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K⁺/π⁺ horn: appropriate statistics for dynamical nonequilibrium nonextensive particle production

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ABSTRACT

We suggest that the incapability of the thermal models in reproducing the horn-like structure of the Kaon-to-pion ratios measured at AGS, SPS, and RHIC energies and then confirmed in the Beam Energy Scan programs is likely emerged from an ad hoc utilization of inappropriate types of statistics, either extensive additive Boltzmann-Gibbs (BG) or the Tsallis-type of nonextensivity. In this regard, we emphasize that any suggested statistical approach for the dynamical and likely nonequilibrium nonextensive particle production, for example, Kaon and pion yields, should not necessarily remain valid over the whole range of the collision energies. We claim that generic (non)extensive statistics with two equivalence classes (c,d) is the most suitable statistical approach to replace all those assumptions and conjectures. Accordingly, the statistical ensemble itself becomes autonomously capable to determine its degree of extensivity or nonextensivity, where these are not necessarily limited to BG or Tsallis. The generic approach covers the entire (c,d)-space, in which both extensive BG and nonextensive Tsallis statistics are characterized by distinct equivalence classes, (1,1) and (0,q), respectively. The energy dependence of the light-, γ₀, and strange-quark occupation factor, γ₁, indicates that the produced particles are best described as a nonequilibrium ensemble, where the remarkable nonmonotonic behavior associating the horn-structure of K⁺/π⁺ ratio is qualitatively reproduced. On other hand, the resulting equivalence classes (c,d) refer to a generic nonextensivity associated with an extended exponential and a Lambert-W_0 exponentially generating distribution function, which obviously emerge from free, short- and long-range correlations over the wide range of the collision energies. With this generic statistical approach, the hadron resonance gas model, ideal thermal model, excellently reproduces the K⁺/π⁺ ratio, including its nonmonotonic horn-like structure. Therefore, we conclude that the utilization of any specific type of extensive or nonextensive statistics must be strictly aligned to the statistical nature of the underlying ensemble, otherwise, the price to be paid becomes unexpectedly huge. Such a universal conclusion apparently goes beyond the reproduction of the horn-like structure of K⁺/π⁺ ratio or particle physics to all possible statistical implications either on stock markets or economy or sociology or biological evolution or weather prediction, etc. This brings about the great deal of the present paper.

1 Introduction

A nonmonotonic horn-like structure of the K⁺/π⁺ ratio was confirmed in various experiments, at AGS, SPS and low RHIC energies¹⁵. At √s_NN ≲ 20 GeV, there exists a monotonic rapid increase followed by a monotonic rapid decrease, i.e., a maximum appears, called horn⁴, while at RHIC and LHC energies ≥ 20 GeV, the K⁺/π⁺ ratio seems to be nearly energy independent, i.e., a universal plateau is formed. Various interpretations have been given in literature¹⁶⁻¹². The thermal models which are either based on extensive additive Boltzmann-Gibbs (BG) and/or on Tsallis-type nonextensive statistics¹³ are not well succeeded in reproducing the nonmonotonic horn-like structure of the K⁺/π⁺ ratio, Fig. 4 and Fig. 3. Despite of this, various extensions to the thermal models are allegedly assumed to deliver a better explanation for the underlying processes in the particle production¹⁴⁻²⁰!

In this regard, we summarize a list of the key attempts to improve the capability of the thermal models, the ideal gas with identified hadron resonances. One of the authors (AT) proposed i) a substantial change in the single-particle distribution function due to a dynamical change in the phase-space volume, especially at the phase transition, whose imprints seem to survive till the chemical freezeout, where the produced particles are finally detected¹¹, ii) integrating nonequilibrium light- and strange-quark occupation factors, γ₀,s, in the phase-space volume¹⁶,²¹,²², iii) applying Tsallis-type nonextensive statistics¹³,²³, and iv) deriving the ideal gas of hadron resonances away from equilibrium by assigning an eigen-volume to each of its constituents²⁴,²⁵. Castorina et al. assumed a suppression in the strange-quark occupation factor, γ₁, that accounts for reducing the production rates of the strange particles in pp, pA, AA collisions, at different energies and found that the strangeness suppression disappears with the onset of the color deconfinement and full-chemical nonequilibrium is subsequently attained²⁶. The occupation factors, γ₀,s, are conjectured to introduce controllers over the numbers of light- and strange-quarks fitting within
the phase-space element, i.e., a similar role as that of the chemical potentials. None of these ingratiation was able to satisfactorily reproduce the various particle ratios, including \( K^+ / \pi^+ \).\(^{21,22,27}\)

Hence, there is a conceptional reason to be explained. The present script is all about this. Apparently, imposing nonequilibrium either through partial- or full-chemical nonequilibrium of the quark occupation factors (either \( \gamma_q \neq 1 \) while \( \gamma_s = 1 \) vice versa or both \( \gamma_q \) and \( \gamma_s \) differ from unity) or through finite eigen-volumes assigned to the constituents of the thermal model, or through various types of interactions added to the ideal gas of hadron resonances\(^{28-30}\) on an equilibrium ensemble of extensive and additive BG of statistics seems conceptionally not proper and practically not successful. Even if the Tsallis-type of nonextensivity is assumed, it is conjectured that in principle the matching of nonequilibrium with nonextensivity would be improved. On the other hand, the price to be paid was huge. This was nothing less than the lost of the physical meaning of various thermodynamic quantities, such as the freezeout temperature\(^{31-33}\). In other words, colleagues who strongly imposed this type of statistics on the particle production have found that the resulting values of the various thermodynamic quantities largely differ from the typical physical ones. To get rid of such a contradiction, two distinct types of thermodynamic quantities were assumed\(^{33}\).

In this regard, we refer to the generic \((c,d)\) statistical approach, i.e., (non)extensive statistics that covers the entire two-dimensional \((c,d)\)-space, in which, for example, Tsallis statistics is represented by a line, at \((c,d) = (0, q)\), where \(q\) is a positive parameter, and Boltzmann statistics is positioned at the point \((c,d) = (1,1)\). With the degree of extensivity and nonextensivity, we mean how far are the resulting \((c,d)\) form the extensivity, i.e., \((1,1)\) and from the Tsallis-type of nonextensivity, i.e., \((0,q)\). We first refer to a study comparing nonequilibrium Tsallis with extensive BG statistics\(^{31}\), with which one is is confronted with various contradictions. One of the authors (AT) instead conducted calculations based on generic nonextensive statistics\(^{34}\). When comparing both studies, we conclude that there are huge and small differences between the resulting freezeout temperatures \(T\) and the chemical potentials \(\mu_B\), for example, between Tsallis and BG statistics, respectively.

AT advocates that the resulting parameters \(T\) and \(\mu_B\) should not differ from the ones characterizing the extensive thermal ensemble, i.e., the physical parameters. As mentioned, there are, on the other hand, other authors who are assuming an emergence of another type of temperatures, while they preserve the chemical potentials\(^{32}\). Ref.\(^{34}\) gives a recent review on all these topics. From this short discussion, a profound extensive study is urged, in which various particle ratios should be utilized in analyzing the dependence of the freezeout parameters, \(T\) and \(\mu_B\), on the generic equivalence classes \((c,d)\).\(^{27}\) To remain within the scope of the present paper, such a study shall be conducted elsewhere.

