Fuzzy Fractional Order Volterra-Fredholm Integro-Differential Equations Using MADM

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Fuzzy Fractional Order Volterra-Fredholm Integro-Differential Equations Using MADM

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Abstract

This paper investigates solutions to the fractional integro-differential equation (FIDE) of fuzziness. Here we consider, In the Caputo sense, we look at the fuzzy fractional Volterra-Fredholm integro-differential equation (CFFVFIDE) and use fixed point theory to prove its existence and uniqueness. Moreover, we apply the modified Adomian decomposition method (MADM) to find a solution to the presented problem, and to support our approach, we offer a few examples. We also give visual analysis to see how the solutions behave.

Keywords: Caputo fractional derivatives, Fuzzy fractional Volterra-Fredholm integro-differential equation, Modified Adomian decomposition method.

2010 MSC: 26A33, 35R09, 34A07, 45J05.

1. Introduction:

Fractional calculus (FC) is a vital branch of applied mathematics. Leibnitz and Newton first introduced the idea of FC in their conversations. After the development of FC, it attracted the interest of physicists, mathematicians, and researchers in applied sciences. The famous mathematician Abel was the first to use fractional derivatives (FD) in 1823 [1], and he also used FC to solve an integral equation (IE) that occurs in the Tautochrone problem.

The Riemann-Liouville (RL) concept’s fractional integral (FI) and fractional derivative (FD) has addressed a few mathematical problems in [2] [3] [1]. Salahshour and colleagues in [5] discovered the interval approximations of the asymptotic solutions of fractional FDEs under the (A-B) derivative. Various models, such as the generalized fractional-order EPQ model, have been built by Rahaman et al. in their study [6]. Researchers encountered ambiguity and uncertainty in several models. Fuzziness has been developed one of the new concepts that are recently being introduced to solve such challenges. The most crucial aspect of fuzzy analysis, which also plays a significant role in mathematics and applied mathematics, is the notion of fuzziness in differential, integral, and integro-differential equations (IDEs).

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Various researchers have studied fuzzy fractional models in recent years and developed novel ideas for fuzzy fractional integral equations (FFIEs), fuzzy fractional differential equations (FFDEs) and fuzzy fractional integro-differential equations (FFIDEs) \[7\]. Then, FFIEs and FFDEs are used to solve a wide range of problems in applied mathematics, including those in physics, electromagnetic, viscoelasticity, medicine, and biology \[8\], as well as geography, chemistry, electrochemistry, economy, and control theory \[9, 10, 11, 12\], where parameters are conferred by fuzzy numbers instead of crisp numbers. The concept of fuzziness and its different operations were first proposed by Zadeh \[13, 14\].

The idea of integrating fuzzy functions was given by Dubois and Prade \[15\]. Following that, Voxman and Goetschel studied basic calculus and worked on the Riemann integral \[16\]. Agarwal et al. \[17\] introduced the idea of Riemann-Liouville fractional differential equations (RLFDEs) with uncertainty for the first time. M -fractional derivatives under interval uncertainty, as well as their properties and applications, have been explored by Salahshour and his colleagues in \[18\]. The article "A novel coronavirus infection system using a FFDE under Caputo's sense" by S. Ahmad and others may be found in \[19\]. To solve systems of FFDEs with fuzzy initial conditions under fuzzy Caputo differentiability (CD), Chakraverty et al. investigated time fractional fuzzy vibration equation of large membranes using double parametric based residual power series method in \[20\]. The concept of fuzziness is addressed in \[21\] using various fractional equations \[22\]. As regards the Fredholm integro-differential problem, Pachpatte has also investigated the existence and uniqueness (EAU) of the solution \[23\]. Sell and Miller have analyzed the second kind of Volterra integral equations \[24\]. The fuzzy Volterra integro-differential equations (FVIDEs) have been studied by Mikaeilvand et al. \[25\].

For the solution of fractional Volterra-Fredholm integro-differential equations (FVFIDEs), Hamoud and Ghadle have published a modified Laplace decomposition approach. Additionally, they have employed MADM to obtain the approximations of fractional integro-differential equation solutions \[26\] presented the Adomian polynomials. Wazwaz \[27\] originally proposed the MADM. This approach presupposes that the function \(\lambda(t)\) may be decomposed into its components, \(\lambda_1(t)\) and \(\lambda_2(t)\). Under this premise, the following is composed as

\[
\lambda(t) = \lambda_1(t) + \lambda_2(t)
\]

This decomposition is appropriate where the function \(\lambda(t)\) has several components that may be divided into two separate parts \[28\]. \(\lambda(t)\) in this situation can be any polynomial or trigonometric function added together. The selection of \(\lambda_1\) be made carefully to increase the method’s efficiency; \(\lambda_1\) can be chosen as a single term of \(\lambda(t)\) or at least many terms, if possible, and \(\lambda_2\) will be made up of the remaining terms of \(\lambda(t)\). This approach minimizes the number of computations and the cost of computing processes of MADM.

It is possible to solve linear and non-linear problems using either decomposition methods (standard or modified), although the modified decomposition method (MADM) provides several advantages over the standard.
Our investigation of the FFVFIDE in the Caputo sense of order $\beta$ of the form:

$$^cD^\beta \tilde{z}(t, \gamma) = h(t) + a(t)\tilde{z}(t, \gamma) + \int_0^t \Psi_1(t, \eta)\mu_1(\tilde{z}(\eta, \gamma))d\eta + \int_0^1 \Psi_2(t, \eta)\mu_2(\tilde{z}(\eta, \gamma))d\eta, \quad (1.1)$$

with the initial condition (IC) $\tilde{z}(0, \gamma) = [\tilde{z}(0, \gamma), \tilde{z}(0, \gamma)]$, for $\gamma \in [0, 1]$; $^cD^\beta$ is the CFDs, and $\tilde{z}(t, \gamma)$ is a fuzzy numbers, $h : U \rightarrow \mathbb{R}$, $\Psi_i : U \times U \rightarrow \mathbb{R}$ for $i = 1, 2$ are continuous functions (CFs), $\mu_i : \mathbb{R} \rightarrow \mathbb{R}$ for $i = 1, 2$ are Lipschitz continuous functions (LCF), here $U = [0, 1]$.

The structure of this paper is as follows: The introduction is presented in Section 1. The fundamentals of FC and IDE are presented in Section 2. It also includes a list of fuzzy set theory concepts. The existence and uniqueness (EAU) of the solution to the problem under consideration are provided in Section 3. The method for locating the proposed equation’s solution is described in Section 4. Some examples and graphs are included in Section 5. While Section 6 contains the paper’s conclusion.

2. Preliminaries:

Definition 2.1 [29, 30]

Fuzzy set $\tilde{z}$ of the real line $\mathbb{R}^1$ is a fuzzy number if $\tilde{z}$ is satisfied convex, normalized, and its membership function is piecewise continuous.

Definition 2.2 [31]

Usually, a fuzzy number $\tilde{z}$ is represented by $\tilde{z} = [\tilde{z}(\gamma), \tilde{z}(\gamma)]$ in its parametric form, satisfying the conditions below.

(I) $\tilde{z}(\gamma)$ is a left continuous, non-decreasing, and bounded between closed intervals $0$ to $1$.

