonlinear dynamics

The nonlinear Fokker-Planck equation is given in non-additive statistical mechanics by [20]

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 \{ [P(x, t)]^{2-q} \}}{\partial x^2} - \frac{\partial [P(x, t) K(x, t)]}{\partial x}. \quad (1)$$

We obtain that Eq. (1) is equivalent to following Itô stochastic differential equation

$$dX = A(X(t), t)dt + \phi(X(t), t) \circ dW, \quad (2)$$

where $A(X(t), t) = K(X(t), t)$ and $\phi(X(t), t) = [P(X(t), t)]^{\frac{1-q}{2}}$. $X(t)$ is a stochastic process defined on

probability space Ω , being the triple (Ω, \mathcal{F}, P) a probability space, where \mathcal{F} is a σ -algebra and P a probability measure, i.e. a function that to every set $A \in \mathcal{F}$ assigns a number in range $[0, 1]$, where $P(\Omega) = 1$, $P(\emptyset) = 0$ and

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n). \quad (3)$$

The random variable X , defined on Ω with the property that for every Borel subset B of \mathbb{R} , the subset of Ω given by $\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\}$ is in the σ -algebra \mathcal{F} . Moreover, $P(X(t), t)$, given X a random variable on a probability space (Ω, \mathcal{F}, P) is the probability measure of X . μ_X assigns to each Borel subset B of \mathbb{R} the mass $\mu_X(B) = P\{X \in B\}$ [14, 15]. Moreover, we have that dW is the Wiener increment, where $W(t)$ is a Wiener process also known as Brownian motion. We have an environment stochastic white noise $\zeta(t)$ that if relate with the Wiener process $W(t)$ by

$$W(t) = \int_0^t \zeta(t') dt'. \quad (4)$$

Although in the literature had been explained differently [21], we have that $W(t)$ which is the integral of $\zeta(t)$, is not the derivative of the Wiener process, $\zeta(t) \neq dW(t)/dt$, being therefore $W(t)$ not differentiable [16]

$$\frac{dW(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta W(\Delta t)}{\Delta t} \sim \frac{\sqrt{\Delta t}}{\Delta t} \rightarrow \infty. \quad (5)$$

The Itô integral for ϕ is given by

$$\int \phi(X(t), t) dW(t) = \text{ms-} \lim_{n \rightarrow \infty} \left\{ \sum_{j=1}^n \phi(t_{j-1}) [W(t_j) - W(t_{j-1})] \right\}, \quad (6)$$

where by ms–lim means square limit. $W(t)$ is a Markovian process, presenting a normal distribution, satisfying the conditions $\langle \zeta(t) \rangle = 0$ and $\langle \zeta(t) \zeta(t') \rangle = \delta(t - t')$. Furthermore, the model Eq. (2) can be used to study the financial market where the prices tend to go for different local minimums in a nonlinear polynomial potential of degree n [17–19].

We can write the Eq. (1) as

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 \left\{ [P(x, t)P(x, t)]^{1-q} \right\}}{\partial x^2} - \frac{\partial [P(x, t)K(x, t)]}{\partial x}. \quad (7)$$

Where we define $A(x, t) = K(x, t)$ and $\phi(X(t), t) = [P(X(t), t)]^{\frac{1-q}{2}}$ to obtain the correspondent Itô stochastic differential equation

$$dX = K(X(t), t)dt + [P(X(t), t)]^{\frac{1-q}{2}} \circ dW, \quad (8)$$

which is the Eq. (2). The solution for $X(t)$ using the Stratonovich integral is

$$X(t) = X(0) + \int_0^t K(s, X_s)ds + \int_0^t \phi(s, X_s)dW_s. \quad (9)$$

This implies that $X(t)$ is the solution of the following modified Itô equation

$$X(t) = X(0) + \int_0^t K(s, X_s)ds + \frac{1}{2} \int_0^t \phi'(s, X_s)\phi(s, X_s)ds + \int_0^t \phi(s, X_s)dW_s, \quad (10)$$

where ϕ' denotes the derivative of $\phi(x, t)$ with respect to x . Therefore, the Eq. (1) in Itô calculus is different of the Stratonovich interpretation.

From the Feynman-Kac theorem, let $h(x)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given, we define the function

$$g(x, t) = E^{t,x} h(X(T)), \quad (11)$$

where

$$E|g(X)| = \int_{-\infty}^{\infty} |g(x)|P(x)dx. \quad (12)$$

Furthermore, we assume $E^{t,x}|h(X(T))| < \infty$ for all t and x , then $g(x, t)$ satisfies the partial differential equation

$$\frac{\partial g(x, t)}{\partial t} + K(x, t)\frac{\partial g(x, t)}{\partial x} + \frac{1}{2}\phi^2(x, t)\frac{\partial^2 g(x, t)}{\partial x^2} = 0 \quad (13)$$

with the terminal condition $g(x, t) = h(x)$ for all x , where we assume that the stochastic process $g(X(t), t)$, $0 \leq t \leq T$ is a martingale.[14, 15].