The relativistic high-energy collisions are assumed to go through several critical processes, such as deconfinement and hadronization\(^{35}\). Radical changes in properties, symmetries, and degrees of freedom (dof) of the QCD matter are apparently depending on the collision energies\(^{36}\). The horn-like structure of the Kaon-to-pion ratio, which could be interpreted as a process normalizing the strange to the light quark flavors in the final state, likely signals an onset of a nonequilibrium state that dynamically enhances the strange quark flavors or suppresses the light quark flavors. Thus, the particle production in the final state is anything but not equilibrium extensive.

The present script suggests that the incapability of thermal models with the extensive additive BG statistics in reproducing the horn-like structure of the \(K^+ / \pi^+\) ratio measured at AGS, SPS, and RHIC energies, Fig. 4 and Fig. 3, is due to a case-of-necessity utilization of inappropriate statistics, with which the type of statistics is a priori fixed regardless the statistical nature of the underlying system. Generic (non)extensive statistics, section 2, autonomously describes the equivalence classes of the generalized entropies in the entire \((c,d)\)-plane\(^{37}\). At the chemical freezeout, where the particle production is ceased and the detection of the produced particles takes place, the statistical ensemble is thus conjectured to be dynamic. Such a situation likely violates the fourth Shannon-Khinchin axiom\(^{33}\). Even if this would not be the exact case and the produced particles would be neither nonequilibrium nor nonextensive. The utilization of generic (non)extensive statistics is still proper, as this does not necessarily fix a certain type of statistics. Fortunately, extensive, nonextensive, equilibrium, and out-of-equilibrium ensemble are properly encoded by the equivalence classes \((c,d)\) including extensive additive BG and nonextensive nonadditive Tsallis statistics, which are characterized by \((c,d) = (1,1)\) and \((c,d) = (0, q)\), respectively.

We first assume partial-chemical nonequilibrium in generic (non)extensive statistics, i.e., \(\gamma_q = 1\), while \(\gamma_s\) is to be determined from the statistical fits of various particle ratios, in a wide range of energies, to the available experimental measurements. With partial-chemical nonequilibrium, we mean a nonequilibrium value assigned to the strange quark occupation factor, while the value assigned to the light quark occupation factor can be equilibrium. This was reported, for instance, in ref.\(^ {38}\). We also assume full-chemical nonequilibrium, i.e., both \(\gamma_q \neq 1\) and \(\gamma_s \neq 1\) are determined from the statistical fits of particle ratios, as well. In the same manner, the energy-dependence of the equivalence classes \((c,d)\) is also determined, Fig. 2. In the present script, the occupation factors \(\gamma_{qs}\), the equivalence classes \((c,d)\), and the freezeout parameters; the temperature \(T\) and the baryon chemical potential \(\mu_B\), are not fitting variables. They are inputs varying with the collision energies. With these inputs, the \(K^+ / \pi^+\) ratio calculated in generic (non)extensive statistics excellently reproduces its measured result including the nonmonotonic horn-like structure, at \(\sqrt{s_{NN}} \lesssim 20\) GeV; the AGS, SPS, and RHIC energies, Fig. 3.

The present paper is organized as follows. The generic (non)extensive approach is introduced in section 2. The energy-
dependence of the equivalence classes \((c, d)\), the light- and strange-quark occupation factors \(\gamma_q\), and the \(K^+/\pi^+\) ratio is presented in section 3.2, 3.1, and 3.3, respectively. Section 4 is devoted to the conclusions. Appendices A, B and C respectively, elaborate mathematical details on the probability distribution, the entropy, and the exponential and logarithm functions in Boltzmann, Tsallis, and generic (non)extensive statistics. The impacts of the inclusion of very high-mass resonances and missing states in the thermal model, the hadron resonance gas model, and the contour plots of the quality of the statistical fits for varying \(\gamma_0 (\gamma_q = 1), \gamma_q (\gamma_q = 1)\), and \(\gamma_{q,t}\).

### 2 Generic (non)extensive statistical approach

In generic (non)extensive statistics, the partition function can be given as\(^{27,34}\)

\[
\ln Z(T, \mu) = \pm V \sum_i \frac{g_i}{(2\pi)^3} \int \ln \left[1 \pm \epsilon_{c,d,r}(x_i)\right] d^3 p,
\]

where \(g\) is the degeneracy factor, \(V\) is the volume of the fireball, \(p\) is the momentum, and \(\pm\) represent fermions and bosons, respectively. The \(i\)-th chemical potential \(\mu_i\) can be constructed as \(\mu_i = B_i \mu_B + S_i \mu_S + \cdots\), with \(S_i\) and \(B_i\) are the corresponding strangeness and baryon quantum numbers and \(\mu_B\) and \(\mu_S\) are the strangeness and baryon chemical potential, respectively. The energy dependence of the partition function, Eq. (1), is given through the chemical potential, which can be interpreted as the stopping power and/or the particle asymmetry\(^{35,39}\).

\[
\begin{align*}
\mu_B(\sqrt{s_{\text{NN}}}) & = \frac{1.245 \pm 0.084 \text{ GeV}}{1 + (0.246 \pm 0.024 \text{ GeV}^{-1}) \sqrt{s_{\text{NN}}}}, \\
\mu_S(\sqrt{s_{\text{NN}}}) & = \frac{(0.138 \pm 0.01) \frac{\mu_B(\sqrt{s_{\text{NN}}})}{T}}{1 - 2.4 \pm 0.1 \left(\frac{T}{T_c}\right)^2 + 2.7 \pm 0.2 \left(\frac{T}{T_c}\right)^3}.
\end{align*}
\]

The extended exponential function \(\epsilon_{c,d,r}(x_i)\)\(^{37,40}\), which substantially differs from both Boltzmann and Tsallis exponential functions, section A, introduces \(T\)- and \(\mu\)-dependence of the partition function, Eq. (1). With \(x_i = [\mu_i - \epsilon_i(p)]/T\) and the dispersion relation of \(i\)-th particle \(\epsilon_i(p) = \sqrt{p^2 + m_i^2}\), reads,

\[
\epsilon_{c,d,r}(x_i) = \exp \left[\frac{-d}{1-c} \left( W_k \left[ B \left(1 - \frac{x_i}{r}\right)^\frac{1}{2} \right] - W_k[B] \right) \right],
\]

where \(W_k\) is the Lambert-\(W_k\) function which has real solutions, at \(k = 0\) with \(d \geq 0\), and at \(k = 1\) with \(d < 0\).

\[
B = \left(\frac{(1-c)r}{1-(1-c)r}\right) \frac{\exp \left[\frac{(1-c)r}{1-(1-c)r}\right]}{1-c}
\]

with \(r = (1-c+cd)^{-1}\) and \(c\) and \(d\) are two exponents defining the equivalence classes for both types of extensive and nonextensive statistical ensembles. The latter are statistical systems violating the fourth Shannon-Khinchin axiom, namely the generalized Shannon additivity, especially in their large size limit, in which two asymptotic properties of the generalized entropies are associated with one scaling function each. Each scaling function is characterized by one exponent defining equivalence relations of the entropic forms, i.e., two entropic forms are equivalent if their exponents are the same.