(II) $\tilde{z}(\gamma)$ is a right continuous, non-increasing and bounded between closed interval $0$ to $1$.

(III) $\tilde{z}(\gamma) \leq \tilde{z}(\gamma)$, where $\gamma$ belongs to closed interval $0$ to $1$.

Definition 2.3 [32]

Here, $R_F = \{\tilde{z} : [0, 1]\}$ is the class of fuzzy subsets on the real axis. $R_F$ is known as the space of fuzzy numbers if $\tilde{z}$ is normal, fuzzy convex, upper semicontinuous, and its closure $cl \{t \in R | \tilde{z}(t, \gamma) > 0\}$ is closed and bounded (or compact).
**Definition 2.4** [33]

The fractional integral of a function \( g(t) \) in the RL type of order \( \beta > 0 \) is described as

\[
J^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \eta)^{\beta - 1} g(\eta) d\eta, \quad t > 0, \beta \in \mathbb{R}^+ \tag{2.1}
\]

\[
J^0 g(t) = g(t) \tag{2.2}
\]

**Definition 2.5** [34]

Let \( \tilde{z} \in C^F[a, b] \cap L^F[a, b] \). In the RL definition of order \( \beta > 0 \) of \( \tilde{z} \), the fuzzy fractional integral is written as

\[
J^\beta \tilde{z}(t, \gamma) = [J^\beta \tilde{z}(t, \gamma), J^\beta \tilde{z}(t, \gamma)] \tag{2.3}
\]

where, \( C^F[a, b] \) is the space of all continuous fuzzy valued function on \([a, b]\), and \( L^F \) is the space of all Lebesgue integrable fuzzy valued function on \([a, b]\). Also

\[
J^\beta \tilde{z}(t, \gamma) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \eta)^{\beta - 1} \tilde{z}(\eta, \gamma) d\eta, \quad t > 0, \beta \in \mathbb{R}^+ \tag{2.4}
\]

and

\[
J^\beta \tilde{z}(t, \gamma) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \eta)^{\beta - 1} \tilde{z}(\eta, \gamma) d\eta, \quad t > 0, \beta \in \mathbb{R}^+ \tag{2.5}
\]

**Definition 2.6** [33]

The FD of \( g(y) \) of order \( \beta \in \mathbb{R}^+ \) is the representation of Caputo type which is expressed as

\[
cD^\beta_t g(t) = J^{m-\beta} D^m g(t)
\]

\[
= \left\{ \frac{1}{\Gamma(m - \beta)} \int_0^t (t - \eta)^{m-\beta-1} \frac{d^m g(\eta)}{d\eta^m} d\eta, \quad m - 1 < \beta < m \right\}
\]

\[
= \frac{d^m g(t)}{d t^m}, \beta = m, m \in \mathbb{N}, \tag{2.7}
\]

\[
= \frac{d^m g(t)}{d t^m}, \beta = m, m \in \mathbb{N}, \tag{2.8}
\]

**Definition 2.7** [35]

Suppose \( \tilde{z} : (t_1, t_2) \to R^F \) and \( I_0 \in (t_1, t_2) \). If an element \( \tilde{z}'(I_0) \in R^F \) exists, then we say \( \tilde{z} \) is differentiable at \( I_0 \), such that

(1) for all \( h > 0 \) sufficiently small, \( \exists \pi(I_0 + h) \odot \tilde{z}(I_0), \exists \pi(I_0) - \odot \tilde{z}(I_0 - h) \) such that

\[
\lim_{h \to 0} \frac{\pi(I_0 + h) \odot \tilde{z}(I_0)}{h} = \lim_{h \to 0} \frac{\pi(I_0) \odot \tilde{z}(I_0 - h)}{h} = \tilde{z}'(I_0) \tag{2.9}
\]
(2) for all \( h > 0 \) sufficiently small, \( \exists \tilde{z}(I_0) \odot \tilde{z}(I_0 + h), \exists \tilde{z}(I_0 - h) \ominus \tilde{z}(I_0) \) such that

\[
\lim_{{h \to 0}} \frac{\tilde{z}(I_0) \odot \tilde{z}(I_0 + h)}{h} = \lim_{{h \to 0}} \frac{\tilde{z}(I_0 - h) \ominus \tilde{z}(I_0)}{-h} = \tilde{z}'(I_0) \quad (2.10)
\]

For convenience, we refer to \( \tilde{z} \) as (1)-differentiable on \((a,b)\) if it is differentiable under the assumption of Case(1), and similarly for (2)-differentiability for Case(2).

Note

We focus on the type (1)-differentiability in this paper.

**Theorem 2.8** [36]

Assume that \( \tilde{z} : R \to R^F \) is a fuzzy function and is represented as \( \tilde{z}(t; \gamma) = [\pi(t; \gamma), \bar{z}(t; \gamma)] \), where \( \gamma \in [0,1] \).

1. If \( \tilde{z} \) is a (1)-differentiable function (DF), then \( z(t; \gamma) \) and \( \bar{z}(t; \gamma) \) are DFs and
   \[ [\tilde{z}'(t)]^\gamma = [\bar{z}(t; \gamma), \pi(t; \gamma)] \]

2. If \( \tilde{z} \) is a (2)-DF, then \( z(t; \gamma) \) and \( \bar{z}(t; \gamma) \) are DFs and
   \[ [\tilde{z}'(t)]^\gamma = [\bar{z}(t; \gamma), \tilde{z}(t; \gamma)] \]

**Theorem 2.9** [33]

Schauder’s Fixed Point Theorem (SFPT): Consider \( A \) is a Banach space and assume \( S \) to be a convex and closed subset of \( A \). If \( T : S \to S \) is a map such that the set \( \{Tx : x \in S\} \) is closed and bounded (or compact) in \( A \), then \( T \) has at least one fixed point (FP) \( Tx^* = x^* \)

**Theorem 2.10** [33]

Banach Contraction Principle (BCP): Let \( A, d \) be a complete metric space, then each contraction mapping \( T : A \to A \) has a unique fixed point (UFP) \( x \) of \( T \) in \( A \), that is \( Tx = x \)

3. **Existence and uniqueness of solution (EAUS)** [26] [37]

The following hypothesis is presented in this part before we analyze the EAUS for equation (1.1)

\( H_1 \): For any \( \tilde{z}_1, \tilde{z}_2 \in C(U, R^F) \) there exist two constants \( c_1, c_2 > 0 \) such that

\[ |\mu_1(\tilde{z}_1(t, \gamma)) - \mu_1(\tilde{z}_2(t, \gamma))| \leq c_1 |\tilde{z}_1 - \tilde{z}_2| \]

and
\[ |\mu_2(\zeta_1(t, \gamma) - \mu_2(\zeta_2(t, \gamma)))| \leq c_2 |\zeta_1 - \zeta_2| \]

**H2:** There are functions \( \Psi_1^*, \Psi_2^* \) that exist for the set of all positive CFs on \( J = \{(t, \eta) \in R \times R : 0 \leq \eta \leq t \leq 1\} \),

\[ \Psi_1^* = \sup_{\eta \in [0,1]} \int_0^t |\Psi_1(t, \eta, \gamma)| d\eta < \infty, \]

and

\[ \Psi_2^* = \sup_{\eta \in [0,1]} \int_0^t |\Psi_2(t, \eta, \gamma)| d\eta < \infty. \]

**H3:** Here, \( a, g : U \to R \) to be two CFs.

**Theorem 3.1** \cite{26, 37}

Let us assume \( H_2 \) and \( H_3 \) should hold, thus. There exists at least one solution to equation (1.1) if \( \frac{||\alpha||}{\Gamma(\beta+1)} < 1 \).