From Eq. (2), we obtain the time development of an arbitrary $f(X(t))$ using the Itô formula [16]

$$\begin{aligned} & f[X(t) + dX(t)] - f[X(t)] \\ &= \frac{\partial f}{\partial x} \left[K(X(t), t)dt + [P(X(t), t)]^{\frac{1-q}{2}} \circ dW \right] \\ &+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \left\{ [P(X(t), t)]^{\frac{1-q}{2}} \right\} + \mathcal{O}(\dots). \end{aligned} \quad (14)$$

Taking the average of both sides in the equation above, we obtain

$$\begin{aligned} \left\langle \frac{df}{dt} \right\rangle &= \left\langle \left[\frac{df}{dx} K(X(t), t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \left\{ [P(X(t), t)]^{\frac{1-q}{2}} \right\} \right] \right\rangle \\ &+ \frac{d}{dt} \left\langle \left\{ [P(X(t), t)]^{\frac{1-q}{2}} \right\} \right\rangle \end{aligned} \quad (15)$$

and using

$$\begin{aligned} \frac{\langle f(X(t)) \rangle}{dt} &= \frac{d}{dt} \int_{-\infty}^{\infty} dx f(x) P(x, t) \\ &= \int_{-\infty}^{\infty} dx f(x) \frac{\partial}{\partial t} [P(x, t)] = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} K(x, t) P(x, t) dx \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial^2 f}{\partial x^2} \left\{ [P(x, t)]^{\frac{1-q}{2}} \right\} P(x, t) dx \\ &+ \int_{-\infty}^{\infty} dx f(x) \frac{\partial P}{\partial t} \end{aligned} \quad (16)$$

we integrate by parts and discard surface terms to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} dx f(x) \frac{\partial}{\partial t} [P(x, t)] &= \int_{-\infty}^{\infty} f(x) \frac{\partial}{\partial x} [K(x, t)P(x, t)] dx \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} f(x) \frac{\partial^2}{\partial x^2} \left[\left\{ [P(x, t)]^{\frac{1-q}{2}} \right\} P(x, t) \right] dx. \end{aligned} \quad (17)$$

and hence

$$\begin{aligned} \frac{\partial}{\partial t} P(x, t) &= -\frac{\partial}{\partial x} [K(x, t)P(x, t)] \\ &+ \frac{1}{2} \frac{\partial^2}{\partial x^2} \left\{ \left\{ [P(x, t)]^{\frac{1-q}{2}} \right\}^2 P(x, t) \right\}. \end{aligned} \quad (18)$$

Thus, we have complete equivalence between a diffusion process defined by a drift coefficient $K(x, t)$ and a diffusion coefficient given as $\phi(x, t) = [P(x, t)]^{1-q}$ in which the diffusion process can be locally approximated by an Itô stochastic differential equation.

Corresponding to Stratonovich stochastic differential equation

$$(S)dX = K^s(X(t), t)dt + [P(X(t), t)]^{\frac{1-q}{2}} dW, \quad (19)$$

with $K^s = K - \frac{1}{2}\phi\partial_x\phi$, and using the correspondence between the Itô stochastic differential equation and Fokker-Planck equation, we have a equivalent Fokker-Planck equation

$$\partial_t P = -\partial_x \{K^s P\} + \frac{1}{2} \partial_x \{ \phi \partial_x [\phi P] \}, \quad (20)$$

which is known as the Stratonovich form of the Fokker-Planck equation[16]. However, it is different from Eq. (1). Therefore, we have that the corresponding nonlinear Fokker-Planck equation in the Stratonovich prescription

is different from nonlinear equation obtained in the Itô prescription, being the Itô stochastic differential equation more usually employed to make the connection with the Fokker-Planck equation, contrary the Ref. [21]. Despite both definitions can be related by the choosing of i by $\tau_i = \alpha t_i + (1 - \alpha)t_{i-1}$, $\alpha \in (0, 1)$ they generate different definitions of stochastic integral (Itô integral and Stratonovich integral) and consequently to different stochastic equations, even though the Itô stochastic differential equation is equivalent to an another Stratonovich equation however, with an additional term [16]. In a general way, the Eq. (1) can be applied in different branches of the science and engineering such as electronics (phase-locked loops); biology (oscillating neurons, firefly flashing rhythm, human sleep-wake cycle); condensed matter physics (Josephson junction), charge density waves.