As for Shannon-Khinchin axioms, if \(p_{ij} \geq 0\), \(p_i = \sum_{j=1}^n p_{ij}\) and \(\sum_i p_i = 1\), where \(i = 1, \ldots, n\) and \(j = 1, \ldots, m_i\), then \(S(p_{11}, \ldots, p_{mn}) = S_1(p_{11}, \ldots, p_{m1}) + \sum_{i=2}^n p_i S_1(p_{i1}, \ldots, p_{im_i}/p_i)\). Accordingly, \(S_n\), the Shannon entropy, is given by \(S_n(P) = \tau \cdot \sum_{i=1}^n p_i \log_2 p_i\) with \(\tau < 0\), i.e., the entropy of a statistical system divided into sub-systems \(A\) and \(B\) is given as \(S_A\) and the expectation value of \(S_B\). But this is conditioned to \(A\); \(S_{nm}(AB) = S_n(A) + S_m(B|A)\), where \(S_m(B|A) = \sum_B p(B) S_m(B|A)\)\(^{34}\). The generalized entropy reads \(S_{c,d}[p] = \sum_{i=1}^n \Gamma(d + 1, 1 - c \log(p_i))\), where the constants \((c,d)\) characterize the universality class of entropy of the system in the thermodynamic limit and also specifies its distribution functions. The fourth Shannon-Khinchin axiom is corresponding to Markovian processes, which are random processes, whose future is independent of the past, i.e., natural stochastic analog of the deterministic processes described by differential equations.

The equivalence classes \((c,d)\) which are two exponents determining two scaling functions with two asymptotic properties, are intrinsic properties of all types of the underlying statistics. For instance, extensive BG and Tsallis-type nonextensive statistics have \((c,d) = (1, 1)\) and \((0, q)\), respectively\(^{40}\). The remaining \((c-d)\) space is inhabited by various types of nonextensivity or superstabilities enabling us to reveal various nonequilibrium phenomena such as critical endpoint, correlations, fluctuations, higher-order cumulants, etc. The key conclusion of the present script is extending the statistical treatment beyond BG extensivity.
and Tsallis-type nonextensivity. Typical statistical systems, such as population, economic progression, and weather, should not be priori constrained to a specific type of statistics.

The present script assumes that the horn-like structure of the Kaon-to-pion ratio likely signals an onset of nonextensive particle production from nonequilibrium processes, such as inelastic interactions, nonequilibrium quark occupation factors, etc. In this regard, it is worthy emphasizing how the state of the chemical freezeout is to be interpreted, especially in an out-of-equilibrium-system? The final number of the produced particles is conjectured to characterize the properties of the fireball, in which the quark-gluon plasma (QGP), the colored deconfined parton state, hadronizes through a critical phenomenon, the parton-hadron phase transition. Thus, the final state of colorless hadrons are assumed to be produced in accordance with the accessible phase space with otherwise equal reaction strength\(^\text{11}\). Accordingly, the chemical nonequilibrium statistical hadronization, for instance, by adding non-unity \(\gamma_q\) and/or \(\gamma_p\) is suitable to describe the particle yields.

As introduced, the incapability of the equilibrium thermal models in reproducing the enhancement in both particle yields, the Kaons and the pions, Fig. 4 and Fig. 3, could be interpreted due to the priori utilization of an inappropriate type of statistics\(^\text{11,24,27,33,34}\), i.e., either extensive BG or Tsallis-type nonextensive statistics. In Eq. (1), both Tsallis and Boltzmann-Gibbs statistics are apparently included, section A. The Tsallis partition function can be obtained, at \((c,d) = (0,q)\), while the BG partition function, at \((c,d) = (1,1)\).

With the microscopic theory introduced in ref.\(^\text{11}\), a recipe was suggested of how non-unity occupation factors of light- and strange-quarks, \(\gamma_q\) and \(\gamma_p\), respectively, are introduced to the partition function of the grand-canonical ensemble in extensive and nonextensive statistics. Accordingly, the generic (non)extensive partition function reads\(^\text{42,43}\)

\[
\ln Z(T,\mu) = \pm V \sum_i \frac{g_i}{(2\pi)^3} \int \ln \left[ 1 \pm \left( \gamma_q^n \right)^{\gamma_q} \left( \gamma_p^n \right)^{\gamma_p} \varepsilon_{c,d,r}(x_i) \right] d^3 p,
\]

where \(n_q\) and \(n_s\) are the number of light and strange quarks, respectively, since we are largely interested on the QCD phase diagram, the contributions of the heavier quarks are conjectured not being as essential as that of light and strange quark flavors. As BG statistics assumes extensive, additive, and equilibrium particle production, \(\gamma_q\) and \(\gamma_p\) could plainly be seen as additional variables imposed on BG statistics. But they are essential ingredients to nonextensive statistics. The nonextensivity should not necessarily be limited to the Tsallis-type. The entire \((c-d)\) space characterizes various types of superstatistics including both types of said extensivity and nonextensivity. In generic (non)extensive statistics, the degree of extensivity and nonextensivity shall subjectively be determined by the changeable statistical nature of the system of interest, at various energies. It is worthy highlighting that \(T\) and \(\mu_B\) are the temperature and the baryon chemical potential of the hadronic system characterized by the underlying statistics, at the chemical freezeout. This is valid for equilibrium as well as for nonequilibrium. With nonequilibrium, we mainly mean a deviation from the equilibrium state caused by eigen volumes, interactions, non-unity quark occupation factors, etc.

In this regard, we recall that the statistical properties of an ensemble consisting of individual objects is given by a single-particle distribution function. The probability of finding an object in infinitesimal region of the phase-space is proportional to the volume of the phase-space element and the single-particle distribution function. The latter is assumed to fulfill the Boltzmann-Vlasov (BV) master equation\(^\text{44}\), which is the semi-classical limit of a time-dependent Hartree-Fock theory, the simplest approximate theories for solving the many-body Hamiltonian\(^\text{45}\), through the Wigner transform of the one-body density matrix\(^\text{46}\) and can be used to study the dynamics of the constituents. At high energies, the phase-space volume and its configurations are apparently modified. So does the single-particle distribution function\(^\text{41}\).

As introduced in the microscopic theory\(^\text{11}\), the occupation factors \(\gamma_{q,s}\) are conjectured to act as controllers over the number of the quarks fitting within the phase-space element, i.e., they play a role similar to that of the chemical potentials \(a = \ln E - \ln T - \ln N\), where \(\lambda = \exp(-a)\). For example, at \(\gamma_q = \gamma_q = 1\), the statistical ensemble refers to a system in chemical and thermal equilibrium, but non-unity \(\gamma_q\) or \(\gamma_p\), the statistical system is modified to partial nonequilibrium, i.e., \(\gamma_q \neq 1\) while \(\gamma_p = 1\). At \(\gamma_q \neq 1\) and \(\gamma_p \neq 1\), the statistical ensemble is fully out-of-equilibrium. If pressure, energy density, entropy and chemical potential are fixed, we can only distinguish between equilibrium and nonequilibrium states by the occupation factors. If \(\gamma\) is allowed to take values differ from unity, then the number of the produced particles differs from that at unity gamma, as \(N = \gamma \sum_i \exp(-\varepsilon_i/T)\). In this regard, we recall that the chemical potential is equivalent the energy needed to be added or removed to insert or take out one particle from the thermal system.