Proof. The self operator \( C(U, R^F) \to C(U, R^F) \) is described by

\[(T\zeta)(t, \gamma) = \zeta_0 + J^\beta(a(t)\zeta(t, \gamma)) + h(t) + \int_0^t \Psi_1(t, \eta, \mu_1(\zeta(\eta, \gamma)) d\eta + \int_0^t \Psi_2(t, \eta, \mu_2(\zeta(\eta, \gamma)) d\eta, \forall \eta \in U \]

To do this, we first prove that the operator \( T \) is completely continuous,

\( T \) is continuous. Assume that a sequence \( \zeta_n \) converges to \( \zeta \) in \( C(U, R^F) \). We have for any \( \zeta_n, \zeta \in C(U, R^F) \), and for each \( t \) belongs to \( U \), then

\[ |T\zeta_n(t, \gamma) - T\zeta(t, \gamma)| = |\zeta_0 + J^\beta(a(t)\zeta_n(t, \gamma)) + h(t) + \int_0^t \Psi_1(t, \eta, \mu_1(\zeta_n(\eta, \gamma)))d\eta + \int_0^t \Psi_2(t, \eta, \mu_2(\zeta_n(\eta, \gamma)))d\eta - \zeta_0 + J^\beta(a(t)\zeta(t, \gamma)) + h(t) + \int_0^t \Psi_1(t, \eta, \mu_1(\zeta(\eta, \gamma)))d\eta + \int_0^t \Psi_2(t, \eta, \mu_2(\zeta(\eta, \gamma)))d\eta| \]

\[ \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |a(s)||\zeta_n(s, \gamma) - \zeta(s, \gamma)||ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left( \int_0^s |\Psi_1(s, \eta, \mu_1((\zeta_n)(\eta, \gamma)))|d\eta + \int_0^s |\Psi_2(s, \eta, \mu_2((\zeta_n)(\eta, \gamma)))|d\eta\right)ds \]

Using the supremum on each side.

\[ ||(T\zeta_n)(t, \gamma) - (T\zeta)(t, \gamma)|| \leq ||a||_\infty ||\zeta_n - \zeta||_\infty \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}ds + \Psi_1^* \|\mu_1(\zeta_n) - \mu_1(\zeta)\|_\infty \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}ds + \Psi_2^* \|\mu_2(\zeta_n) - \mu_2(\zeta)\|_\infty \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}ds \]
we know, $\Psi_1$ and $\Psi_2$ are continuous, $\int_0^t (t-s)^{\beta-1} ds$, is bounded.

So it is determined that $||(T\tilde{z}_n)(t,\gamma) - (T\tilde{z})(t,\gamma)||_\infty \to 0$ as $n \to \infty$. Then $T$ is continuous.

Here, $T$ is mapping from the bounded set into bounded sets in $C(U,R^F)$. for all $\delta > 0$, and $p > 0$ where $p$ is there exist a positive constant such that $\tilde{z} \in B_\delta = \{ \tilde{z} \in C(U,R^F) : ||\tilde{z}_\infty \leq \delta \}$, one has $||\tilde{z}_\infty \leq p$.

Let $\omega_1 = sup_{\tilde{z} \in U \times [0,\delta]} \mu_1(\tilde{z}(t,\gamma)) + 1$ and $\omega_2 = sup_{\tilde{z} \in U \times [0,\delta]} \mu_2(\tilde{z}(t,\gamma)) + 1$. Also for each $t \in U$ and for any $\tilde{z} \in B_\gamma$, thus we have

\[
| (T\tilde{z})(t,\gamma) | = | \tilde{z}_0 + J^\beta (a(t)\tilde{z}(t,\gamma)) + h(t) + \int_0^t \Psi_1(t,\eta)\mu_1(\tilde{z}(\eta,\gamma)) d\eta + \int_0^1 \Psi_2(t,\eta)\mu_2(\tilde{z}(\eta,\gamma)) d\eta |
\]

\[
\leq | \tilde{z}_0 | + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} | a(s) | \tilde{z}(s,\gamma) | ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} | h(s) | ds
\]

\[
+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left( \int_s^t | \Psi_1(s,\eta) | \mu_1((\tilde{z}(\eta,\gamma))) \right) | d\eta + \int_0^1 | \Psi_2(s,\eta) | \mu_2((\tilde{z}(\eta,\gamma))) | d\eta | ds
\]

Again taking supremum on both sides, then we obtain

\[
||(T\tilde{z})(t,\gamma)||_\infty \leq | \tilde{z}_0 | + ||a||_\infty ||\tilde{z}_\infty ||_\infty \frac{t^\beta}{\Gamma(\beta + 1)} + ||h||_\infty \frac{t^\beta}{\Gamma(\beta + 1)} + \frac{\Psi_1^* \omega_1 t^\beta}{\Gamma(\beta + 1)} + \frac{\Psi_2^* \omega_2 t^\beta}{\Gamma(\beta + 1)}
\]

\[
\leq \left( | \tilde{z}_0 | + ||a||_\infty \delta + ||h||_\infty + \Psi_1^* \omega_1 + \Psi_2^* \omega_2 \right) \frac{t^\beta}{\Gamma(\beta + 1)}
\]

\[
\leq p.
\]

Thus, for every $\tilde{z} \in B_\gamma$, we have $||T\tilde{z}|| \leq p$, which implies that $TB_\gamma \subset B_p$.

$T$ is mapping from bounded sets into equi-continuous sets of $C(U,R^F)$. Consider $B_\delta$ be referred above in the same way for all $\tilde{z} \in B_\delta, t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we get

\[
| (T\tilde{z})(t_1,\gamma) - (T\tilde{z})(t_2,\gamma) | = | \tilde{z}_0 + \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2-s)^{\beta-1} a(s)\tilde{z}(s,\gamma) ds + \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2-s)^{\beta-1} h(s) ds
\]

\[
+ \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2-s)^{\beta-1} \left( \int_s^{t_2} \Psi_1(s,\eta)\mu_1((\tilde{z}(\eta,\gamma))) \right) d\eta + \int_0^1 \Psi_2(s,\eta)\mu_2((\tilde{z}(\eta,\gamma))) d\eta | ds
\]

\[
- | \tilde{z}_0 + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1-s)^{\beta-1} a(s)\tilde{z}(s,\gamma) ds + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1-s)^{\beta-1} h(s) ds
\]

\[
+ \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1-s)^{\beta-1} \left( \int_s^{t_1} \Psi_1(s,\eta)\mu_1((\tilde{z}(\eta,\gamma))) \right) d\eta + \int_0^1 \Psi_2(s,\eta)\mu_2((\tilde{z}(\eta,\gamma))) d\eta | ds
\]
Thus we get

\[
\leq \frac{1}{\Gamma(\beta)} \left| \int_{t_2}^{t_1} (t_2 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds \right|
\]

\[
+ \frac{1}{\Gamma(\beta)} \left| \int_{t_2}^{t_1} (t_2 - s)^{\beta - 1} h(s) ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} h(s) ds \right|
\]

