Study of markovian queueing model with F-policy, impatience customers, feedback and re-service, server vacation and breakdown

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Abstract A system with two heterogeneous servers with impatient customers and finite capacity is considered. The admission control of customers is based on F-policy. This paper is based on M/M/2 queues, and presents the study of two heterogeneous servers where one server is the primary server and other is secondary. Customers are primarily served by the main server, who also takes customer’s feedback into account. If customers are unsatisfied and opt for further service, the second server will serve them. The main server considered is unreliable, that is, it may breakdown at any time. The main server will go for vacation whenever there is no customer to serve. To analyze the performance of the model, we calculated the measures of performance and did the cost analysis. Impact of F-policy on system’s estimated profit is shown. At last, conclusions are drawn.

Keywords M/M/2 queueing model · Breakdown · Impatient customers · F-policy · Feedback · Re-service

1 Introduction

Queuing theory has advanced significantly in recent years; we encounter numerous queueing systems in our daily lives, some of which have infinite capacity and others which have finite capacity. This article’s suggested queueing model is applicable to a variety of fields, including manufacturing, production, communication, transportation, and networks. Restaurants, railway ticket counters, post offices, contact centers, gas pumps, ATMs, parks, and other public places use the finite capacity queueing paradigm. Public dealing centers have
a limited capacity, thus F-policy appears to be an important technique to manage consumer arrival at such locations.

Working for a long period of time server may become counterproductive due to overheating, cyber-attacks, spare breakdown, service degradation or many more reasons. Vacation is incorporated into the system to prevent such situations. Vacations increase server performance, safeguard against unavoidable power surges, prolong system life, and are cost-effective for the system.

In today’s fast-paced world, no one is willing to wait longer than their projected wait time. When a server is slow or unavailable, customers lose patience. Impatient clients are characterized by two major characteristics: balking and reneging. Anytime a customer is unsure whether to join the line or not, incidences of balking result. Balking is the action of a customer who resolves their conundrum by choosing to depart the system without enrolling. Reneging occurs when a customer initially enters the line but eventually disperses. For the system to be more realistic, both the server’s vacation policies and the customers’ impatient behavior must be considered.

On a daily basis, whether consciously or unconsciously, we deal with numerous kinds of servers. System tries to function with several types of servers; to determine which server is best to utilize, system solicits customer feedback. Feedback enables the system to understand service quality. Each customer may not be content with the service provided by each sort of server; some customers may favour fast-serving servers, while others may desire quality. If a client is unsatisfied for any reason, re-service may be offered in order to gain the customer’s loyalty.

Literature review of the paper is provided in section 2. Section 3 and 3.1 provides a description and schematic illustration of the model. The model’s practical applications are provided in section 4. In section 5 the model’s mathematical analysis is carried out using state transition diagrams. Stationary probabilities and performance measures are derived in section 6 and 7 respectively. Cost, revenue and profit functions are constructed in section 8. Section 9 shows the numerical analysis of model where impact of various factors on performance measurements is shown in section 9.1, economic analysis in section 9.2 and the impact of F-policy on the economics of the system in section 9.3. Finally the conclusion is drawn in section 10.

2 State of art

In the course of our daily activities, we frequently see queuing systems with constrained capacities at places like post offices, ticket booths, corporate warehouses, gas stations, and restaurants, among others. F-policy is added to such systems to prevent overcrowding and minimize consumer inconvenience. During F-policy, when the system hits its maximum capacity, new customers are barred from entering; however, this restriction is lifted whenever the length of the queue falls below a certain threshold. Pioneer study on F-policy was done by Gupta [40] in which he derived the steady state probabilities for customers
on M/M/1/k queue under F policy. This model of Gupta was generalized by Wang [39] including the server breakdown. Analysis of M/M/2/k queues under F-policy was done by Wang [38]. Finite queue length single server queues under F-Policy with re-trial attempts were analyzed by Jain [18]. The latest survey on F-policy is presented by Jain [17].

It becomes vital to bring vacations to the system in order to reduce system costs and maintain appropriate operation. Vacation functions as a system-supporting tool. A server may take a vacation depending on the queue status, that is, anytime there are no customers in the system, or may take a vacation regardless of the queue status even if there are customers in the system who are waiting to be served. Different types of vacation policies and models were explained by Tian [30] and Takagi [22] in their books. Vijayshree [29] analyzed an M/M/c model subjects to single exponential vacation where all servers will go to vacation whenever the system become empty. Sridhar [25] examined a M/M/2 queue where, after the first vacation, one server remains constantly available while the other goes on a working vacation whenever the system is empty. Sapkota [23] examined an M/M/c vacation queueing model with impatient consumers, where the server begins the vacation after an arbitrary duration of waiting. Levy and Yechiali [37] examined the queues where, following a vacation, the server could either come back to the queue and wait for a customer to arrive or keep vacationing until it discovers at least one customer in the queue. A multi-server vacation queueing model is analyzed by Dudin et.al. [13] where the servers are self-sustainable and can take vacation independently of system manager, by terminating the service. Other M/M/2 queueing system with vacations have been discussed in [36], [19] and [7]. A comparative study of vacation models is done by Chakravarthy [10].

Most realism models make it easy to see how impatient customers can be. Customers can lose patience if they have to wait a long time in queue or the queue anticipates a long wait. The literature describes three types of impatient behavior: balking, reneging, and jockeying. In this study, reneging and balking behaviors are taken into account. Reneging is the act of a client choosing to leave a queueing system before getting service. Robert [32] introduced the reneging phenomenon for the first time in single channel queues. Balking is the act of prospective customers coming at a queueing system choosing not to enter the system. Haight [14] was the first to think about balking customers. Haight [15] also studied queues that had reneging. Ancker and Gafarian [2] [3] conducted a brief investigation of consumer impatient behavior in various queueing situations. The analysis of finite queue length M/M/1 model with balking and reneging customers and multiple server vacation is done by Zhang et. al. [31]. Impatience behavior of customers is analyzed by Saikia [33] in multi-server markovian queues where the impatience is state dependent. M/M/2/N queue with balking, reneging and Bernoulli feedback is analyzed by Bouchentouf [8] where reneged customer may retain to the system. An M/M/2/N and M/M/c/N queuing models were established by Som [27], [28] that took into account consumer impatience, encouraged arrivals, and retention of reneged customers. Altman and Yechiali [11] analyzed an another types
of single and multi-server queues with different vacations where customers become impatient owing to server’s absence on arrivals. Five different types of queueing problems with server breakdown were analyzed by Avi-Itzhak [4]. A steady state analysis of two server queue with heterogeneous servers is presented by Yang [35] where one server may take vacation and other server may breakdown. The cost analysis of finite length multi-server queueing system with impatient customers and server breakdown is done by Wang and Chang [34]. A study of queueing model with server breakdown, vacation and back up server is presented by Chakravarthy [11] where the back up server works as main server during the breakdown and vacation of main server. Jain [16] analyzed a single server queueing model with working vacations where server may face multiple types of breakdowns.

To improve the quality of service it become necessary to know about the customer’s experience in the queue. Feedback is the prominent tool to know about customers satisfaction. In the businesses like restaurant, petrol pump, health care etc, customers fulfillment become more important. Perceiving this, different organizations can be seen providing feedback service facilities to discontented customers. Takas [41] analyzed the single server feedback queue where the customers after being served either joins the queue with probability p or leaves the system with probability 1-p. A finite feedback service facility is considered by Begum [6] in study of batch arrival queueing system with unreliable server. Singla [24] provided a re-service facility to the unsatisfied customers in retrial queueing model with two servers and feedback where feedback customers may join the orbit with retrial customers and re-try for service. Optional re-service is also provided by Boutchentouf [9] in Markovian multi-server feedback queue with impatient customers and variant of multiple vacations. Comparison of M/M/2 and M/M/1 queueing models having particular service interruption and feedback customers was done by Mahla [20].