In the thermodynamic limit, the number density can then be deduced from the partition function, Eq. (4)\(^\text{27,47}\),

\[
n(T,\mu) = \pm \sum_i \frac{g_i}{2\pi^2} \int_0^\infty \frac{(1-c)^{-1} (\gamma_q^n)^{\gamma_q} (\gamma_p^n)^{\gamma_p} \varepsilon_{c,d,r}(x_i) \left[ B(1 - \frac{n_q}{\gamma_q})^2 \right]^{\gamma_q} \left[ 1 + B(1 - \frac{n_q}{\gamma_q})^2 \right] \left[ 1 + W_0 \left[ B(1 - \frac{n_q}{\gamma_q})^2 \right] \right]^{\gamma_p} d^3 p.
\]

The chemical freezeout in both extensive BG\(^\text{6,7,48-50}\) and generic (non)extensive statistics\(^\text{27}\) is thermodynamically conditioned along the entire QCD phase diagram. In the final state, we take into account that the various hadron resonances decay either to
stable particles, \( n_i^{\text{direct}} \), or to other resonances \( n_j \), so that a specific particle yield, such as Kaon and pion, is determined as
\[
\langle n_i^{\text{final}} \rangle = \langle n_i^{\text{direct}} \rangle + \sum_{j \neq i} b_{j \rightarrow i} \langle n_j \rangle,
\]
where \( b_{j \rightarrow i} \) is the branching ratio for the decay of \( j \)-th resonance to \( i \)-th particle. Eq. (6) explains that the average number of the \( i \)-th particle, for example \( K^+ \) or \( \pi^+ \), is constructed from the averaged number of this \( i \)-th particle, called the direct particle, calculated from Eq. (5) and the sum of the averaged number of this \( i \)-th particle generated from the various decay channels of \( j \)-th resonances according to Eq. (5) weighted with the corresponding branching ratio.

In the present work, we first assume partial-chemical nonequilibrium, i.e., \( \gamma_q = 1 \), while \( \gamma \) is determined from the statistical fits of the generic (non)extensive calculations of the \( K^+ / \pi^+ \) ratio, in a wide range of center-of-mass energies \( \sqrt{s_{\text{NN}}} \) to the available experimental measurements, top panel of Fig. 1. We also assume full-chemical nonequilibrium, i.e., both \( \gamma_q \neq 1 \) and \( \gamma \neq 1 \) are determined from the same statistical fits of the \( K^+ / \pi^+ \) ratio, Fig. 1 panels (b) and (c). The degree of extensivity and nonextensivity is autonomously determined by the statistical ensemble itself. The results shall be discussed in the next section.

### 3 Results

By statistical fits of various particle ratios, at the chemical freezeout, at which the various quantum numbers, including baryon, strangeness, and electric charge, are conserved, whose parameters \( T \) and \( \mu_B \) are well expressed as functions of the collision energies and accordingly thermodynamically conditioned, for example to constant entropy density \( s / T^3 = 7 \), in either extensive BG\(^{25,48-50} \) or generic (non)extensive statistics\(^{27} \), the energy dependence of the light- and strange-quark occupation factors \( \gamma_q \) and \( \gamma \), section 3.1 and that of the equivalence classes \( (c, d) \), section 3.2, have been determined in generic (non)extensive statistics, at various center-of-mass energies \( \sqrt{s_{\text{NN}}} \). The three sets of parameters, namely the freezeout parameters, the quark occupation factors, and the equivalence classes, enter our calculations for \( K^+ / \pi^+ \) ratio as inputs. At a given \( \sqrt{s_{\text{NN}}} \), which is related to \( \mu_B \) and the freezeout temperature \( T \), the corresponding quark occupation factors, and the equivalence classes are substituted in Eq. (5), which together with Eq. (6) which determines the number density of a specific particle yield.

#### 3.1 Nonequilibrium quark occupation factors

In partial- and full-chemical nonequilibrium, i.e., \( \gamma_q \neq 1 \) while \( \gamma = 1 \), different particles ratios measured, at various \( \sqrt{s_{\text{NN}}} \), are fitted to their calculations with generic (non)extensive statistics\(^{27,36,43,51} \). At a given \( \sqrt{s_{\text{NN}}} \), the corresponding equivalence classes \( (c, d) \), section 3.2, and the freezeout parameters \( T \) and \( \mu_B \) are substituted in the expression for the number density of a specific particle yield. With the resulting fitting parameters either \( \gamma_q \) at \( \gamma_q = 1 \) for partial-chemical nonequilibrium or \( \gamma \) and \( \gamma_q \) for full-chemical nonequilibrium the various particles ratios calculated in generic (non)extensive statistics are fitted to the measured ones. The overall energy dependence of both \( \gamma_q \) and \( \gamma \), Fig. 1, can be summarized as
\[
\gamma_i (\sqrt{s_{\text{NN}}}) = \alpha \exp (-\beta \sqrt{s_{\text{NN}}}) \sin (\nu \sqrt{s_{\text{NN}}} + \eta) + \zeta,
\]
where the values of the parameters \( \alpha, \beta, \nu, \eta, \zeta \), Table 1, distinguish between partial- (\( \gamma_q = 1 \), while \( \gamma \) is the fitting variable) and full-chemical nonequilibrium (both \( \gamma_q = 1 \) and \( \gamma \) are the fitting variables). The different particle ratios including \( K^+ / \pi^+ \) (Fig. 3), \( P / \pi^+ \) and \( \Lambda / \pi^- \) (Fig. 7) measured at various collision energies are utilized in this statistical fit. It is important to highlight that the calculations for \( \gamma \) and \( \gamma_q \) performed in our numerical code of the hadron resonance gas model are conducted, at the quark level of each hadron and resonance. At this level of calculations, our numerical code enables us to deal with the quark constituents, only. Thus, \( \gamma_i \) and \( \gamma_q \) for the various strange and non-strange hadrons and resonances could vary from one constituent to another.