\[
+ \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} \left( \int_{t_0}^{s} \Psi_s(s, \eta) \mu_1(\bar{z}(\eta, \gamma)) ds \right) d\eta + \int_{t_0}^{t_1} \Psi_s(s, \eta) \mu_2(\bar{z}(\eta, \gamma)) d\eta ds
\]

\[
- \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} \left( \int_{t_0}^{s} \Psi_s(s, \eta) \mu_1(\bar{z}(\eta, \gamma)) ds \right) d\eta + \int_{t_0}^{t_1} \Psi_s(s, \eta) \mu_2(\bar{z}(\eta, \gamma)) d\eta ds
\]

\[
= \frac{1}{\Gamma(\beta)} \left| \int_{t_2}^{t_1} (t_2 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds \right|
\]

\[
+ \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds + \int_{t_0}^{t_1} (t_2 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds
\]

\[
- \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds + \int_{t_0}^{t_1} (t_2 - s)^{\beta - 1} h(s) ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} h(s) ds
\]

\[
\leq \frac{1}{\Gamma(\beta)} \left| \int_{t_2}^{t_1} (t_2 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} a(s) \bar{z}(s, \gamma) ds \right|
\]

\[
+ \frac{1}{\Gamma(\beta)} \left| \int_{t_1}^{t_2} (t_2 - s)^{\beta - 1} h(s) ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} h(s) ds \right|
\]

\[
+ \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} \left( \int_{t_0}^{s} \Psi_s(s, \eta) \mu_1(\bar{z}(\eta, \gamma)) ds \right) d\eta + \int_{t_0}^{t_1} \Psi_s(s, \eta) \mu_2(\bar{z}(\eta, \gamma)) d\eta ds
\]

\[
- \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} \left( \int_{t_0}^{s} \Psi_s(s, \eta) \mu_1(\bar{z}(\eta, \gamma)) ds \right) d\eta + \int_{t_0}^{t_1} \Psi_s(s, \eta) \mu_2(\bar{z}(\eta, \gamma)) d\eta ds
\]

Thus we get

\[
\left| (T\bar{z})(t_1, \gamma) - (T\bar{z})(t_2, \gamma) \right| \leq \frac{1}{\Gamma(\beta)} \left( \int_{t_2}^{t_1} (t_2 - s)^{\beta - 1} | a(s) | | \bar{z}(s, \gamma) | ds + \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1} | a(s) | | \bar{z}(s, \gamma) | ds \right)
\]

\[
+ \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1} \left( \int_{t_0}^{s} \Psi_s(s, \eta) \mu_1(\bar{z}(\eta, \gamma)) ds \right) d\eta + \int_{t_0}^{t_1} \Psi_s(s, \eta) \mu_2(\bar{z}(\eta, \gamma)) d\eta ds
\]

\[
= I_1 + I_2 + I_3
\]

where

\[
I_1 = \frac{1}{\Gamma(\beta)} \left( \int_{t_2}^{t_1} (t_2 - s)^{\beta - 1} | a(s) | | \bar{z}(s, \gamma) | ds + \int_{t_0}^{t_1} (t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1} | a(s) | | \bar{z}(s, \gamma) | ds \right)
\]

\[
\leq \frac{1}{\Gamma(\beta + 1)} |a(s)|_\infty \delta + \frac{(t_1)^\beta}{\Gamma(\beta + 1)} |a(s)|_\infty \delta + \frac{1}{\Gamma(\beta + 1)} |a(s)|_\infty \delta - \frac{(t_2)^\beta}{\Gamma(\beta + 1)} |a(s)|_\infty \delta
\]

\[
= \frac{|a(s)|_\infty \delta}{\Gamma(\beta + 1)} (2(t_2 - t_1)^\beta - ((t_1)^\beta - (t_2)^\beta)) \leq \frac{|a(s)|_\infty \delta}{\Gamma(\beta + 1)} 2(t_2 - t_1)^\beta
\]
Which means that
\[ I_1 \leq \frac{\|a(s)\|_{\infty} \delta}{\Gamma(\beta + 1)} 2(t_2 - t_1)^\beta \]  
(3.1)

In the same manner, we can find
\[ I_2 = \frac{1}{\Gamma(\beta)} \left( \int_{t_1}^{t_2} (t_2 - s)^{\beta - 1} |h(s)| \, ds + \int_0^{t_1} (t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1} |h(s)| \, ds \right) \]
\[ \leq \frac{\|h(s)\|_{\infty} \delta}{\Gamma(\beta + 1)} 2(t_2 - t_1)^\beta \]

which implies that
\[ I_2 \leq \frac{\|h(s)\|_{\infty} \delta}{\Gamma(\beta + 1)} 2(t_2 - t_1)^\beta \]  
(3.2)

\[ I_3 = \frac{1}{\Gamma(\beta)} \left( \int_{t_1}^{t_2} (t_2 - s)^{\beta - 1} \left( \int_0^{\delta} \Psi_1(s, \eta) \mu_1((\bar{z})(\eta, \gamma)) \right) \, d\eta + \int_0^1 \Psi_2(s, \eta) \mu_2((\bar{z})(\eta, \gamma)) \, d\eta \right) \, ds \]
\[ + \int_0^{t_1} (t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1} \left( \int_0^{\delta} \Psi_1(s, \eta) \mu_1((\bar{z})(\eta, \gamma)) \right) \, d\eta + \int_0^1 \Psi_2(s, \eta) \mu_2((\bar{z})(\eta, \gamma)) \, d\eta \]
\[ \leq \frac{\Psi^* \omega_1 + \Psi^* \omega_2}{\Gamma(\beta + 1)} 2(t_2 - t_1)^\beta \]

Which implies that
\[ I_3 \leq \frac{\Psi^* \omega_1 + \Psi^* \omega_2}{\Gamma(\beta + 1)} 2(t_2 - t_1)^\beta \]  
(3.3)

The three terms on the right remain unchanged by the \( z \in B_\delta \) and tend to zero as \( t_2 - t_1 \to 0 \).

This indicates that,
\[ |(T\bar{z})(t_1, \gamma) - (T\bar{z})(t_2, \gamma)| \to 0. \]

As a result, the set \( TB_\delta \) is equal-continuous. We deduce that \( T \) is continuous using the Arzela'-Ascoli theorem.

To prove their existence of closed, convex, and bounded subset (CCABS), \( B_{\delta'} = \{ \bar{z} \in C(U, R^F) : \|\bar{z}\|_{\infty} \leq \delta' \} \), such that \( TB_{\delta'} \subseteq B_{\delta'} \). For every positive integer \( \delta' \), \( B_{\delta'} \) is CCABS of \( C(U, R^F) \).

For any positive integer, let if \( \delta' \), there exist \( \bar{z}_{\delta'} \in B_{\delta'} \), such that \((T\bar{z}_{\delta'}) \neq TB_{\delta'}\) then
\[ \delta' < \left( \frac{\|\bar{z}_0\| + \|\bar{a}\|_{\infty} \delta + \|h\|_{\infty} + \Psi^* \omega_1 + \Psi^* \omega_2}{\Gamma(\beta + 1)} \right) \]

Taking \( \delta' \) on both sides and limit \( \delta' \to \infty \), we have
\[ \frac{\|\bar{a}\|_{\infty}}{\Gamma(\beta + 1)} > 1 \]

It fails our assumption that is \( \frac{\|\bar{a}\|_{\infty}}{\Gamma(\beta + 1)} < 1 \). Then we have \( TB_{\delta'} \subseteq B_{\delta'} \), for all positive integer \( \delta' \).