We can investigate the existence and uniqueness of solutions for nonlinear differential equations by utilizing the well-known existence and uniqueness theorem for stochastic differential equations [14]. Let $T > 0$ and $K(x, t) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \phi(x, t) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ be measurable functions satisfying

$$|K(x, t)| + |\phi(x, t)| \leq C(1 + |x|); x \in \mathbb{R}^n, t \in [0, T]$$

for some constant C and such that

$$|b(x, t) - b(y, t)| + |(\phi(x, t) - \phi(y, t))| \leq D|x - y|; \\ x, y \in \mathbb{R}^n, t \in [0, T] \quad (21)$$

for some constant D . Let Z be a random variable which is independent on the σ -algebra \mathcal{F}_∞^m generated by $W_s(\cdot)$, $s \leq 0$ and such that the expectation $E[|Z|^2] < \infty$ Then the stochastic differential equation

$$dX = A(X(t), t)dt + \phi(X(t), t)dW, \quad 0 \leq t \leq T, X_0 = Z \quad (22)$$

has a unique t -continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to filtration \mathcal{F}_t^Z generated by Z and $W_s(\cdot)$; $s \leq t$ and

$$E \left[\int_0^T |X_t|^2 dt \right] < \infty. \quad (23)$$

Dynamics of the COVID-19 propagation

In Fig. 1, we display the plot of number of infected by COVID-19 in Brazil as function of time (days). By fit of least minimum squares of the set of data of Brazilian health agencies, we estimate the curve for propagation of COVID-19 given by a polynomial of t of form (solid-black-line) $N(t) = -7653.3 + 1047t - 50.923t^2 + 1.01t^3$ by April 13th within the range considered. However, from April 13th, the adjustment is given for a polynomial given

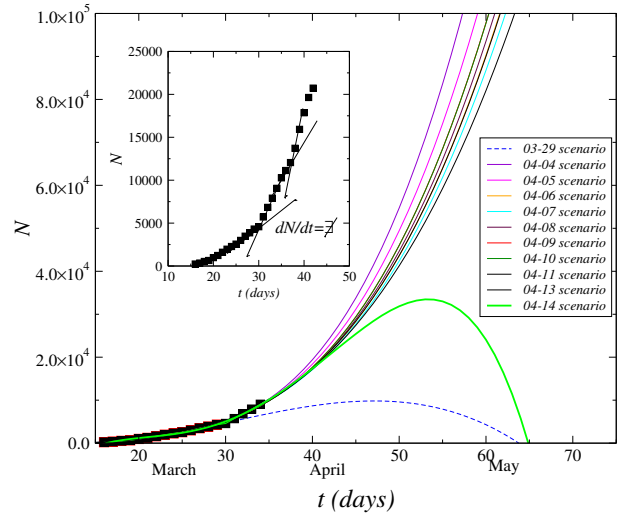


FIG. 1. Fit of the number of cases of COVID-19 $N(t)$. The squares (in black) are the number of cases of COVID-19 registered in the Brazil in March 2020. We obtain the polynomial adjustment in the range considered.

by $N(t) = 47988 + 7341t + 407t^2 + 9.7t^3 + 0.08t^4$. This sudden change in the behavior might be due to population isolation measures and a uncertainty in the official data. Therefore, we add the random term together with nonlinear terms with aim to simulate the effect of uncertainty of the official data of health agencies.

$$dN(t) = (f(t) + \alpha N(t))dt + B(N(t), t) \circ dW \quad (24)$$

where $f(t)$ is a polynomial of n degree. Furthermore, we can see an increase of inclination of the curves from March 30th that is the effect of quarantine measures adopted.

In Fig. 2, we display the behavior for $f(t)$ given as $f(t) \simeq 3t^2 + 102t - 1047$ in March 30th, for different values of coupling $B(N(t), t) = \beta$ constant (additive white noise term) in Eq. (24). The peak position of $N(t)$ varies a little with we increase of the random in the model indicating that an environmental noise must change the peak of the curve of the number of cases. Furthermore, we obtain a strong oscillation in the curve with the increasing of β indicating a oscillation in the number of cases in range considered.