<table>
<thead>
<tr>
<th></th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \nu )</th>
<th>( \eta )</th>
<th>( \zeta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>partial ( \gamma )</td>
<td>10.373 ± 0.625</td>
<td>0.361 ± 0.009</td>
<td>0.233 ± 0.003</td>
<td>5.362 ± 0.014</td>
<td>0.751 ± 0.018</td>
</tr>
<tr>
<td>full ( \gamma )</td>
<td>2.913 ± 0.259</td>
<td>0.262 ± 0.062</td>
<td>-0.257 ± 0.018</td>
<td>4.353 ± 0.081</td>
<td>0.804 ± 0.025</td>
</tr>
<tr>
<td>( \gamma_q )</td>
<td>-5.264 ± 0.256</td>
<td>0.381 ± 0.018</td>
<td>0.373 ± 0.017</td>
<td>4.88 ± 0.071</td>
<td>1.037 ± 0.041</td>
</tr>
</tbody>
</table>

**Table 1.** Fitting parameters with the statistical uncertainties of the dependence of \( \gamma_{i,q} \) on \( \sqrt{s_{\text{NN}}} \), Eq. (7). The experimental dataset consists of various particle ratios including \( K^+ / \pi^+ \) (Fig. 3), \( P / \pi^+ \) and \( \Lambda / \pi^- \) (Fig. 7). We distinguish between partial-(\( \gamma_q = 1 \) while \( \gamma \) varies) and full-chemical nonequilibrium (both \( \gamma_{i,q} \) differ from unity).

The dependence of \( \gamma_q \) and \( \gamma \) on the nucleon-nucleon center-of-mass energy \( \sqrt{s_{\text{NN}}} \) is presented in Fig. 1. The solid symbols refer to the present calculations with generic (non)extensive statistics, Eq. (5), and their statistical fits to the measured \( K^+ / \pi^+ \) ratio, while the open symbols represent the calculations from the extensive hadron resonance gas (HRG) model based on extensive additive BG statistics\(^{21,22} \). Top panel of Fig. 1 shows \( \gamma \) in dependence on \( \sqrt{s_{\text{NN}}} \) as calculated in generic
Figure 1. Top panel (partial-chemical nonequilibrium): $\gamma_s$ is determined as a function of $\sqrt{s_{NN}}$, whereas $\gamma_q = 1$. Bottom panel (full-chemical nonequilibrium): both $\gamma_q$ (left) and $\gamma_s$ (right) are variables depending on $\sqrt{s_{NN}}$. Solid symbols with errors depict the statistical fits of our calculations to the measured $K^+/\pi^+$ with the generic (non)extensive calculations. Open symbols with errors represent the statistical fits based on extensive BG statistics. The solid and double-dotted curves show Eq. (7) of the present study and refs. 21, 22, respectively.

In top panel of Fig. 1, at $\gamma_q = 1$ and varying $\gamma_s$, we notice that $\gamma_s$ exponentially increases, at $\sqrt{s_{NN}} \lesssim 7$ GeV. The same behavior was quantitatively observed in the extensive additive BG calculations in ref. 21. The rate of the increase with increasing $\sqrt{s_{NN}}$ in the present calculations is larger than in extensive additive BG statistics. At higher energies ($7 \lesssim \sqrt{s_{NN}} \lesssim 20$ GeV), $\gamma_s$ rapidly decreases. Also here, the rate of decreasing $\gamma_s$ with generic (non)extensive statistics is larger than that with extensive additive BG statistics. At $\sqrt{s_{NN}} > 20$ GeV, $\gamma_s$ becomes nearly energy independent. At this plateau, the resulting $\gamma_s$ from both types of statistics, present work and ref. 21, is in good agreement with each other, so that we find $\gamma_s \approx 0.78 \pm 0.01$. We conclude that with the global assumption that $\gamma_q = 1$, the resulting $\gamma_s$ equally encodes the partial-chemical nonequilibrium in both types of statistics, at $\sqrt{s_{NN}} > 20$ GeV. At lower energies, both types of statistics significantly disagree with each other, especially at $\sqrt{s_{NN}} \simeq 7$ GeV. Here, the generic (non)extensive $\gamma_s$ is nearly 40% larger than the extensive BG $\gamma_s$, i.e., the partial-chemical nonequilibrium in strange quark occupation factor is $\sim 40\%$ larger. In other words, the signals for the strangeness enhancement, for instance, at the peak (horn) of the Kaon-to-pion ratio, are more than 40% overcast shrouded when measurements, such as the multiplicity of the produced particles, are restricted to extensive additive BG statistics. Hence, a recommendation to the fellow experimentalists can be now addressed. Applying generic (non)extensive statistics, especially in Run II of the STAR beam energy scan program 52, 53, and the future facilities, FAIR and NICA, likely exposes essential experimental signals. Equally recommending that a priori stacking to extensive additive BG statistics probably hides essential statistical features.
Bottom panel of Fig. 1 shows nonequilibrium $\gamma_q$ (left) and $\gamma_c$ (right). In the bottom left panel of Fig. 1, we notice that, at $\sqrt{s_{NN}} \lesssim 7$ GeV, varying $\gamma_q$ calculated from generic (non)extensive statistics rapidly decreases with increasing $\sqrt{s_{NN}}$. $\gamma_q$ deduced from the extensive additive HRG calculations also rapidly decrease with increasing $\sqrt{s_{NN}}^{22}$. It is apparent that the rate of change in the earlier is more than that of the latter. Also here, we find that both maxima of generic (non)extensive and extensive additive BG statistics are likely positioned, at $\sqrt{s_{NN}} \simeq 7$ GeV. At $7 \lesssim \sqrt{s_{NN}} \lesssim 20$ GeV, $\gamma_q$ from both types of statistics increase. At high energies, $\gamma_q$ from both types of statistics become nearly energy independent. The heights of both plateaus are not coincident. The generic (non)extensive draws a lower plateau than that of the extensive additive BG results. Both values of resulting $\gamma_q$ are greater that unity.

For the energy dependence of $\gamma_c$, bottom right panel of Fig. 1, we also find that, at $\sqrt{s_{NN}} \lesssim 7$ GeV, while generic (non)extensive $\gamma_c$ increases, the extensive additive BG $\gamma_c$ decreases with increasing energy. The rate of the earlier is larger than that of the latter. While both minimum and maximum are nearly positioned, at $\sqrt{s_{NN}} \simeq 7$ GeV, their depth and height are not equal. The height of the generic (non)extensive $\gamma_c$ is larger than the depth of the extensive BG $\gamma_c$. Also here, $\gamma_c$ becomes energy independent, at $\sqrt{s_{NN}} > 20$ GeV. The heights of both plateaus are not coincident. The results of generic (non)extensive give a lower plateau than the extensive additive BG. While the extensive additive BG $\gamma_c \simeq 0.98 \pm 0.1$, i.e., very close to the equilibrium unity, the generic (non)extensive $\gamma_c \simeq 0.81 \pm 0.01$ referring to a large nonequilibrium. To draw a conclusion about the degree of (non)equilibrium, the resulting $\gamma_q$, bottom left panel, should also be taken into consideration.

Both bottom panels illustrate that a) the resulting values of $\gamma_q$ and $\gamma_c$ obtained from generic (non)extensive statistics are lower than their values in extensive additive BG statistics, b) at $\sqrt{s_{NN}} \lesssim 7$ GeV, there is an exception that the generic (non)extensive $\gamma_c$ reaches maximum, c) the span of the values that $\gamma_c$ and $\gamma_q$ take in extensive additive BG statistics is narrower than in generic (non)extensive statistics, and d) at $\sqrt{s_{NN}} \gtrsim 20$ GeV, both generic (non)extensive extensive additive BG $\gamma_q$, are nearly energy independent, where $\gamma_q > 1$ and $\gamma_c < 1$.