3 Model Description

The model considered is a two servers (Server 1 and Server 2) queueing model with finite capacity where customers arrive to the system according to Poisson distribution with arrival rate $\lambda$. The arrived customers are served by server 1 with service rate $\mu_1$. Customers have to submit feedback regarding the quality of service through a feedback facility. Customers who are pleased will leave favorable reviews, while those who are dissatisfied will leave negative ones. $p$ is the probability that a review will be positive, and $q = (1 - p)$ is the probability that a review will be negative. For dissatisfied clients, a re-service facility is offered where they can opt to re-service with probability $f$ or leave the system with probability $1 - f$. The customer will depart the system after receiving service from server 2. Both servers’ service times are distributed exponentially. Server 1 is susceptible to breakdown at any time with rate $\alpha$, and it will be fixed right away with rate $\beta$. If there isn’t a system-required customer for Server 1 to serve, Server 1 will take a vacation. Server 1 will
be back from vacation with a rate $\xi$. Server 2 is reliable and will constantly be accessible within the system. No additional arrivals are permitted into the system after $N$ consumers. Arrivals are now again permitted when there are $F$ clients left in the system. Customers may balk or renege with probability $k$ and rate $g$ respectively. Breakdown time, repair time and vacation time are exponentially distributed. The inter arrival time, service time, breakdown time and vacation time are mutually independent. If there are $n$ customers in the system then the joining probability is $k_n$ and the balking probability is $(1 - k_n)$, where $k_n$ is defined according to Choudhary [12] as,

$$k_n = \begin{cases} 1, & n = 0, \\ 1 - \left(\frac{1}{2}\right)^{N-n+2}, & 1 \leq n \leq N. \end{cases}$$

Let $r$ be the reneging parameter then reneging rate $g_n$ is defined as

$$g_n = \begin{cases} 0, & n = 0, 1 \\ (n-1)r, & 1 < n \leq N. \end{cases}$$

3.1 Schematic Representation of model

The schematic representation of model is presented in Fig. 3.1

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**Fig. 1** Schematic Representation of model
4 Real-life applications of the model

The concept that is being given is comparable to numerous real-world models, including marketing, public dealing centers, food delivery apps, and others. When food is ordered through a food delivery app, the app function offers the customer a feedback facility at the conclusion of the service. Customer dissatisfaction could be brought on by a variety of problems, including sub-par cuisine, clumsy packing, longer than anticipated wait times during service, or any other problem. Therefore, a re-service facility is offered to the dissatisfied clients where another order or extra item is provided in order to prevent customer discontent. Similar situations occur when customers browse for products; after receiving their purchases, they have three options: retain them, return them, or exchange them. The satisfied consumers will keep the product, the dissatisfied ones who want re-service will swap the product, and the disgruntled ones will return the merchandise. In this case, keeping the product will be regarded as the customer providing positive feedback, returning the product is same to the customer leaving the system without selecting the provision of re-service, and swapping the product will be regarded as the dissatisfied customer desiring re-service.

5 Mathematical Analysis

Let $N(t)$ denotes the number of customers in the queue of server 1 at time $t$, $K(t)$ denotes the number of customers in the queue of server 2 at time $t$, $I(t)$ denotes the state of server at time $t$ and $J(t)$ denotes if the arrivals are allowed or not in the system at time $t$.

$I(t) = \begin{cases} 
0, \text{ the server 1 is in normal state at time } t, \\
1, \text{ the system is in the breakdown state at time } t, \\
2, \text{ the server 1 is in the vacation state at time } t.
\end{cases}$

$J(t) = \begin{cases} 
0, \text{ arrivals are allowed in the system at time } t, \\
1, \text{ arrivals are not allowed in the system at time } t.
\end{cases}$

$(N(t), K(t), J(t))_{I(t)}$ is a continuous time Markov chain. The state transition diagram of model is presented in two parts, Fig 2 and Fig 3. This state transition diagram is for the states with system capacity $(N)=6$ and $F=2$. Fig 2 shows the transition of the state up-to when the number of customers in the system is less than $N$ and Fig 3 shows the transition of states when number of customers in the system reaches $N$ and no further arrivals are allowed till the customers count reaches $F$. Let us denote $(N(t), K(t), 0)_{I(t)}$ as $(N(t), K(t))_{I(t)}$ and $(N(t), K(t), 1)_{I(t)}$ as $(N(t), K(t))'_{I(t)}$ in state transition diagram.
5.1 State transition diagram

The block-partitioned form of Matrix geometric method is obtained as follows:

\[
Q = 
\begin{pmatrix}
A_0 & B_0 \\
C_1 & A_1 & B_1 \\
C_2 & A_2 & B_2 \\
... & ... & ... \\
C_{F-1} & A_{F-1} & B_{F-1} \\
C_F & A_F & B_F \\
C_{F+1} & A_{F+1} & B_{F+1} \\
... & ... & ... \\
C_{N-1} & A_{N-1} & B_{N-1} \\
C_N & A_N & B_N
\end{pmatrix}
\]

Where

\[
A_0 = \begin{pmatrix}
-\lambda & 0 & 0 \\
\beta & -(\lambda + \beta) & 0 \\
\xi & 0 & -(\lambda + \xi)
\end{pmatrix},
B_0 = (\lambda I Z_3)
\]

\[
E_n = \begin{cases}
E_n Z^{(3)} & \text{for } 1 \leq n \leq F-1, \\
(E_n Z^{(3)})^t & \text{for } n = F-1,
\end{cases}
\]

\[
B_n = \begin{cases}
(E_n Z^{(3)})^t & \text{for } F \leq n < N - 1, \\
(Z^3) & \text{for } n = N - 1.
\end{cases}
\]

\[
A_n = \begin{cases}
D_{n1} & \text{for } 1 \leq n \leq F-1, \\
(Z_{3(n+1)}) & \text{for } F \leq n \leq N - 1, \\
D_{n2} & \text{for } n = N,
\end{cases}
\]

where \(Z^{(3)}\) is zero matrix of order \(3(n+1) \times 3(n+2)\), and \(Z_{m,n}\) is zero matrix of order \(m \times n\) and \(Z_n\) is zero matrix of order \(n \times n\).

\[
E_n = \begin{pmatrix}
[\lambda k_n I O] \\
[\lambda k_n I O] \\
[\lambda k_n I O]
\end{pmatrix}
\]

\[
E'_{n} = \begin{pmatrix}
[\lambda k_n I O] \\
[\lambda k_n I O] \\
[\lambda k_n I O]
\end{pmatrix}
\]

and \(O\) is the column vector of zero entries and \(O\) is row vector of zero entries.