In Brief, we propose the stochastic model for the time evolution of the number of cases of the COVID-19 in the Brazil based in the set of data Brazilian health agencies. Our results propose a polynomial behavior for the time dynamics of the propagation COVID-19 through of days. Despite the model reported here is based in Brazilian data only, we believe that the model must be adequate the behavior of the time evolution in other countries too. Furthermore, we obtain the correspondence between the stochastic differential equation and nonlinear Fokker-Planck equation obtained in the non-additive sta-

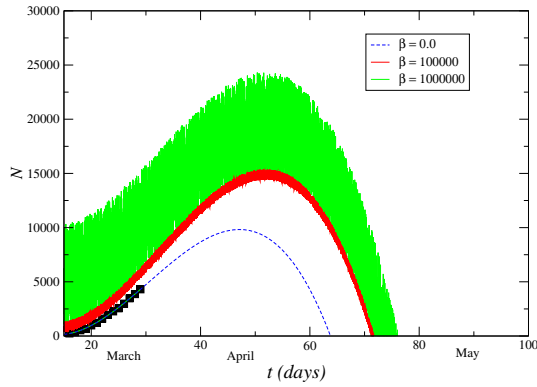


FIG. 2. Effect of the white noise term on fit of the number of cases of COVID-19. We obtain an increase of random on curve of the polynomial fit to the data.

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Figures

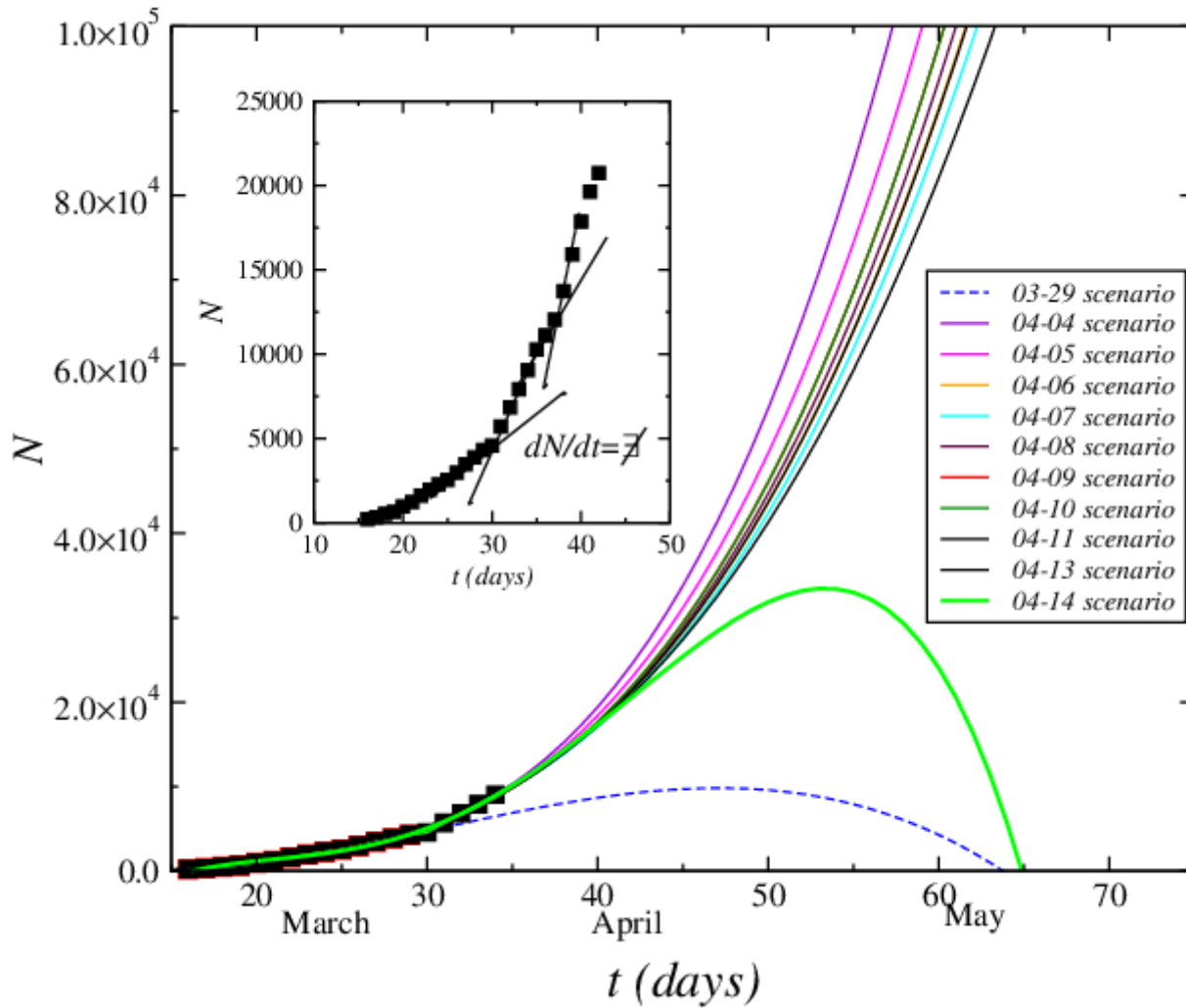


Figure 1

Fit of the number of cases of COVID-19 $N(t)$. The squares (in black) are the number of cases of COVID-19 registered in the Brazil in March and April 2020. We obtain the polynomial adjustment in the range considered.