### 3.2 Equivalence classes $(c,d)$ and nonextensivity

In Fig. 2, the resulting equivalence classes $(c,d)$, at various collision energies$^{27}$, are respectively depicted as squares and circles. Those results could be well parameterized by the following expressions:

\[
c(\sqrt{s_{NN}}) = \left[ a_1 + (\sqrt{s_{NN}})^{-b_1} \right]^{-c_1} d_1 \sqrt{s_{NN}}, \tag{8}
\]

\[
d(\sqrt{s_{NN}}) = \left[ a_2 - (\sqrt{s_{NN}})^{-b_2} \right]^{c_2} \exp \left( \frac{d_2}{\sqrt{s_{NN}}} \right), \tag{9}
\]

where $a_1 = 1.005 \pm 0.01, b_1 = 3.242 \pm 0.106, c_1 = 3.247 \pm 0.24, d_1 = 6.526 \times 10^{-06} \pm 1.003 \times 10^{-06}, a_2 = 1.947 \pm 0.0006, b_2 = 0.008 \pm 0.00015, c_2 = 3.493 \pm 0.0739, \text{and} d_2 = 0.506 \pm 0.019$. Both expressions are respectively depicted as dashed and solid curves in Fig. 2.

At AGS energies, we find that $d$ seems to start from a large value (nearly unity) and rapidly decreases, while $c$ starts from $\sim 0.94$ and exponentially increases with increasing energy. At $\sqrt{s_{NN}} \simeq 4 \pm 1$ GeV, the values of both equivalence classes become nearly equal, $(c,d) \simeq (0.97, 0.97)$. This is a phenomenological finding based on Fig. 2, merely. There is another intersection at $\sqrt{s_{NN}} \simeq 400 \pm 50$ GeV, where the values of both equivalence classes become again nearly equal, i.e., $(c,d) = (0.985, 0.985)$. The physical importance of this finding can be first discussed, if this result shall be confirmed in future studies. For the time being, we would say that there exist two distinguishable regions, at some values of $\sqrt{s_{NN}}$, where $(c,d)$ flip from relative higher to relative lower values and vice versa. At SPS and low RHIC energies, another remarkable conclusion can be drawn. While $c$ reaches a maximum value, at $\sqrt{s_{NN}} \simeq 10 \pm 1$ GeV, $d$ continues its decrease and approaches its minimum $d \sim 0.93 \pm 0.02$, at $\sqrt{s_{NN}} \simeq 20 \pm 2$ GeV. At RHIC and LHC energies, $c$ sets on a wide plateau, at $c \simeq 0.985$. At $\sqrt{s_{NN}} \gtrsim 20$GeV, $d$ switches to an exponential increase with increasing energy. $c$ turns to set on such an exponential increase, at the LHC energies, where both equivalence classes exceed the unity. This means that the typical statistical approach, at LHC energies, should be the one violating the second Shannon-Khinchin axiom. The nature of such a statistical approach is not yet invented, at least it is not fitting within the scope of the present study, which focuses on the energy range typical for the horn-like structure of the Kaon-to-pion ratio.

As discussed, the degree of extensivity and nonextensivity is to be determined by the equivalence classes $(c,d)$, whose energy-dependence was best parameterized by Eqs. (8) and (9), respectively, and depicted in Fig. 2. With this we mean how the values of $(c,d)$ vary from $(1,1)$, extensive additive BG statistics. We notice that the resulting equivalence classes are positive and close to unity. This means that:

1. underlying statistics is neither extensive additive BG, especially at AGS and SPS energies, nor Tsallis-type nonextensive,
2. at AGS, SPS and RHIC energies, the statistical system holds the first, second, and third Shannon-Khinchin axioms but apparently violates the fourth one,
3. at LHC energies, both second and fourth Shannon-Khinchin axioms are likely violated, and

4. the corresponding entropy is likely extended exponential and Lambert-W₀ exponential generating function \( W₀(x) = \sum_{n=1}^{\infty} (-n)^{n-1} x^n / n! \).

Thus, we emphasize that a priori assumption of any type of statistics and applying this over the entire range of the collision energies, would not be fitting with the standard of evidence.

In this regard, we summarize the four Shannon-Khinchin axioms as follows.

- First, the Shannon entropy \( S_n(p) = \tau \cdot \sum_{k=1}^{n} p_k \log_2 p_k \), where \( \tau < 0 \), is continuous in \( \Delta_n \), i.e., the entropy continuously depends on \( p \).

- Second, for the uniform distribution \( U_n = (1/n, \cdots, 1/n) \in \Delta_n \), the Shannon entropy is maximum, i.e., \( S_n(p) \leq S_n(U_n) \) for any \( p \in \Delta_n \).

- Third, the Shannon entropy is expandable; \( S_{n+1}(p_1, p_2, \cdots, p_n, 0) = S_n(p_1, p_2, \cdots, p_n) \) for all \( (p_1, p_2, \cdots, p_n) \in \Delta_n \), i.e., the requirement that adding a zero-probability state to a system does not change the entropy.

- Fourth, the Shannon entropy is separable or additive; \( S(p_{11}, \cdots, p_{1w}, p_{21}, \cdots, p_{2w}, p_{w1}, \cdots, p_{ww}) = \sum_{d=1}^{w} p_1 \cdot S(p_{1d}/p_1, p_{2d}/p_1, \cdots, p_{wd}/p_1) \), where \( p_i = \sum_{j=1}^{w} p_{ij} \).

To suggest a graphical illustration of the violation of the fourth Shannon-Khinchin axiom in the nuclear collision, we would recall the Bjorken space-time evolution of the relativistic partonic matter. We mean that here the initial conditions (past) seem to affect the final state (future), violating stochastic Markovian processes.

Even the slight variation of the equivalence classes from unity as depicted in Fig. 2 allows the resulting \((c,d)\) to greatly modify the statistical nature of the underlying system. This obviously derives the entire statistical system away from extensivity, additivity, and equilibrium. Boltzmann statistics is sharply located at \((c,d) = (1,1)\). Any deviation from \((c,d) = (1,1)\) apparently replaces Boltzmann distribution, entropy, exponential, and logarithm functions by their counterparts characterizing generic (non)extensive statistics, Appendix A. The extended exponential and Lambert-W₀ exponentially generating distribution function, concluded in the present study, are substantially different from the ordinary exponential and distribution functions, review Appendix A for details.
3.3 $K^+/\pi^+$ ratio

Figure 3 depicts the $K^+/\pi^+$ ratio as a function of $\sqrt{s_{NN}}$. Symbols refer to the experimental results, at energies ranging from 5 GeV to 5.44 TeV. The dotted curve shows the generic (non)extensive calculations, at partial-chemical nonequilibrium, i.e., varying $\gamma_s$ while $\gamma_q = 1$. The solid curve illustrates the results, at full-chemical nonequilibrium, i.e., varying $\gamma_s$ and $\gamma_q$. The double-dashed curve gives the extensive additive BG calculations, at partial-chemical nonequilibrium, while the dashed curve presents the extensive additive BG calculations, at full-chemical nonequilibrium. The incapability of the thermal model, the equilibrium extensive additive HRG model, i.e., $\gamma_q = \gamma_s = 1$, in reproducing both the horn-like structure of the $K^+/\pi^+$ ratio, at AGS, SPS, and low RHIC energies, and the $K^+/\pi^+$ ratio, at RHIC and LHC energies, is shown by the long-dashed curve.