\[
D_{n1} = \begin{pmatrix}
D_{n11} & D_{n12} & Z_{n+1} \\
D_{n21} & D_{n22} & Z_{n+1} \\
D_{n31} & Z_{n+1} & D_{n33}
\end{pmatrix},
D_{n2} = \begin{pmatrix}
D'_{n11} & D'_{n12} & Z_{n+1} \\
D'_{n21} & D'_{n22} & Z_{n+1} \\
D'_{n31} & Z_{n+1} & D'_{n33}
\end{pmatrix}
\]

\[
D'_{n2} = \begin{pmatrix}
D''_{n11} & D''_{n12} & Z_{n+1} \\
D''_{n21} & D''_{n22} & Z_{n+1} \\
D''_{n31} & Z_{n+1} & D''_{n33}
\end{pmatrix}
\]

for \(1 \leq n \leq F-1\), \(F \leq n \leq N - 1\), and \(n = N\).
Fig. 2 State transition diagram 1
Fig. 3 State transition diagram 2
All the block matrices of $D_{n_1}$ and $D_{n_2}$ are square matrices of order $(n + 1)$.

$$
D_{n_{11}} = \begin{pmatrix}
(\lambda k_n + \mu_1 + \alpha + r) & \mu_1 qf \\
-\lambda k_n + \mu_1 + \mu_2 + r + \alpha + r) & \mu_1 qf \\
-\lambda k_n + \mu_1 + \mu_2 + r + \alpha + r) & \mu_1 qf \\
\end{pmatrix}
$$

$$
D_{n_{12}} = \begin{pmatrix}
\alpha I & 0 \\
0 & \beta I \\
\end{pmatrix}, \quad D_{n_{21}} = \beta I, \quad D_{n_{31}} = \xi I
$$

$$
D_{n_{22}} = \begin{pmatrix}
(\lambda k_n + r + \beta) & 0 \\
0 & (\lambda k_n + \mu_2 + r + \beta) \\
\end{pmatrix}
$$

$$
D_{n_{33}} = \begin{pmatrix}
-\lambda k_n + r + \xi) & 0 \\
0 & -\lambda k_n + \mu_2 + r + \xi) \\
\end{pmatrix}
$$

$$
D'_{n_{11}} = \begin{pmatrix}
-(\mu_1 + \alpha + r) & \mu_1 qf \\
-(\mu_1 + \mu_2 + \alpha + r) & \mu_1 qf \\
\end{pmatrix}
$$

$$
D''_{n_{11}} = D'_{n_{11}}, \quad D''_{n_{21}} = (\beta I)_{n \times (n+1)}, \quad D''_{n_{31}} = (\xi I)_{n \times (n+1)},
$$

$$
D''_{n_{22}} = \begin{pmatrix}
-(r + \beta) & 0 \\
0 & -(\mu_2 + \beta) \\
\end{pmatrix}, \quad D''_{n_{12}} = \begin{pmatrix}
\alpha I \\
0 \\
\end{pmatrix}_{n+1 \times n},
$$

$$
D''_{n_{33}} = \begin{pmatrix}
-(r + \xi) & 0 \\
0 & -(\mu_2 + \xi) \\
\end{pmatrix}, \quad D''_{n_{23}} = \begin{pmatrix}
-(r + \beta) & 0 \\
0 & -(\mu_2 + r + \beta) I \\
\end{pmatrix}_{n \times n},
$$

$$
D''_{n_{33}} = \begin{pmatrix}
-(r + \xi) & 0 \\
0 & -(\mu_2 + r + \xi) I \\
\end{pmatrix}_{n \times n},
$$

$$\begin{cases}
C_{n_1}, & \text{for } 1 \leq n \leq F-1, \\
(C_{n_1}, C_{n_1}), & \text{for } n = F, \\
(C_{n_1}, Z_{(n+1),3(n)}^F), & \text{for } F \leq n < N - 1, \\
(Z_{(n+1),3(n)}^F, C_{n_1}), & \text{for } n = N.
\end{cases}
$$

$$C_{n_1} = \begin{pmatrix}
C_{n_1,1} & Z^2 & C_{n_1,3} \\
Z^2 & C_{n_2,1} & Z^2 \\
Z^2 & Z^2 & C_{n_3,1}
\end{pmatrix}_{3(n+1) \times 3(n)}, \quad C'_{n_1} = \begin{pmatrix}
C'_{n_1,1} & Z^2 & C'_{n_1,3} \\
Z^2 & C'_{n_2,1} & Z^2 \\
Z^2 & Z^2 & C'_{n_3,1}
\end{pmatrix}_{3(n+1) \times 3(n)}
$$

$Z^2$ is zero matrix of order $(n+1) \times (n)$. 

\[ C'_{n13} = C_{n13}, \quad C'_{n11} = C_{n11}, \quad C'_{n22} = C'_{n33} = \begin{pmatrix} r & r \\ \mu_2 & \mu_2 \\ & \ddots & \ddots \\ & & \mu_2 & r \end{pmatrix}_{(n+1) \times n}, \]

\[ C_{n11} = \begin{pmatrix} \mu_1(1 - qf) + r & \mu_1(1 - qf) + r \\ \mu_2 & \mu_2 \\ & \ddots & \ddots \\ & & \mu_2 & \mu_2 \end{pmatrix}_{(n+1) \times n}, \]

\[ C_{n13} = \begin{pmatrix} Z_{n-1} & 0 \\ 0 & \mu_1(1 - qf) + r \end{pmatrix}_{(n+1) \times n}, \quad C_{n22} = C_{n33} = \begin{pmatrix} r & r \\ \mu_2 & \mu_2 \\ & \ddots & \ddots \\ & & \mu_2 & r \end{pmatrix}_{(n+1) \times n}. \]

6 Stationary Analysis

For steady state solution, define steady state vector \( \psi = \{\psi_0, \psi_1, \psi_2, \ldots, \psi_N\} \) and \( e \) as the column matrix with all elements set as one. Then we have, \( \psi Q = 0, \) and \( \sum_{i=0}^{\infty} \psi_i e = 1. \)

Which gives

\[ \psi_0 A_0 + \psi_1 C_1 = 0, \]
\[ \psi_{i-1} B_{i-1} + \psi_i A_i + \psi_{i+1} C_{i+1} = 0, \quad \text{for} \quad 1 \leq i \leq N - 1, \]
\[ \psi_{N-1} B_{N-1} + \psi_N A_N = 0. \]

As equations have repeating structure, so solution by Baumann [5] and Neuts [21] is

\[ \psi_{n+1} = \psi_n R_n. \]

The equations will be converted to the form

\[ \psi_{N-1}(B_{N-1} + R_{N-1}A_N) = 0, \]
\[ \psi_{i-1}(B_{i-1} + R_{i-1}A_i + R_{i-1}R_iC_{i+1}) = 0, \quad \text{for} \quad 1 \leq i \leq N - 1, \]
\[ \psi_0(A_0 + R_0C_1) = 0. \]
which results
\[ R_{N-1} = -B_{N-1}A_N^{-1}, \]
\[ R_{n-1} = -B_n(A_n + R_nC_n)^{-1}, \quad n = N-1, N-2, \ldots 1. \]

By using the normalizing condition \( \psi_0 + \psi_1 + \cdots + \psi_N = 1 \)
\[ \psi_0 = \frac{1}{(1 + R_0 + R_0R_1 + \cdots + R_0R_1 \cdots R_{N-1})}. \]

7 Performance measures

– Expected number of customers in the system
\[ E_n = \sum_{m=0}^{F-1} \psi_m (e)^{3(m+1)} + \sum_{m=F}^{N-1} \psi_m (e)^{2(3(m+1))} + \psi_N (e)^{(3N+1)}. \]

– Probability of servers being on normal busy state
\[ P_n = \sum_{m=0}^{F-1} \psi_m \begin{pmatrix} e_{(m+1)} \\ O_{2(m+1)} \end{pmatrix}^{3(m+1)} + \sum_{m=F}^{N-1} \psi_m \begin{pmatrix} e_{(m+1)} \\ O_{2(m+1)} \end{pmatrix}^{2(3(m+1))} + \psi_N \begin{pmatrix} e_{(N+1)} \\ O_{2(N)} \end{pmatrix}^{(3N+1)}. \]

where \( O_n \) is the zero matrix of order \( n \times 1 \).