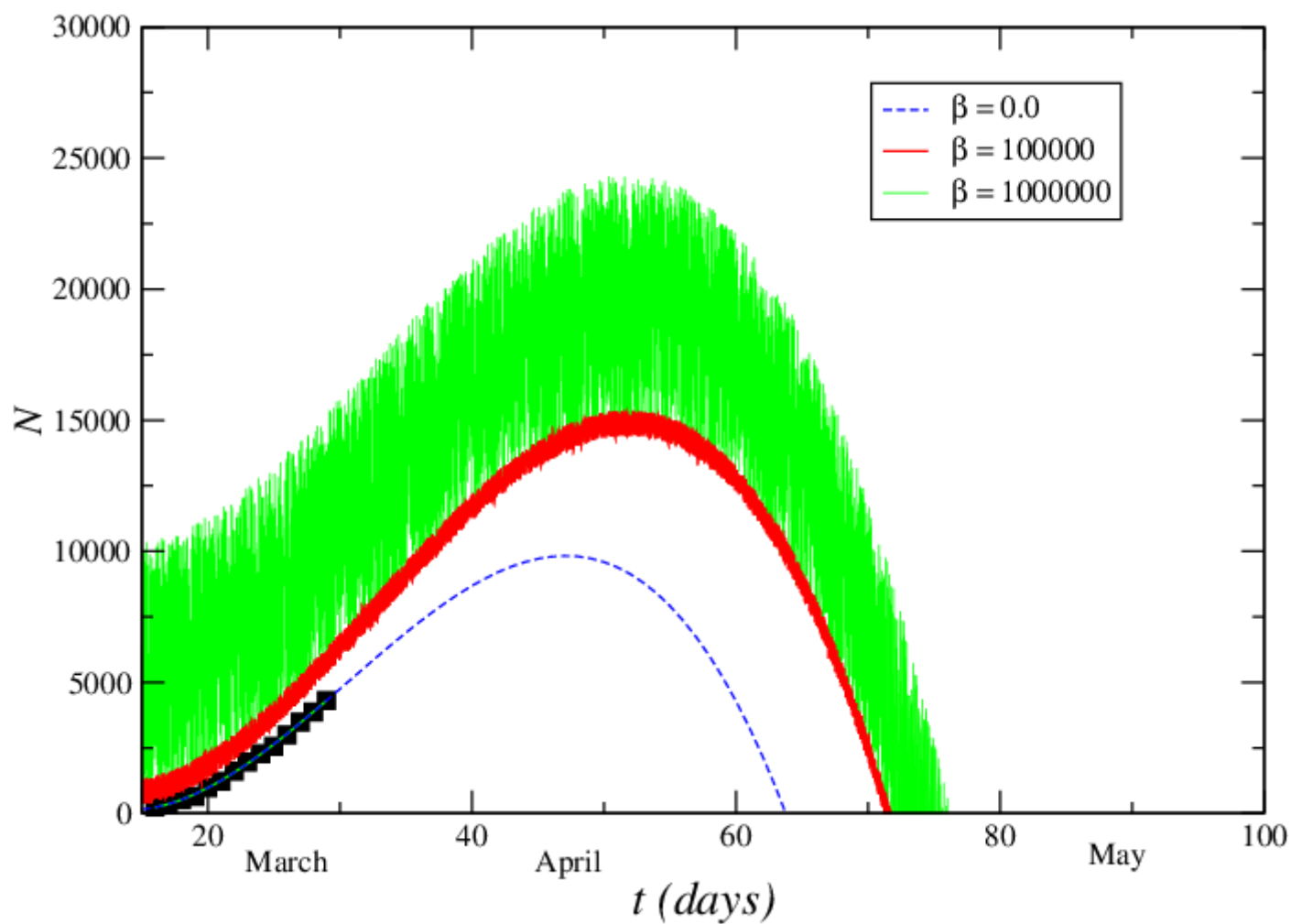


Figure 2

Effect of the white noise term on fit of the number of cases of COVID-19. We obtain an increase of random on curve of the polynomial fit to the data.