There is at least one solution to equation (1.1) according to the Schauder’s fixed point theorem (SFPT).

The conclusion is that the problem has a solution. Using the Banach contraction principle (BCP), we now verify the uniqueness of this solution.
Theorem 3.2 [26, 37]

Let us assume that \( H_1 \) to \( H_3 \) holds. If

\[
\left( \frac{\|a\|_{\infty} + \Psi_1^* c_1 + \Psi_2^* c_2}{\Gamma(\beta + 1)} \right) < 1,
\]

(3.4)

which gives a unique solution to equation (1.1).

Proof. If \( \bar{z} \) satisfies the initial condition (IC), then \( \bar{z} \) is the solution to the equation (1.1), is

\[
\bar{z}(t, \gamma) = \bar{z}_0 + J^\beta \left( h(t) + a(t)\bar{z}(t, \gamma) + \int_0^t \Psi_1(t, \eta)\mu_1(\bar{z}(\eta, \gamma)) d\eta + \int_0^t \Psi_2(t, \eta)\mu_2(\bar{z}(\eta, \gamma)) d\eta \right), \text{ for } z \in U
\]

Define \( T \) according to Theorem 3.1. If \( \bar{z} \in C(U < R^3) \) is a FP of \( T \), then \( \bar{z} \) is the solution of the equation (1.1). Let \( \bar{z}_1, \bar{z}_2 \in C(U < R^3) \), then

\[
| (T\bar{z}_1)(t, \gamma) - (T\bar{z}_2)(t, \gamma) | = | \bar{z}_0 + J^\beta \left( h(t) + a(t)\bar{z}_1(t, \gamma) + \int_0^t \Psi_1(t, \eta)\mu_1(\bar{z}(\eta, \gamma)) d\eta + \int_0^t \Psi_2(t, \eta)\mu_2(\bar{z}(\eta, \gamma)) d\eta \right) |
\]

\[
- \left( \bar{z}_0 + J^\beta \left( h(t) + a(t)\bar{z}_2(t, \gamma) + \int_0^t \Psi_1(t, \eta)\mu_1(\bar{z}(\eta, \gamma)) d\eta + \int_0^t \Psi_2(t, \eta)\mu_2(\bar{z}(\eta, \gamma)) d\eta \right) \right) |
\]

\[
\leq \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} | a(s) \| \bar{z}_1(s, \gamma) - \bar{z}_2(s, \gamma) \| ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}
\]

\[
\left( \int_0^s | \Psi_1(s, \eta) \| \mu_1((\bar{z}_1)(\eta, \gamma)) - \mu_1((\bar{z}_2)(\eta, \gamma)) \| d\eta \right) \int_0^1 | \Psi_2(s, \eta) \| \mu_2((\bar{z}_1)(\eta, \gamma)) - \mu_2((\bar{z}_2)(\eta, \gamma)) \| d\eta \right) ds
\]

Taking supremum both hand side

\[
\|T\bar{z}_1 - T\bar{z}_2\|_{\infty} \leq \frac{\|a\|_{\infty} \| \bar{z}_1 - \bar{z}_2 \|_{\infty} + \frac{\Psi_1^* c_1}{\Gamma(\beta + 1)} \| \bar{z}_1 - \bar{z}_2 \|_{\infty} + \frac{\Psi_2^* c_2}{\Gamma(\beta + 1)} \| \bar{z}_1 - \bar{z}_2 \|_{\infty}}{\Gamma(\beta + 1)}
\]

since \( \left( \frac{\|a\|_{\infty} + \Psi_1^* c_1 + \Psi_2^* c_2}{\Gamma(\beta + 1)} \right) < 1 \), we obtain \( \|T\bar{z}_1 - T\bar{z}_2\|_{\infty} = \|\bar{z}_1 - \bar{z}_2\|_{\infty} \).

Hence, \( T \) is a contraction. Thus, by BCP \( T \) has a UFP \( \bar{z} \in C(U, R^3) \).

4. MADM [26, 37]

Consider the following equation: Applying the operator \( J^\beta \) to both sides of Equation (1.1), we obtain

\[
\bar{z}(t, \gamma) = \bar{z}_0 + J^\beta \left( h(t) + a(t)\bar{z}(t, \gamma) + \int_0^t \Psi_1(t, \eta)\mu_1(\bar{z}(\eta, \gamma)) d\eta + \int_0^t \Psi_2(t, \eta)\mu_2(\bar{z}(\eta, \gamma)) d\eta \right)
\]

Adomian’s technique specifies the solution \( \bar{z}(t, \gamma) \) as a series, which is

\[
\bar{z} = \sum_{n=0}^{\infty} \bar{z}_n
\]

(4.1)
and the non-linear terms $\mu_1$ and $\mu_2$ are decomposed as

$$\mu_1 = \sum_{n=0}^{\infty} E_n \text{ and } \mu_2 = \sum_{n=0}^{\infty} K_n \quad (4.2)$$

Where $E_n$ and $K_n$ are Adomian polynomials given by [26, 37]

$$E_n = \frac{1}{n!} d^n \left[ \mu_1 \left( \sum_{t=0}^{\infty} \rho^t \tilde{z}_t \right) \right]_{\rho=0},$$

$$K_n = \frac{1}{n!} d^n \left[ \mu_2 \left( \sum_{t=0}^{\infty} \rho^t \tilde{z}_t \right) \right]_{\rho=0}.$$

Therefore

$$E_0 = \mu_1(\tilde{z}_0),$$

$$E_1 = \tilde{z}_1 \mu_1'(\tilde{z}_0),$$

$$E_2 = \tilde{z}_2 \mu_1'(\tilde{z}_0) + \frac{1}{2} \tilde{z}_1^2 \mu_1''(\tilde{z}_0),$$

$$E_3 = \tilde{z}_3 \mu_1'(\tilde{z}_0) + \tilde{z}_1 \tilde{z}_2 \mu_1''(\tilde{z}_0) + \frac{1}{3} \tilde{z}_1^3 \mu_1'''(\tilde{z}_0),$$

and

$$K_0 = \mu_2(\tilde{z}_0),$$

$$K_1 = \tilde{z}_1 \mu_2'(\tilde{z}_0),$$

$$K_2 = \tilde{z}_2 \mu_2'(\tilde{z}_0) + \frac{1}{2} \tilde{z}_1^2 \mu_2''(\tilde{z}_0),$$

$$K_3 = \tilde{z}_3 \mu_2'(\tilde{z}_0) + \tilde{z}_1 \tilde{z}_2 \mu_2''(\tilde{z}_0) + \frac{1}{3} \tilde{z}_1^3 \mu_2'''(\tilde{z}_0),$$

The components $\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, ...$ are determined recursively as below [26, 37]

$$\tilde{z}_0(t, \gamma) = \tilde{z}_0 + J^\beta(h(t)),$$

$$\tilde{z}_{t+1}(t, \gamma) = J^\beta(a(t)\tilde{z}_t(t, \gamma)) + J^\beta \left( \int_0^t \Psi_1(t, \eta) E_t d\eta + \int_0^t \Psi_2(t, \eta) K_t d\eta \right).$$