– Probability of server 1 being on breakdown state
\[ P_b = \sum_{m=0}^{F-1} \psi_m \begin{pmatrix} O_{(m+1)} \\ e_{(m+1)} \end{pmatrix}^{3(m+1)} + \sum_{m=F}^{N-1} \psi_m \begin{pmatrix} O_{(m+1)} \\ e_{(m+1)} \end{pmatrix}^{2(3(m+1))} + \psi_N \begin{pmatrix} O_{(N+1)} \\ e_{(N)} \end{pmatrix}^{(3N+1)}. \]

– Probability of server 1 being on vacation state
\[ P_v = \sum_{m=0}^{F-1} \psi_m \begin{pmatrix} O_{2(m+1)} \\ e_{(m+1)} \end{pmatrix}^{3(m+1)} + \sum_{m=F}^{N-1} \psi_m \begin{pmatrix} O_{2(m+1)} \\ e_{(m+1)} \end{pmatrix}^{2(3(m+1))} + \psi_N \begin{pmatrix} O_{2(N+1)} \\ e_{(N)} \end{pmatrix}^{(3N+1)}. \]
– Probability of server 2 being idle

\[ P_i = \sum_{m=0}^{F-1} \psi_m \left( \frac{1}{3^{(3m+2)}} \right) \frac{1}{3^{(m+1)}} + \sum_{m=F}^{N-1} \psi_m \left( \frac{1}{2^{(3(m+1))}} \right) + \psi_N \left( \frac{1}{(3N+1)} \right). \]

– Expected balking rate

\[ E_{br} = \sum_{m=0}^{F-1} \lambda (1-k_m) \psi_m \left( e \right) \frac{1}{3^{(m+1)}} + \sum_{m=F}^{N-1} \lambda (1-k_m) \psi_m \left( e \right) \left( \frac{e}{2^{(3(m+1))}} \right) + \psi_N \lambda (1-k_N) \left( e \right) \left( \frac{1}{3^{(m+1)}} \right). \]

– Expected reneging rate

\[ E_{rr} = \sum_{m=0}^{F-1} g_m \psi_m \left( e \right) \frac{1}{3^{(m+1)}} + \sum_{m=F}^{N-1} g_m \psi_m \left( e \right) \left( \frac{e}{2^{(3(m+1))}} \right) + g_N \psi_N \left( e \right) \left( e \right) \left( \frac{1}{3^{(m+1)}} \right). \]

– Average customer loss

\[ A_{loss} = E_{br} + E_{rr}. \]

– Effective arrival rate

\[ \lambda_e = \lambda - A_{loss}. \]

– Expected waiting time

\[ E_w = \frac{E_n}{\lambda_e}. \]

8 Economic Analysis

In this section cost, profit and revenue functions are constructed, the symbols and notations taken into consideration are

– \( C_1 \equiv \) Cost per unit time when server 1 is on normal busy period and server 2 is not idle.
– \( C_2 \equiv \) Cost per unit time when server 1 is on normal busy period and server 2 is idle.
– \( C_2 \equiv \) Cost per unit time when server 1 is on breakdown state.
– \( C_4 \equiv \) Cost per unit time when server 1 is on vacation state.
– \( C_5 \equiv \) Cost per unit time when a customer joins the queue and waits for service.
– \( C_6 \equiv \) Cost per service per unit time when server 1 is on normal state and server 2 is not idle.
– \( C_7 \equiv \) Cost per service per unit time when server 1 is on vacation state.
– \( C_8 \equiv \) Cost per service per unit time when server 1 is on breakdown state.
\[ \Delta \equiv \text{Cost per service per unit time when server 1 is on normal state and server 2 is idle.} \]
\[ C_10 \equiv \text{Cost per unit time when a customer balks.} \]
\[ C_{11} \equiv \text{Cost per unit time when a customer reneges.} \]
\[ C_{12} \equiv \text{Fixed server purchase cost per unit time.} \]
\[ R_{v1} \equiv \text{Revenue earned by providing service to a normal customer.} \]
\[ R_{v2} \equiv \text{Revenue earned by providing service to a re-service customer.} \]

Let \( \Delta \) be the total expected cost per unit time of the system. Then,
\[
\Delta = C_1 (P_n - P_i) + C_2 P_i + C_4 P_b + C_5 E_{N_0} + C_6 (\mu_1 + \mu_2) + C_7 \mu_2 + C_8 \mu_2 + C_9 \mu_1 + C_{10} E_{br} + C_{11} E_{rr} + C_{12}.
\]

Let the total expected revenue per unit time is \( R \)
\[
R = R_{v1} (\mu_1 (P_n - P_i) + \mu_1 P_i) + R_{v2} (\mu_2 (P_n - P_i) + \mu_2 P_b + \mu_2 P_v).
\]

Then total expected profit per unit time will be \( \mathcal{P} = R - \Delta \)

### 9 Numerical Analysis

In this section, numerical analyses of a few performance measures with variable parameters are described with the use of tables and diagrams.

#### 9.1 Effect of different parameters on the performance measures of the system

##### 9.1.1 Effect of arrival rate

To observe the effect of arrival rate, let the fixed parameters are: \( \mu_1 = 4, \mu_2 = 2, \alpha = 0.3, \beta = 2, \xi = 1.1, p = 0.5, f = 0.8, r = 0.2, N = 8, F = 4 \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_n )</td>
<td>0.80668</td>
<td>2.04529</td>
<td>3.22239</td>
<td>4.03225</td>
<td>4.53465</td>
<td>4.84962</td>
<td>5.05621</td>
</tr>
<tr>
<td>( P_n )</td>
<td>0.68403</td>
<td>0.63511</td>
<td>0.66259</td>
<td>0.69725</td>
<td>0.72347</td>
<td>0.74165</td>
<td>0.75428</td>
</tr>
<tr>
<td>( P_i )</td>
<td>0.11432</td>
<td>0.15638</td>
<td>0.14692</td>
<td>0.13375</td>
<td>0.12547</td>
<td>0.12075</td>
<td>0.11803</td>
</tr>
<tr>
<td>( P_v )</td>
<td>0.20164</td>
<td>0.20850</td>
<td>0.19048</td>
<td>0.16898</td>
<td>0.15104</td>
<td>0.13758</td>
<td>0.12768</td>
</tr>
<tr>
<td>( E_{br} )</td>
<td>0.53095</td>
<td>0.37824</td>
<td>0.31327</td>
<td>0.27196</td>
<td>0.24137</td>
<td>0.21785</td>
<td>0.19939</td>
</tr>
<tr>
<td>( E_{rr} )</td>
<td>0.06141</td>
<td>0.24944</td>
<td>0.46003</td>
<td>0.61294</td>
<td>0.70997</td>
<td>0.77152</td>
<td>0.81215</td>
</tr>
</tbody>
</table>

As observed from the Table 1 and Fig 4, the queue length increases quickly for the initial values of arrival rate, but after that, the increment slows down with introduction of F-policy. \( P_n \) increases with arrival rate. As server go to vacation whenever there are no customers in the system, so with increment in
arrival rate $P_v$ decreases, $P_i$ also decreases, $P_h$ initially rises quickly and then falls off gradually. As both the rate of balking and reneging depend on how many customers are in the system, therefore they both rise with the rise in arrival rate.