In the HRG model; an ideal gas of point-like hadrons characterized by the Hagedorn mass spectrum which is consisting of a discrete (so-far measured) and a continuous (still missing hadron states), various higher-order thermodynamic quantities have been analyzed. In a previous study, one of the authors (AT) concluded that the missing states are not considerably contributing to the lower-order thermodynamic quantities, especially the first-order ones, the particle number or the particle multiplicity. Thus, the inclusion of missing states would not improve the thermal model capability in reproducing the horn-like structure of the $K^+/\pi^+$ ratio. Such a conclusion shall be systematically verified in Appendix B.

![Figure 3](image_url)

**Figure 3.** The experimental results of the $K^+/\pi^+$ ratio are given in dependence on $\sqrt{s_{NN}}$. The solid and dotted curves depict the generic (non)extensive calculations, at full- ($\gamma_q$ and $\gamma_s \neq 1$) and partial-chemical nonequilibrium ($\gamma_q \neq 1$), respectively. The double-dashed and dashed curves represent the extensive additive BG results, at full- and partial-chemical nonequilibrium. The long-dashed curve shows the HRG calculations, at full-chemical equilibrium, i.e., $\gamma_q = \gamma_s = 1$.

We find that the generic (non)extensive calculations in the entire range of collision energies excellently agree with the experimental results. The horn-like structure of the Kaon-to-pion ratio is perfectly reproduced. Comparing to the other types of statistics, the generic (non)extensive approach is well capable in describing the rapid increase, at $\sqrt{s_{NN}} \lesssim 7$ GeV and also the fast decrease, at $7 \lesssim \sqrt{s_{NN}} \lesssim 20$ GeV. At $\sqrt{s_{NN}} \gtrsim 20$ GeV, a fine structure of the plateau is also well predicted. Because of the wide gap between top RHIC and LHC energies, this is an essential estimation. Another finding is that both partial- and full-chemical nonequilibrium results significantly vary, especially at $\sqrt{s_{NN}} \lesssim 7 \pm 1$ GeV. This emphasizes that the light quark occupation factor, $\gamma_q$, becomes significant, at $\sqrt{s_{NN}} \lesssim 7 \pm 1$ GeV. As depicted in the bottom right panel of Fig. 1, at this energy, $\gamma_q$ reaches a minimum, while $\gamma_s$ reaches a maximum. The span between both values increases when moving from partial- to full-chemical nonequilibrium. So does the horn of the Kaon-to-pion ratio. Thus, we conclude that the horn-like structure of the Kaon-to-pion ratio is likely due to

- extended exponential and Lambert-W0 exponentially generating distribution function, i.e., generic (non)extensive statistics, and
- full-chemical nonequilibrium quark occupation factors.
The full-chemical nonequilibrium predicts a further enhancement in the $K^+/\pi^+$ ratio, especially at $3 \lesssim \sqrt{s_{NN}} \lesssim 10$ GeV. At these energies, a detailed beam energy scan by the STAR and CBM experiments, for instance, is recommended to refine the conclusion whether partial- or full-chemical nonequilibrium excellently reproduces the nonmonotonic horn-like structure of the $K^+/\pi^+$ ratio, at AGS, SPS and low RHIC energies.

4 Conclusion

The present paper introduces generic (non)extensive statistics. The physical motivation is not just to propose a approximate solution to a long-standing problem, the failure of thermal models in reproducing the nonmonotonic horn-like structure of the $K^+/\pi^+$ ratio. The assumption that the particle production could be analyzed by extensive additive equilibrium statistics badly contradicts the nature of the high-energy particle production, which is a dynamical critical process evolving through prompt symmetry changes and phase transitions. On one hand, this could be tolerated as long as no other alternative exists. The nonextensive Tsallis approach as a very special case of nonextensivity seems to fail, as well. The present paper introduces a first-principle solution based on a proper statistical approach, namely statistics whose extensivity and nonextensivity match well with the statistical nature of the particle production and the degree of extensivity and nonextensivity is autonomously determined by the statistical system itself.

Various ingredients, such as modifying single-particle distribution function\(^{11}\), assuming nonequilibrium light and strange quark occupation factors in the phase-space volume, i.e., $\gamma_{q,s} \neq [18,21,22]$, and integrating an excluded volume to the pointlike ideal gas of hadron resonances\(^{24,25}\), have been allegedly introduced, to improve the capability of the thermal models, which were either based on extensive additive BG statistics\(^{1,6-12}\) or on Tsallis-type nonextensive statistics\(^{13}\). All these attempts were not fully succeeded in characterizing the particle ratios including the nonmonotonic horn-like structure of the $K^+/\pi^+$ ratio, which likely signals an onset of a nonextensive particle production process and reflects consequences of critical nonequilibrium nonextensive particle production.

The present script assumes that the incapability of the thermal models in reproducing the horn-like structure (long-dashed curve in Fig. 3) is due to an ad hoc inappropriate type of statistics; either extensive additive BG or Tsallis-type nonextensive statistics or their extensions. Inserting nonequilibrium ingredients such as modified single-particle distribution function or nonequilibrium occupation factors $\gamma_{q,s}$ or finite volume of the constituents of the ideal gas to extensive additive BG statistics seems to be conceptionally fruitless. But these might be fruitful in nonextensivity statistics. Relative to the entire $(c-d)$ space of (non)extensivity, the Tsallis-type nonextensivity is limited to a line, where $c = 0$ and $d = q$. On the other hand, generic (non)extensive statistics covers the entire space of the two equivalence classes $(c,d)$. With this statistical approach, the ensemble itself determines its statistical extensivity and nonextensivity and accordingly the proper statistical approach.

To summarize our findings, the present script suggests that

1. neither Boltzmann nor Tsallis statistics alone can be the proper statistical approach, in the entire range of collision energies,
2. there are various types of nonextensivity depending on the collision energies, and
3. there is a rapid change in nonextensivity, an onset of a nonextensive particle production process, at low collision energy.

We conclude that the excellent reproduction of the horn-like structure of the $K^+/\pi^+$ ratio requires:

1. extended exponential and Lambert-$W_0$ exponentially generated distribution rather than BG or Tsallis and
2. partial- or full-chemical nonequilibrium occupation factors.

While both partial- and full-chemical nonequilibrium in generic (non)extensive statistics seem to excellently reproduce the measured $K^+/\pi^+$ ratio, at all energies, the full-chemical nonequilibrium is likely significant, at $3 \lesssim \sqrt{s_{NN}} \lesssim 10$ GeV. To refine the conclusion whether partial- or full-chemical nonequilibrium is essential for the nonmonotonic horn-structure of the $K^+/\pi^+$ ratio, a detailed beam energy scan by the STAR and CBM experiments, for instance, is highly recommended.