It is proposed that part $\lambda_1$ be assigned to $\tilde{z}_0$, while $\lambda_2$ be taken with the other terms to define $\tilde{z}_1$. As a result, we get to the following refined recursive relation, [27, 37]
\[ \tilde{z}_0(t, \gamma) = z_0 + \lambda_1(t), \]
\[ \tilde{z}_1(t, \gamma) = \lambda_2(t) + J^\beta (a(t)\tilde{z}_0(t, \gamma)) + J^\beta \left( \int_0^t \Psi_1(t, \eta)E_0 d\eta + \int_0^1 \Psi_2(t, \eta)K_0 d\eta \right), \]
\[ \vdots \]
\[ \tilde{z}_{i+1}(t, \gamma) = J^\beta (a(t)\tilde{z}_i(t, \gamma)) + J^\beta \left( \int_0^t \Psi_1(t, \eta)E_i d\eta + \int_0^1 \Psi_2(t, \eta)K_i d\eta \right). \]

5. Illustrative examples

Example 5.1

Let us consider the following CFFVFIDE is

\[ cD^\frac{3}{5} \tilde{z}(t, \gamma) = -\frac{(t^3 - 4)}{5} \tilde{z}(t, \gamma) + \int_0^t e^t \eta^3 \tilde{z}(t, \eta)d\eta - \int_0^1 (4 - t^3)\eta^4 \tilde{z}(t, \eta)d\eta, \]
\[ \tilde{z}(0, \gamma) = [\tilde{z}(0, \gamma), \tilde{z}(0, \gamma)] = [2(\gamma - 1), 2(1 - \gamma)] \]

Under type (1) differentiability, the above equation becomes:

\[ cD^\frac{3}{5} \tilde{z}(t, \gamma) = -\frac{(t^3 - 4)}{5} \tilde{z}(t, \gamma) + \int_0^t e^t \eta^3 \tilde{z}(t, \eta)d\eta - \int_0^1 (4 - t^3)\eta^4 \tilde{z}(t, \eta)d\eta \]
\[ \tilde{z}(0, \gamma) = [2(\gamma - 1), 2(1 - \gamma)] \]

First consider \( \tilde{z}(t, \gamma) \), and apply the operator \( J^\frac{3}{5} \) to both sides of equation (5.2)

\[ \tilde{z}(t, \gamma) = \tilde{z}(0, \gamma) - J^\frac{3}{5} \left( \frac{(t^3 - 4)}{5} \tilde{z}(t, \gamma) \right) + J^\frac{3}{5} \left( \int_0^t e^t \eta^3 \tilde{z}(t, \eta)d\eta - \int_0^1 (4 - t^3)\eta^4 \tilde{z}(t, \eta)d\eta \right) \]
\[ = 2(\gamma - 1) - J^\frac{3}{5} \left( \frac{(t^3 - 4)}{5} \tilde{z}(t, \gamma) \right) + J^\frac{3}{5} \left( \int_0^t e^t \eta^3 \tilde{z}(t, \eta)d\eta - \int_0^1 (4 - t^3)\eta^4 \tilde{z}(t, \eta)d\eta \right) \]

We have \( h(t) = 0 \). Suppose \( \lambda(t) = J^\frac{3}{5} (h(t)) \). So \( \lambda(t) = J^\frac{3}{5} (h(t)) = 0 \). Now applying the MADM
\( \lambda(t) = \lambda_1(t) + \lambda_2(t) = 0 \). Now

\[
\begin{align*}
\ddot{z}(t, \gamma) &= \dot{z}(0, \gamma) + \lambda_1(t) = 2(\gamma - 1) \\
\ddot{z}_1(t, \gamma) &= \lambda_2(t) - J^\frac{3}{5} \left( \frac{(t^3 - 4)}{5} (2(\gamma - 1)) \right) + J^\frac{3}{5} \left( \int_0^t e^\eta^3 (2(\gamma - 1)) \, d\eta - \int_0^1 (4 - t^3) \eta^4 (2(\gamma - 1)) \, d\eta \right) \\
&= -J^\frac{3}{5} \left( \frac{(t^3 - 4)}{5} (2(\gamma - 1)) \right) + J^\frac{3}{5} \left( e^{\frac{t^4}{4}} (2(\gamma - 1)) \right) - J^\frac{3}{5} \left( \frac{(4 - t^3)}{5} (2(\gamma - 1)) \, d\eta \right) \\
&= J^\frac{3}{5} \left( e^{\frac{t^4}{4}} (2(\gamma - 1)) \right) \\
&= \frac{12}{\Gamma(\frac{28}{5})} t^\frac{23}{5}
\end{align*}
\]

Similarly, we can determine the remaining terms and get the solution is,

\[ \ddot{z} = \sum_{n=0}^{\infty} \ddot{z}_n = 2(\gamma - 1) + \frac{12}{\Gamma(\frac{28}{5})} t^\frac{23}{5} + \ldots \]

Accordingly, we can find

\[ \ddot{z} = \sum_{n=0}^{\infty} \ddot{z}_n = 2(1 - \gamma) + \frac{12}{\Gamma(\frac{28}{5})} t^\frac{23}{5} + \ldots \]

Next, we plot fuzzy solutions in various scenarios in Figures 1–3. Here, the uncertain parameter \( \gamma \) is taking different points that is \( \gamma = 0.5, 0.7, 0.8, 0.85, \) and 0.9. Then we get other fuzzy upper and lower solutions presented in Figure 4

Similarly by taking various points of \( t \) that is \( t = 0.2, 0.4, 0.5, 0.6, 0.7, \) and 0.8. After that, we obtain different fuzzy upper and lower solutions, shown in Figure 5.

Three-dimensional upper and lower fuzzy solutions are plotted in Figure 6.

**Example 5.2**

The following CFFVFIDE is defined by \[37\]

\[
\begin{align*}
^cD^\frac{3}{5} \ddot{z}(t, \gamma) &= \frac{2}{\Gamma(\frac{13}{5})} t^\frac{8}{5} + \frac{(t^3 e^t)}{3} \ddot{z}(t, \gamma) + \int_0^t e^\eta^3 \ddot{z}(t, \eta) \, d\eta + \int_0^1 e^\eta^3 \ddot{z}(t, \eta) \, d\eta \\
\ddot{z}(0, \gamma) &= [\ddot{z}(0, \gamma), \ddot{z}(0, \gamma)] = [2(\gamma - 1), 2(1 - \gamma)]
\end{align*}
\]