Fig. 4 Effect of $\lambda$ on different performance measures

Fig. 5 Effect of $\xi$ on different performance measures

9.1.2 Effect of vacation rate

To observe the effect of vacation rate, let the fixed parameters are: $\lambda = 4, \mu_1 = 4, \mu_2 = 2, \alpha = 0.3, \beta = 2, p = 0.5, f = 0.8, r = 0.2, N = 8, F = 4.$

Table 2

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>0.2</th>
<th>0.5</th>
<th>0.8</th>
<th>1.1</th>
<th>1.4</th>
<th>1.7</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n$</td>
<td>0.4549</td>
<td>0.6070</td>
<td>0.6651</td>
<td>0.6973</td>
<td>0.7186</td>
<td>0.7344</td>
<td>0.7408</td>
</tr>
<tr>
<td>$P_h$</td>
<td>0.0923</td>
<td>0.1208</td>
<td>0.1296</td>
<td>0.1358</td>
<td>0.1355</td>
<td>0.1361</td>
<td>0.1362</td>
</tr>
<tr>
<td>$P_v$</td>
<td>0.4527</td>
<td>0.2722</td>
<td>0.2051</td>
<td>0.1609</td>
<td>0.1459</td>
<td>0.1295</td>
<td>0.1170</td>
</tr>
<tr>
<td>$P_i$</td>
<td>0.1789</td>
<td>0.2380</td>
<td>0.2602</td>
<td>0.2720</td>
<td>0.2793</td>
<td>0.2844</td>
<td>0.2882</td>
</tr>
<tr>
<td>$E_{br}$</td>
<td>0.1348</td>
<td>0.1252</td>
<td>0.1184</td>
<td>0.1139</td>
<td>0.1108</td>
<td>0.1086</td>
<td>0.1070</td>
</tr>
<tr>
<td>$E_{rr}$</td>
<td>0.7216</td>
<td>0.6574</td>
<td>0.6293</td>
<td>0.6129</td>
<td>0.6022</td>
<td>0.5945</td>
<td>0.5887</td>
</tr>
</tbody>
</table>

As soon as the server returns from vacation mode, it will resume working in normal mode, this causes $P_n$ to rise. If server works more in normal mode so queue length drops generating a decrement in $E_{br}$ and $E_{rr}$. $P_v$ decreases with increase in vacation rate as a result $P_n$ and $P_h$ increase, $P_i$ also increases as illustrated from Table 2 and Fig 5.
9.1.3 Effect of service rate

To observe the effect of service rate, let the fixed parameters are: \( \lambda = 4, \mu_2 = 2, \alpha = 0.3, \beta = 2, \xi = 1.1, p = 0.5, f = 0.8, r = 0.2, N = 8, F = 4. \)

Table 3

<table>
<thead>
<tr>
<th>( \mu_1 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_n )</td>
<td>5.2589</td>
<td>4.8283</td>
<td>4.3752</td>
<td>4.0322</td>
<td>3.8086</td>
<td>3.6680</td>
<td>3.5779</td>
</tr>
<tr>
<td>( P_n )</td>
<td>0.8637</td>
<td>0.8308</td>
<td>0.7684</td>
<td>0.6972</td>
<td>0.6332</td>
<td>0.5809</td>
<td>0.5392</td>
</tr>
<tr>
<td>( P_b )</td>
<td>0.1308</td>
<td>0.1322</td>
<td>0.1334</td>
<td>0.1337</td>
<td>0.1332</td>
<td>0.1322</td>
<td>0.1311</td>
</tr>
<tr>
<td>( P_v )</td>
<td>0.0053</td>
<td>0.0368</td>
<td>0.0980</td>
<td>0.1689</td>
<td>0.2334</td>
<td>0.2867</td>
<td>0.3295</td>
</tr>
<tr>
<td>( E_{br} )</td>
<td>0.3800</td>
<td>0.3766</td>
<td>0.3274</td>
<td>0.2719</td>
<td>0.2269</td>
<td>0.1935</td>
<td>0.1689</td>
</tr>
<tr>
<td>( E_{rr} )</td>
<td>0.8520</td>
<td>0.1475</td>
<td>0.1287</td>
<td>0.1139</td>
<td>0.1044</td>
<td>0.0988</td>
<td>0.0955</td>
</tr>
</tbody>
</table>

The Table 3 and Fig 6 shows that when the service rate rises, \( E_n \) decreases, the likelihood of being in a normal condition declines, and the likelihood of being on vacation rises. There are insignificant changes in breakdown state probabilities. As \( E_n \) drops, so does the \( E_{br} \) as well as the \( E_{rr} \). The likelihood that the second server would remain idle diminishes as server one serves more customers.

![Fig. 6 Effect of \( \mu_1 \) on different performance measures](image1)

![Fig. 7 Effect of \( \mu_2 \) on different performance measures](image2)

9.1.4 Effect of service rate of Server 2

To observe the effect of service rate, let the fixed parameters are: \( \lambda = 4, \mu_1 = 4, \alpha = 0.3, \beta = 2, \xi = 1.1, p = 0.5, f = 0.8, r = 0.2, N = 8, F = 4. \)

As illustrated from the Table 4 and Fig 7, there is slight change in the \( P_n \) with increasing \( \mu_2 \) while \( P_v \) decreases but \( P_b \) increases. There is a quick increment
Table 4

<table>
<thead>
<tr>
<th>$\mu_2$</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_n$</td>
<td>0.7184</td>
<td>0.6637</td>
<td>0.6784</td>
<td>0.6972</td>
<td>0.7114</td>
<td>0.7215</td>
<td>0.7288</td>
</tr>
<tr>
<td>$P_b$</td>
<td>0.0471</td>
<td>0.0879</td>
<td>0.1155</td>
<td>0.1337</td>
<td>0.1463</td>
<td>0.1533</td>
<td>0.1621</td>
</tr>
<tr>
<td>$P_v$</td>
<td>0.2343</td>
<td>0.2482</td>
<td>0.2600</td>
<td>0.1689</td>
<td>0.1422</td>
<td>0.1230</td>
<td>0.1090</td>
</tr>
<tr>
<td>$P_i$</td>
<td>0.0172</td>
<td>0.1009</td>
<td>0.1959</td>
<td>0.2719</td>
<td>0.3294</td>
<td>0.3734</td>
<td>0.4079</td>
</tr>
<tr>
<td>$E_{br}$</td>
<td>0.1097</td>
<td>0.1263</td>
<td>0.1204</td>
<td>0.1139</td>
<td>0.1092</td>
<td>0.1060</td>
<td>0.1038</td>
</tr>
<tr>
<td>$E_{rr}$</td>
<td>0.7881</td>
<td>0.7153</td>
<td>0.6528</td>
<td>0.6129</td>
<td>0.5874</td>
<td>0.5704</td>
<td>0.5585</td>
</tr>
</tbody>
</table>

in $P_i$ with rising $\mu_2$. $E_n$ decreases gradually with increasing $\mu_2$ so does the $E_{br}$ and $E_{rr}$.

9.1.5 Effect of breakdown rate

To observe the effect of breakdown rate, let the fixed parameters are: $\lambda = 4, \mu_1 = 4, \mu_2 = 2, \beta = 2, \xi = 1.1, p = 0.5, f = 0.8, r = 0.2, N = 8, F = 4$

Table 5

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.2</th>
<th>0.5</th>
<th>0.8</th>
<th>1.1</th>
<th>1.4</th>
<th>1.7</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n$</td>
<td>0.7152</td>
<td>0.6629</td>
<td>0.6158</td>
<td>0.5734</td>
<td>0.5353</td>
<td>0.5012</td>
<td>0.4705</td>
</tr>
<tr>
<td>$P_b$</td>
<td>0.1058</td>
<td>0.1859</td>
<td>0.2560</td>
<td>0.3173</td>
<td>0.3710</td>
<td>0.4181</td>
<td>0.4595</td>
</tr>
<tr>
<td>$P_v$</td>
<td>0.1788</td>
<td>0.1510</td>
<td>0.1281</td>
<td>0.1092</td>
<td>0.0935</td>
<td>0.0806</td>
<td>0.0698</td>
</tr>
<tr>
<td>$P_i$</td>
<td>0.2780</td>
<td>0.2602</td>
<td>0.2436</td>
<td>0.2283</td>
<td>0.2143</td>
<td>0.2015</td>
<td>0.1897</td>
</tr>
<tr>
<td>$E_{br}$</td>
<td>0.1109</td>
<td>0.1194</td>
<td>0.1265</td>
<td>0.1325</td>
<td>0.1374</td>
<td>0.1416</td>
<td>0.1451</td>
</tr>
<tr>
<td>$E_{rr}$</td>
<td>0.6019</td>
<td>0.6334</td>
<td>0.6608</td>
<td>0.6847</td>
<td>0.7055</td>
<td>0.7235</td>
<td>0.7393</td>
</tr>
</tbody>
</table>

If the server breaks down too frequently, it will spend more time in the breakdown state, increasing As seen from Table 5 and Fig 8, The $P_b$ increases. As the server remains in the breakdown state longer, fewer consumers are served, and the likelihood of the server returning to the normal state or vacation state decreases which will result increment in $E_n$ and so in $E_{br}$ and $E_{rr}$. $P_i$ also decrease with increase in breakdown rate.

9.1.6 Effect of repair rate

To observe the effect of repair rate, let the fixed parameters are: $\lambda = 4, \mu_1 = 2, \mu_2 = 2, \alpha = 0.3, \xi = 1.1, p = 0.5, f = 0.8, r = 0.2, N = 8, F = 4$

As soon as the server is repaired, it will start operating normally again, decreasing $P_b$, $E_n$ and increasing $P_n$ illustrated from Table 5 and Fig 9. $P_v$ and $P_i$ increases gradually. Queue length decreases with increasing repair rate so does the $E_{br}$ and $E_{rr}$.
Fig. 8 Effect of $\alpha$ on different performance measures

Fig. 9 Effect of $\beta$ on different performance measures

Table 6

<p>| $\beta$ | 0.4  | 0.8  | 1.2  | 1.6  | 2.0  | 2.4  | 2.8  |</p>
<table>
<thead>
<tr>
<th></th>
<th>$E_n$</th>
<th>$P_n$</th>
<th>$P_b$</th>
<th>$P_v$</th>
<th>$P_i$</th>
<th>$E_{br}$</th>
<th>$E_{rr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>4.6189</td>
<td>0.4668</td>
<td>0.4295</td>
<td>0.1636</td>
<td>0.1705</td>
<td>0.1402</td>
<td>0.7274</td>
</tr>
<tr>
<td>0.8</td>
<td>4.3370</td>
<td>0.5905</td>
<td>0.2746</td>
<td>0.1347</td>
<td>0.2214</td>
<td>0.1301</td>
<td>0.6722</td>
</tr>
<tr>
<td>1.2</td>
<td>4.1875</td>
<td>0.6462</td>
<td>0.2026</td>
<td>0.1510</td>
<td>0.2468</td>
<td>0.1226</td>
<td>0.6430</td>
</tr>
<tr>
<td>1.6</td>
<td>4.0949</td>
<td>0.6774</td>
<td>0.1609</td>
<td>0.1615</td>
<td>0.2620</td>
<td>0.1174</td>
<td>0.6250</td>
</tr>
<tr>
<td>2.0</td>
<td>4.0323</td>
<td>0.6972</td>
<td>0.1337</td>
<td>0.1689</td>
<td>0.2719</td>
<td>0.1139</td>
<td>0.6129</td>
</tr>
<tr>
<td>2.4</td>
<td>3.9871</td>
<td>0.7018</td>
<td>0.1145</td>
<td>0.1746</td>
<td>0.2789</td>
<td>0.1113</td>
<td>0.6042</td>
</tr>
<tr>
<td>2.8</td>
<td>3.9531</td>
<td>0.7206</td>
<td>0.1002</td>
<td>0.1790</td>
<td>0.2839</td>
<td>0.1095</td>
<td>0.5977</td>
</tr>
</tbody>
</table>

9.1.7 Effect of feedback probability

To observe the effect of feedback probability, let the fixed parameters are:
$\lambda = 4, \mu_1 = 4, \mu_2 = 2, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8, F = 4$

Table 7

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_n$</td>
<td>$P_n$</td>
<td>$P_b$</td>
<td>$P_v$</td>
<td>$P_i$</td>
<td>$E_{br}$</td>
<td>$E_{rr}$</td>
</tr>
<tr>
<td>0.2</td>
<td>4.3911</td>
<td>0.7167</td>
<td>0.0998</td>
<td>0.1835</td>
<td>0.1434</td>
<td>0.1248</td>
<td>0.6818</td>
</tr>
<tr>
<td>0.3</td>
<td>4.2729</td>
<td>0.7050</td>
<td>0.1096</td>
<td>0.1854</td>
<td>0.1789</td>
<td>0.1213</td>
<td>0.6589</td>
</tr>
<tr>
<td>0.4</td>
<td>4.1524</td>
<td>0.6988</td>
<td>0.1290</td>
<td>0.1803</td>
<td>0.2216</td>
<td>0.1177</td>
<td>0.6358</td>
</tr>
<tr>
<td>0.5</td>
<td>4.0323</td>
<td>0.6973</td>
<td>0.1309</td>
<td>0.1690</td>
<td>0.2720</td>
<td>0.1139</td>
<td>0.6129</td>
</tr>
<tr>
<td>0.6</td>
<td>3.9157</td>
<td>0.6993</td>
<td>0.1338</td>
<td>0.1524</td>
<td>0.3297</td>
<td>0.1101</td>
<td>0.5910</td>
</tr>
<tr>
<td>0.7</td>
<td>3.8063</td>
<td>0.7035</td>
<td>0.1483</td>
<td>0.1320</td>
<td>0.3944</td>
<td>0.1063</td>
<td>0.5707</td>
</tr>
<tr>
<td>0.8</td>
<td>3.7067</td>
<td>0.7087</td>
<td>0.1645</td>
<td>0.1090</td>
<td>0.4653</td>
<td>0.1029</td>
<td>0.5525</td>
</tr>
</tbody>
</table>

As observed from Table 7 and Fig 10, $E_n$ falls slowly as the probability of receiving positive feedback rises, which also causes $E_{br}$ and $E_{rr}$ to decline. The probability that a second server would be idle rises as the probability
of receiving positive response rises, while $P_v$ falls and $P_b$ rises. $P_n$ slightly changes as it gently decreases at first and then increases.