For type (1) differentiability, the above equation defined as

\[
\begin{align*}
^cD^\frac{3}{5} \ddot{z}(t, \gamma) &= \frac{2}{\Gamma(\frac{13}{5})} t^\frac{8}{5} + \frac{(t^3 e^t)}{3} \ddot{z}(t, \gamma) + \int_0^t e^\eta^3 \ddot{z}(t, \eta) \, d\eta + \int_0^1 e^\eta^3 \ddot{z}(t, \eta) \, d\eta \\
\ddot{z}(0, \gamma) &= [2(\gamma - 1), 2(1 - \gamma)]
\end{align*}
\]

\[
\begin{align*}
^cD^\frac{3}{5} \ddot{z}(t, \gamma) &= \frac{2}{\Gamma(\frac{13}{5})} t^\frac{8}{5} - \frac{(t^3 e^t)}{3} \ddot{z}(t, \gamma) + \int_0^t e^\eta^3 \ddot{z}(t, \eta) \, d\eta + \int_0^1 e^\eta^3 \ddot{z}(t, \eta) \, d\eta \\
\ddot{z}(0, \gamma) &= [2(\gamma - 1), 2(1 - \gamma)]
\end{align*}
\]

\[ \ddot{z}(0, \gamma) = [\ddot{z}(0, \gamma), \ddot{z}(0, \gamma)] = [2(\gamma - 1), 2(1 - \gamma)] \]
First consider $z(t, \gamma)$, and apply the operator $J^{\frac{2}{5}}$ to both sides of equation (5.5)

$$
\dot{z}(t, \gamma) = z(0, \gamma) + J^{\frac{2}{5}} \left( \frac{2}{(\Gamma(\frac{13}{5}))} t^{\frac{2}{5}} - \frac{(t^{\frac{2}{5}}e^{t})}{3} \dot{z}(t, \gamma) + \int_{0}^{t} e^{t} \eta^{2} z(t, \eta) d\eta + \int_{0}^{1} \eta z(t, \eta) d\eta \right)
$$

$$
= 2(\gamma - 1) + J^{\frac{2}{5}} \left( \frac{2}{(\Gamma(\frac{13}{5}))} t^{\frac{2}{5}} \right) + J^{\frac{2}{5}} \left( \frac{(t^{\frac{2}{5}}e^{t})}{3} \dot{z}(t, \gamma) \right) + J^{\frac{2}{5}} \left( \int_{0}^{t} e^{t} \eta^{2} z(t, \eta) d\eta \right) + J^{\frac{2}{5}} \left( \int_{0}^{1} \eta z(t, \eta) d\eta \right)
$$

We have $h(t) = \frac{2}{(\Gamma(\frac{13}{5}))} t^{\frac{2}{5}}$. Suppose $\lambda(t) = J^{\frac{2}{5}}(h(t))$. So using the definition of $J^{\beta}$

$$
\lambda(t) = J^{\frac{2}{5}} \left( \frac{2t^{\frac{2}{5}}}{(\Gamma(\frac{13}{5}))} \right)
$$

$$
= \frac{2}{\Gamma(\frac{13}{5})} \frac{(t - \eta)^{\frac{13}{5}}}{\eta^{\frac{13}{5}}} d\eta
$$

$$
= t^{2}
$$

Now apply the MADM, $\lambda(t) = \lambda_{1}(t) + \lambda_{2}(t) = t^{2}$. So that $\lambda_{1}(t) = 0$ and $\lambda_{2}(t) = t^{2}$. Now

$$
\dot{z}_{0}(t, \gamma) = z(0, \gamma) + \lambda_{1}(t) = 2(\gamma - 1)
$$

$$
\dot{z}_{1}(t, \gamma) = \lambda_{2}(t) + J^{\frac{2}{5}} \left( \frac{(t^{\frac{2}{5}}e^{t})}{3} \dot{z}_{0}(t, \gamma) \right) + J^{\frac{2}{5}} \left( \int_{0}^{t} e^{t} \eta^{2} \dot{z}_{0}(t, \eta) d\eta \right) + J^{\frac{2}{5}} \left( \int_{0}^{1} \eta \dot{z}_{0}(t, \eta) d\eta \right)
$$

$$
= t^{2} + J^{\frac{2}{5}} \left( \frac{(t^{\frac{2}{5}}e^{t})}{3} (2(\gamma - 1)) \right) + J^{\frac{2}{5}} \left( \int_{0}^{t} e^{t} \eta^{2} (2(\gamma - 1)) d\eta \right) + J^{\frac{2}{5}} \left( \int_{0}^{1} \eta (2(\gamma - 1)) d\eta \right)
$$

$$
= t^{2} + \frac{2(\gamma - 1)e^{t}t^{\frac{2}{5}}\Gamma(\frac{3}{5})}{3} + \frac{2(\gamma - 1)2e^{t}t^{\frac{2}{5}}}{\Gamma(\frac{13}{5})} + \frac{2(\gamma - 1)t^{\frac{2}{5}}}{2\Gamma(\frac{7}{5})}
$$

Likewise, we can find the other terms and find a solution that is

$$
\dot{z} = \sum_{n=0}^{\infty} \dot{z}_{n} = 2(\gamma - 1) + t^{2} + \frac{2(\gamma - 1)e^{t}t^{\frac{2}{5}}\Gamma(\frac{3}{5})}{3} + \frac{2(\gamma - 1)2e^{t}t^{\frac{2}{5}}}{\Gamma(\frac{13}{5})} + \frac{2(\gamma - 1)t^{\frac{2}{5}}}{2\Gamma(\frac{7}{5})} + ...
$$

Similarly, we obtain the solution

$$
\ddot{z} = \sum_{n=0}^{\infty} \ddot{z}_{n} = 2(1 - \gamma) + t^{2} + \frac{2(1 - \gamma)e^{t}t^{\frac{2}{5}}\Gamma(\frac{3}{5})}{3} + \frac{2(1 - \gamma)2e^{t}t^{\frac{2}{5}}}{\Gamma(\frac{13}{5})} + \frac{2(1 - \gamma)t^{\frac{2}{5}}}{2\Gamma(\frac{7}{5})} + ...
$$

Next, we plot fuzzy solutions in various scenarios in Figures 4–5. Here, the uncertain parameter $\gamma$ is taking different points that is $\gamma = 0.2, 0.4, 0.5, 0.6, 0.7, \text{ and } 0.8$. Then we get other fuzzy upper and lower solutions presented in Figure 4.

Similarly by taking various points of $t$ that is $t = 0.2, 0.4, 0.5, 0.6, 0.7, \text{ and } 0.8$. After that, we obtain different fuzzy upper and lower solutions, shown in Figure 5.

Three-dimensional upper and lower fuzzy solutions are plotted in Figure 6.
Example 5.3

Consider the following CFFVIDE is \[37\]

\[
cD^\frac{1}{2}z(t, \gamma) = \frac{t^\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{2}{\Gamma(\frac{5}{2})} t^\frac{3}{2} - \frac{(1 - t^2)}{2} z(t, \gamma) + \int_0^t e^t \eta z(t, \eta) d\eta + \int_0^1 (1 - t^2) \eta \tilde{z}(t, \eta) d\eta
\]

(5.7)

\[
\tilde{z}(0, \gamma) = [\tilde{z}(0, \gamma), \bar{z}(0, \gamma)] = [2(\gamma - 1), 2(1 - \gamma)]
\]

Type (1) differentiability corresponding to the above equation:

\[
cD^\frac{1}{2}\bar{z}(t, \gamma) = \frac{t^\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{2}{\Gamma(\frac{5}{2})} t^\frac{3}{2} - \frac{(1 - t^2)}{2} \bar{z}(t, \gamma) + \int_0^t e^t \eta \bar{z}(t, \eta) d\eta + \int_0^1 (1 - t^2) \eta \tilde{z}(t, \eta) d\eta
\]