![Fig. 10](image)  
**Fig. 10** Effect of $p$ on different performance measures

![Fig. 11](image)  
**Fig. 11** Effect of $f$ on different performance measures

### 9.1.8 Effect of re-joining probability

To observe the effect of re-joining probability, let the fixed parameters are: $\lambda = 4$, $\mu_1 = 4$, $\mu_2 = 2$, $\alpha = 0.3$, $\beta = 2$, $\xi = 1.1$, $p = 0.5$, $r = 0.2$, $N = 8$, $F = 4$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n$</td>
<td>0.7126</td>
<td>0.7094</td>
<td>0.7061</td>
<td>0.7029</td>
<td>0.7002</td>
<td>0.6982</td>
<td>0.6973</td>
</tr>
<tr>
<td>$P_b$</td>
<td>0.1967</td>
<td>0.1847</td>
<td>0.1732</td>
<td>0.1624</td>
<td>0.1522</td>
<td>0.1426</td>
<td>0.1338</td>
</tr>
<tr>
<td>$P_v$</td>
<td>0.0906</td>
<td>0.1060</td>
<td>0.1207</td>
<td>0.1347</td>
<td>0.1476</td>
<td>0.1592</td>
<td>0.1690</td>
</tr>
<tr>
<td>$P_i$</td>
<td>0.5220</td>
<td>0.4746</td>
<td>0.4292</td>
<td>0.3860</td>
<td>0.3453</td>
<td>0.3072</td>
<td>0.2720</td>
</tr>
<tr>
<td>$E_b$</td>
<td>0.1005</td>
<td>0.1025</td>
<td>0.1046</td>
<td>0.1068</td>
<td>0.1091</td>
<td>0.1115</td>
<td>0.1139</td>
</tr>
<tr>
<td>$E_r$</td>
<td>0.5404</td>
<td>0.5504</td>
<td>0.5613</td>
<td>0.5731</td>
<td>0.5857</td>
<td>0.5991</td>
<td>0.6129</td>
</tr>
</tbody>
</table>

As seen from Table 8 and Fig 11, As re-joining probability increases, the unsatisfied customers opted for re-service will also increase resulting an increment in $E_n$, $E_b$, and $E_r$. $P_b$ decreases while $P_v$ increase. As $f$ increases so customers in the queue for server 2 will increase hence $P_i$ will also increase.

### 9.1.9 Effect of reneging probability

To observe the effect of reneging probability, let the fixed parameters are: $\lambda = 4$, $\mu_1 = 4$, $\mu_2 = 2$, $\alpha = 0.3$, $\beta = 2$, $\xi = 1.1$, $p = 0.5$, $f = 0.8$, $N = 8$, $F = 4$.

Observed from Table 9 and Fig 12, Increase in reneging probability increases
the expected reneging rates, consequently the queue length reduces, expected balking rate depending on queue length also decrease. \( P_b \) and \( P_v \) grow while \( P_n \) decreases.

![Effect of r on different performance measures](image)

**Fig. 12** Effect of \( r \) on different performance measures

### 9.1.10 Effect of system capacity

To observe the effect of system capacity, let the fixed parameters are: \( \lambda = 4, \mu_1 = 4, \mu_2 = 2, \alpha = 0.3, \beta = 2, \xi = 1.1, p = 0.5, f = 0.8, r = 0.2, F = 4. \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_n )</td>
<td>3.3684</td>
<td>3.7037</td>
<td>4.0323</td>
<td>4.3526</td>
<td>4.6627</td>
<td>4.9597</td>
<td>5.2399</td>
</tr>
<tr>
<td>( P_n )</td>
<td>0.6709</td>
<td>0.6851</td>
<td>0.6973</td>
<td>0.7077</td>
<td>0.7166</td>
<td>0.7243</td>
<td>0.7308</td>
</tr>
<tr>
<td>( P_b )</td>
<td>0.1382</td>
<td>0.1356</td>
<td>0.1338</td>
<td>0.1325</td>
<td>0.1315</td>
<td>0.1308</td>
<td>0.1303</td>
</tr>
<tr>
<td>( P_v )</td>
<td>0.1909</td>
<td>0.1793</td>
<td>0.1690</td>
<td>0.1599</td>
<td>0.1519</td>
<td>0.1449</td>
<td>0.1389</td>
</tr>
<tr>
<td>( P_i )</td>
<td>0.2750</td>
<td>0.2717</td>
<td>0.2720</td>
<td>0.2740</td>
<td>0.2767</td>
<td>0.2797</td>
<td>0.2827</td>
</tr>
<tr>
<td>( E_{br} )</td>
<td>0.2055</td>
<td>0.1503</td>
<td>0.1139</td>
<td>0.0880</td>
<td>0.0686</td>
<td>0.0537</td>
<td>0.0419</td>
</tr>
<tr>
<td>( E_{rr} )</td>
<td>0.4818</td>
<td>0.5480</td>
<td>0.6129</td>
<td>0.6764</td>
<td>0.7380</td>
<td>0.7969</td>
<td>0.8526</td>
</tr>
</tbody>
</table>
As seen from Table 10 and Fig 13, Increasing the system capacity will increase the queue length and so $E_{br}$. Expected balking rates will decrease with increase in N as balking rates are inversely proportional to system capacity. $P_b$ will rise, whilst $P_v$ and $P_b$ will fall.

![Fig. 13 Effect of N on different performance measures](image1)

![Fig. 14 Effect of F on different performance measures](image2)

9.1.11 Effect of admission control policy number F

To observe the effect of admission control policy number F, let the fixed parameters are: $\lambda = 4, \mu_1 = 4, \mu_2 = 2, \alpha = 0.3, \beta = 2, \xi = 1.1, p = 0.5, f = 0.8, r = 0.2, N = 12.$

<table>
<thead>
<tr>
<th>F</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_n$</td>
<td>4.3521</td>
<td>4.5151</td>
<td>4.6027</td>
<td>4.8023</td>
<td>4.9374</td>
<td>5.0688</td>
<td>5.1957</td>
</tr>
<tr>
<td>$P_n$</td>
<td>0.7143</td>
<td>0.7141</td>
<td>0.7166</td>
<td>0.7202</td>
<td>0.7240</td>
<td>0.7276</td>
<td>0.7308</td>
</tr>
<tr>
<td>$P_v$</td>
<td>0.1363</td>
<td>0.1330</td>
<td>0.1315</td>
<td>0.1308</td>
<td>0.1303</td>
<td>0.1300</td>
<td>0.1298</td>
</tr>
<tr>
<td>$P_i$</td>
<td>0.1493</td>
<td>0.1529</td>
<td>0.1519</td>
<td>0.1490</td>
<td>0.1457</td>
<td>0.1424</td>
<td>0.1393</td>
</tr>
<tr>
<td>$P_b$</td>
<td>0.2749</td>
<td>0.2746</td>
<td>0.2767</td>
<td>0.2797</td>
<td>0.2830</td>
<td>0.2866</td>
<td>0.2907</td>
</tr>
<tr>
<td>$E_{br}$</td>
<td>0.0619</td>
<td>0.0650</td>
<td>0.0686</td>
<td>0.0731</td>
<td>0.0791</td>
<td>0.0874</td>
<td>0.0995</td>
</tr>
<tr>
<td>$E_{rr}$</td>
<td>0.6776</td>
<td>0.7091</td>
<td>0.7380</td>
<td>0.7655</td>
<td>0.7922</td>
<td>0.8183</td>
<td>0.8435</td>
</tr>
</tbody>
</table>

Observed from Table 11 and Fig 14, with increase in F, queue length increases slowly, resulting an modest increment in expected balking and reneging rates. $P_b$ drops when $P_n$ likewise rises. $P_v$ is nearly constant. $P_i$ grows gradually as F rises.
9.2 Economic Analysis

The impact of various parameters on expected cost, revenue and profit is explored here. To analyze this, the fixed cost parameters for the cost, revenue and profit function explained in section 8 are $C_1 = 30, C_2 = 40, C_3 = 25, C_4 = 15, C_5 = 30, C_6 = 10, C_7 = 10, C_8 = 20, C_{10} = 30, C_{11} = 20, C_{12} = 50, R_{v1} = 250, R_{v2} = 150$ taken according to Mahla [20].

9.2.1 Effect of $\lambda$, $\mu_1$, $\mu_2$ and $p$

The fixed parameters are

- $\mu_1 = 4, \mu_2 = 2, p = 0.5, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8$ for varying $\lambda$.
- $\lambda = 4, \mu_2 = 2, p = 0.5, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8$ for varying $\mu_1$.
- $\lambda = 4, \mu_1 = 4, p = 0.5, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8$ for varying $\mu_2$.
- $\lambda = 10, \mu_1 = 6, \mu_2 = 4, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8$ for varying $p$.

And the results concluded from the graphs are as

- As seen from 15 with an increase in $\lambda$, costs rise gradually. A significant increase in arrival rate leads to a large number of customers in the system, and as a result, the total expected cost increases. Revenue curve exhibits a similar behavior to profit curve, rising quickly at first and then gradually increasing. When $\lambda$ is large enough, customers using the system may renge due to the long queue length, implying a modest rise in the revenue.

- Service rate for server 1 increases, so does the expected number of customers getting served per unit time which will increase the expected cost, and a higher number of customers getting served raises the overall expected revenue. Thus, the profit will also increase as seen from 15.

- Service rate for the second server grows means a greater number of re-service will be provided, resulting in an increase in the system’s expected cost. Additionally, by providing re-services, revenue will be generated, increasing the expected revenue as well. Since the rise in revenue is higher than the re-service cost, leads to more profit.

- With increasing positive feedback probability, the expected number of customers in the system and the probability of customers being in a normal state vary minimally, as seen from Table 7 and Fig 10; therefore, cost curve changes to a small extent. If the probability of positive feedback increases, the number of customers getting re-served decreases, and the revenue generated by re-service decreases, lowering the system’s expected profit.
9.2.2 Effect of $f$, $\alpha$, $\beta$ and $\xi$

The fixed parameters are

- $\lambda = 10$, $\mu_1 = 6$, $\mu_2 = 4$, $\alpha = 0.3$, $\beta = 2$, $\xi = 1.1$, $p = 0.5$, $r = 0.2$, $N = 8$, $F = 4$ for varying $f$.
- $\lambda = 10$, $\mu_1 = 6$, $\mu_2 = 2$, $\beta = 1$, $\xi = 1.1$, $p = 0.5$, $f = 0.8$, $r = 0.2$, $N = 8$, $F = 4$ for varying $\alpha$.
- $\lambda = 10$, $\mu_1 = 6$, $\mu_2 = 2$, $\alpha = 0.3$, $\xi = 1.1$, $p = 0.5$, $f = 0.8$, $r = 0.2$, $N = 8$, $F = 4$ for varying $\beta$.
- $\lambda = 4$, $\mu_1 = 2$, $\mu_2 = 2$, $\alpha = 0.3$, $\beta = 2$, $p = 0.5$, $f = 0.8$, $r = 0.2$, $N = 8$, $F = 4$ for varying $\xi$.

- If the server breaks down occurs more frequently, that leads to increased $P_b$, $E_n$, $E_{br}$ and $E_{rr}$ as shown in table 5 and Fig 8, which will raise the expected cost of the system. As the server spends more time in the breakdown state, fewer customers get serviced, leading in a fall in expected revenue and, eventually, expected profit.
Fig. 16  Cost, Profit and Revenue for variable parameters

- As soon as the server get repaired, it will resume operating normally, resulting a decrement in $P_b$, $E_n$ as demonstrated from table 6 and Fig 9, that will reduce the cost of the system. With an increase in $\beta$, there will be increment in $P_n$. Additionally, as the server operates more in normal state, more customers will be served, increasing the system’s expected revenue and, ultimately, its profit, as shown in 16.

- With increase in $\xi$ there will be increment in $P_n$ and decrement in $P_v$ and $E_n$ as seen from table 2 and Fig 5 resulting reduction in the cost. When the server returns from vacation, it will service more customers, increasing the system’s expected revenue and, in turn, its expected profit, as seen from Fig 16.

- With increase in the re-joining probability, there are slight changes in $E_n$, $P_n$ and $E_{br}$ as illustrated in table 8 and Fig 11 so, there are very small changes in the cost of the system. If re-joining probability increases then the revenue generated by re-service customers will also increase resulting
an increment in the total revenue of the system as shown in Fig [16].

**Fig. 17** Cost, Profit and Revenue for variable parameters

- As indicated in table [10], the expected number of customers in the system rises as the system’s capacity increases, which raises the expected cost of the system. A large number of expected customers mean a large number of customers served, which will increase the revenue of system. Because the growth in revenue is greater than the growth in costs, profit will also rise.

- With increasing reneging rate, the expected reneging customers will increase that will increase the cost of the system. A large number of reneging customers will reduce the queue length and the probability of server 1 being on normal state, resulting in fewer customers being served and lowering the system’s expected revenue.

- With increasing the F, expected number of customers in the system increases, increasing the cost of the system. But there is very few difference
in the $P_1$, $P_2$, $P_3$, and $P_4$ as illustrated in table 11 and Fig 14 so, the expected revenue of the system remain almost same, resulting an decrement in the profit.

9.3 Effect of F-policy on profit of the system

The model we considered here is the model with F-policy (say it model 1) for controlling arrivals of customers; to determine whether the applied F-policy is economically efficient for the system, consider a fresh model without F-policy by fixing $N=F$ (say it model 2). To observe the economic difference between both models, the profit comparison of both models with changing parameters is done here. The cost parameters to draw graphs for the profit of both models are taken according to section 9.2 and other fixed parameters are

- $\mu_1 = 6, \mu_2 = 4, p = 0.5, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8$ for varying $\lambda$.
- $\lambda = 6, \mu_2 = 4, p = 0.5, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8$ for varying $\mu_1$.
- $\lambda = 6, \mu_1 = 4, \mu_2 = 2, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2, N = 8$ for varying $p$.
- $\lambda = 10, \mu_2 = 6, \mu_1 = 4, p = 0.5, \alpha = 0.3, \beta = 2, \xi = 1.1, f = 0.8, r = 0.2$ for varying $N$.

As illustrated in Fig 18, a comparative analysis is carried out for both models with varying parameters $\lambda, \mu_1, p$ and $N$. It is observed that model 1 is economically more efficient than model 2, which means F-policy introduction adds economic efficiency to the system.

10 Conclusion

In this study analysis of two heterogeneous servers with feedback customers with impatient behavior is done where service interruption as breakdown of server is taken. The system’s performance indicators are provided. Then, using the results of the performance study, a cost model is created to show how different system parameters affect various attributes as well as the system’s overall predicted cost, revenue, and profit. Effect of F-policy on expected profit of the system is concluded by considering two different models. It will be worthwhile to take a multi-server queueing system into consideration for future development. For further work, it will be interesting to consider a multi-server queueing system with multiple working vacations and server breakdown for each server. Additionally, this research can be expanded to a related model that takes numerous kinds of arrival and service distribution into account.
Fig. 18 Effect of different parameters on the profit of the system

References


31. Zhang, Y., Yue, D., Yue, W.: Analysis of an M/M/1/N queue with balking, reneging and server vacations
Figures

Figure 0