(5.8)

\[
\bar{z}(0, \gamma) = 2(\gamma - 1)
\]

\[
cD^\frac{1}{2}\bar{z}(t, \gamma) = \frac{t^\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{2}{\Gamma(\frac{5}{2})} t^\frac{3}{2} - \frac{(1 - t^2)}{2} \bar{z}(t, \gamma) + \int_0^t e^t \eta \bar{z}(t, \eta) d\eta + \int_0^1 (1 - t^2) \eta \tilde{z}(t, \eta) d\eta
\]

(5.9)

First consider \( \tilde{z}(t, \gamma) \), and apply the operator \( J^\frac{1}{2} \) to both sides of equation (5.8)

\[
\tilde{z}(t, \gamma) = \tilde{z}(0, \gamma) + J^\frac{1}{2} \left( \frac{t^\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{2}{\Gamma(\frac{5}{2})} t^\frac{3}{2} - \frac{(1 - t^2)}{2} \tilde{z}(t, \gamma) + \int_0^t e^t \eta \tilde{z}(t, \eta) d\eta + \int_0^1 (1 - t^2) \eta \tilde{z}(t, \eta) d\eta \right)
\]

\[
= 2(\gamma - 1) + J^\frac{1}{2} \left( \frac{t^\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{2}{\Gamma(\frac{5}{2})} t^\frac{3}{2} \right) - J^\frac{1}{2} \left( \frac{(1 - t^2)}{2} \tilde{z}(t, \gamma) \right) + J^\frac{1}{2} \left( \int_0^t e^t \eta \tilde{z}(t, \eta) d\eta \right) + J^\frac{1}{2} \left( \int_0^1 (1 - t^2) \eta \tilde{z}(t, \eta) d\eta \right)
\]

We have \( h(t) = \left( \frac{t^\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{2}{\Gamma(\frac{5}{2})} t^\frac{3}{2} \right) \). Suppose \( \lambda(t) = J^\frac{1}{2}(h(t)) \). So using the definition of \( J^\beta \)

\[
\lambda(t) = J^\frac{1}{2} \left( \frac{t^\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{2}{\Gamma(\frac{5}{2})} t^\frac{3}{2} \right)
\]

\[
= \frac{1}{\Gamma(\frac{3}{2}) \Gamma(\frac{5}{2})} \int_0^t (t - \eta)^{\frac{1}{2}} \eta^\frac{3}{2} d\eta - \frac{2}{\Gamma(\frac{3}{2}) \Gamma(\frac{5}{2})} \int_0^t (t - \eta)^{\frac{1}{2}} \eta^\frac{3}{2} d\eta
\]

\[
= t - t^2
\]

Now apply the MADM,

\[
\lambda(t) = \lambda_1(t) + \lambda_2(t)
\]

\[
= t - t^2
\]
\( z_0(t, \gamma) = \hat{z}(0, \gamma) + \lambda_1(t) = 2(\gamma - 1) + t \)

\[
\begin{align*}
\dot{z}_1(t, \gamma) &= \lambda_2(t) - J^\frac{1}{2} \left( \frac{1}{2} \dot{z}_0(t, \gamma) \right) + J^\frac{1}{2} \left( \int_0^t e^\eta \dot{z}_0(t, \gamma) d\eta \right) + J^\frac{1}{2} \left( \int_0^t (1 - t^2) \dot{z}_0(t, \gamma) d\eta \right) \\
&= -t^2 - J^\frac{1}{2} \left( \frac{1 - t^2}{2} (2(\gamma - 1) + t) \right) + J^\frac{1}{2} \left( \int_0^t e^\eta (2(\gamma - 1) + t) d\eta \right) + J^\frac{1}{2} \left( \int_0^t (1 - t^2) \dot{z}_0(t, \gamma) d\eta \right) \\
&= -t^2 - \frac{t^2}{2 \Gamma(\frac{3}{2})} + \left( \frac{1}{2} + \frac{e^{t^2}}{3} \right) 6 t^2 \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})} + \left( \frac{e^{t^2} (\gamma - 1)}{2} - \frac{1}{3} \right) 2 t^2 \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})} + \frac{t^2}{3 \Gamma(\frac{3}{2})} 
\end{align*}
\]

The remaining terms can be found in the same way and getting the solution is

\[
\begin{align*}
z &= \sum_{n=0}^{\infty} z_n = 2(\gamma - 1) + t - t^2 - \frac{t^2}{2 \Gamma(\frac{3}{2})} + \left( \frac{1}{2} + \frac{e^{t^2}}{3} \right) 6 t^2 \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})} + \left( \frac{e^{t^2} (\gamma - 1)}{2} - \frac{1}{3} \right) 2 t^2 \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})} + \frac{t^2}{3 \Gamma(\frac{3}{2})} + \ldots
\end{align*}
\]

In the same manner, we can find

\[
\begin{align*}
\overline{z} &= \sum_{n=0}^{\infty} \overline{z}_n = 2(1 - \gamma) + t - t^2 - \frac{t^2}{2 \Gamma(\frac{3}{2})} + \left( \frac{1}{2} + \frac{e^{t^2}}{3} \right) 6 t^2 \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})} + \left( \frac{e^{t^2} (1 - \gamma)}{2} - \frac{1}{3} \right) 2 t^2 \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})} + \frac{t^2}{3 \Gamma(\frac{3}{2})} + \ldots
\end{align*}
\]

In addition, we illustrate fuzzy solutions in various scenarios, as shown in Figures 7-9. Here, the uncertain parameter \( \gamma \) is taking different points that is \( \gamma = 0.2, 0.4, 0.5, 0.6, 0.7, \) and 0.8. Then we get other fuzzy upper and lower solutions presented in Figure 7. Similarly by taking various points of \( t \) that is \( t = 0.2, 0.4, 0.5, 0.6, 0.7, \) and 0.8. After that, we obtain different fuzzy upper and lower solutions, shown in Figure 8.

Three-dimensional upper and lower fuzzy solutions are plotted in Figure 9.

6. Conclusion

We looked at FFVFIDEs under Caputo’s derivative in this paper. Using the fixed point theory, we have demonstrated the existence and uniqueness of the FFIDE. The MADM determines the approximate solution to the problems. We have worked through the illustrative examples to understand better how MADM can be utilized more effectively in a fuzzy environment. To investigate the behavior, we have constructed two-dimensional and three-dimensional graphs correlating to each illustrative example and showing it at different points of uncertainty.

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Declaration of Competing Interest

In this work, there are no conflicts of interest.

References


Figure 1: Fuzzy solutions for the problem, where the uncertainty parameter $\gamma$ varies for problem 5.1.

Figure 2: Fuzzy solutions for the problem at different values of $t$ for problem 5.1.
Figure 3: Upper and lower solutions for the present example 5.1.

Figure 4: Fuzzy solutions for the problem 5.2, where the uncertainty parameter $\gamma$ varies.
Figure 5: Fuzzy solutions for the problem 5.2 at different values of $t$.

Figure 6: Upper and lower solutions for the present problem 5.2.
Figure 7: Fuzzy solutions for the problem, where the uncertainty parameter $\gamma$ varies for problem 5.3.

Figure 8: Fuzzy solutions for the problem at different values of $t$ for problem 5.3.
Figure 9: Upper and lower solutions for the present problem 5.3