The methodology applied in the present study goes as follows. First, the partial- and full-chemical nonequilibrium, $\gamma_{q}$ or/and $\gamma_{s}$ have been determined, at different energies. Second, in the same manner, the energy-dependence of $(c,d)$ are also estimated. In calculating the $K^+/\pi^+$ ratio, the quark occupation factors $\gamma_{q,s}$, the equivalence classes $(c,d)$, and the freezeout parameters $(T,\mu_B)$, are no longer fitting variables. They are inputs, at various energies. We found that the $K^+/\pi^+$ ratio calculated in generic (non)extensive statistics excellently well reproduces the experimental results, including the horn-like structure. The nonmonotonic energy-dependence of $\gamma_{q,s}$ and that of $(c,d)$ together are essential. In this regard, we assure that other particle ratios such proton-to-pion and Lambda-to-pion are also perfectly reproduced, Appendix C.
A Probability distribution, entropy, exponential and logarithm functions in Boltzmann, Tsallis and generic (non)extensive statistics

For a reliable comparison between Boltzmann, Tsallis, and generic (non)extensive statistics, we first recall the corresponding probability distributions

**Extensive BG:**  \[ p_i(x) = \frac{\exp(-x)}{\mathcal{Z}(x)} = \frac{1}{\mathcal{Z}(x)} \sum_{n=0}^{\infty} \frac{x^n}{n!}, \]  

**Nonextensive Tsallis:**  \[ p_i^{(q)}(x) = \frac{\exp(q)(-x)}{\mathcal{Z}^{(q)}(x)} = \frac{1}{\mathcal{Z}^{(q)}(x)} \left[ 1 - x + \frac{q}{2} x^2 - \frac{q^2}{3!} x^3 + \cdots \right] = \frac{1}{\mathcal{Z}^{(q)}(x)} \left[ 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} Q_{n-1}(q) \right] = \frac{1}{\mathcal{Z}^{(q)}(x)} \left[ 1 + (1 - q)x^{1/(1-q)} \right], \]  

**(Non)extensive:**  \[ p_i^{(c,d,r)}(x) = \frac{\exp^{(c,d,r)}(-x)}{\mathcal{Z}^{(c,d,r)}(x)} = \exp \left[ \frac{d}{1-q} \left\{ W_k \left[ B \left( 1 - \frac{x}{a} \right)^{1/d} \right] - W_k[B] \right\} - \mathcal{R} p_i \right] \]  

where \( \mathcal{Z}(x) \) is the corresponding canonical partition function, \( Q_{n-1}(q) := \Pi_{i=0}^{n}[iq-(i-1)] \), and the asymptotic expansion of 0-th branch Lambert-W function is given as

\[ W_{k=0}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{n-1}}{n!} x^n. \]

The probability distribution function can be obtained from the integral of the probability density function determining the probability of an individual **event** to take place in a given interval. Also, the probability distribution function can be derived from the maximization of the corresponding entropy under appropriate constraints

**Extensive BG:**  \[ s[p] = - \kappa \sum_{i=1}^{\Omega} p_i \ln p_i, \]  

**Nonextensive Tsallis:**  \[ s^{(q)}[p] = \frac{\kappa}{1-q} \sum_{i=1}^{\Omega} \left( p_i^{(q)} - p_i \right), \]  

**(Non)extensive:**  \[ s^{(c,d,r)}[p] = \kappa \sum_{i=1}^{\Omega} \left[ c \Gamma(d + 1, 1-c \log p_i) - \mathcal{R} p_i \right], \]

where \( \Omega \) is the number of micro-states or processes. \( \kappa, \mathcal{A}, \) and \( \mathcal{R} \) are arbitrary parameters. The incomplete gamma-function is given as

\[ \Gamma(a,x) = \int_{x}^{\infty} dt t^{a-1} \exp(-t) = \Gamma(a) - \frac{x^a}{a \exp(x)} \sum_{i=0}^{\infty} \frac{x^i(a)!}{(a-i)!}. \]

The differences between Boltzmann, Tsallis, and generic (non)extensive statistical statistics can be characterized by the mathematical meaning of the corresponding exponential and logarithm functions; namely for Boltzmann \( \exp(x) \) and \( \ln(x) \), for Tsallis \( \exp^{(q)}(x) \) and \( \ln^{(q)}(x) \), and for generic (non)extensive functions \( \exp^{(c,d,r)}(x) \) and \( \ln^{(c,d,r)}(x) \). For example, the Tsallis exponential and logarithm functions, respectively read

\[ \exp^{(q)}(-x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n-1}}{n!} x^n = 1 - x + \frac{q}{2!} x^2 - \frac{q^2}{3!} x^3 + \cdots, \]

\[ \ln^{(q)}(x) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n-1}}{n} (x-1)^n = (x-1) - \frac{q}{2} (x-1)^2 + \frac{q^2}{3} (x-1)^3 - \cdots. \]

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For generic (non)extensive exponential and logarithm functions, generalization of BG \( \exp(x) \) and \( \ln(x) \) gives\(^{36}\)

\[
\exp^{(c,d,r)}(x) = \exp \left\{ \frac{-d}{1-c} \left[ B_k \left( (1-x/r)^{1/d} \right) - B_k(B) \right] \right\},
\]

\[
\ln^{(c,d,r)}(x) = \frac{r x^{c-1}}{1 - \frac{1 - (1-c) r}{r d} \log(x)} d.
\]

It is now in order to comment on the interpretations of the Tsallis nonextensivity parameter \( q \). First, the assumption that \( q > 1 \) manifests fluctuations in \( 1/\lambda \) of the so-called Levy exponential distribution, \( \exp(-x/\lambda) \), is apparently superficial\(^{65}\). The nature of such fluctuations was not determined yet. On the other hand, such fluctuations should be also accompanied by a remarkable change in the phase space, itself\(^{11}\). Likewise, we assess the connection between the departure of \( q \) from its extensive value, the unity, from above and the large fluctuations of temperature. Second, the claim on the existence of fundamental differences between BG- and Tsallis-statistics leads to an \textit{ad hoc} prerequisite that the thermodynamic temperature, the one deduced from BG statistics, differs from the one determined in Tsallis statistics, \( T_{\text{Tsallis}} = T_{\text{BG}} + (q-1)K \), where the parameter \( K \) should be modeled as a function of the energy transfer between the source and the surrounding\(^{31}\). This \textit{phenomenological} interpretation is closely related to blind improper implementation of Tsallis-algebra by replacing \( \exp(x) \) with \( \exp_q(x) \), Eq. (16) and \( \ln(x) \) with \( \ln_q(x) \), Eq. (16). It is worthy highlighting that the reproduction of the transverse momentum distributions was misinterpreted and accordingly the resulting temperature\(^{56}\). As discussed, there should be no fundamental difference between BG- and Tsallis-statistics, unless the earlier is extensive and the latter is nonextensive statistical framework! Otherwise, one assumes that the type of statistics applied modifies the underlying ensemble, which in turn characterizes a certain physical system.

The present script suggests that Boltzmann and Tsallis statistics are subdomains of generic (non)extensive statistics. The Tsallis nonextensive parameter \( q \) is also included in BG; \( q = 1 \), as well as in generic (non)extensive statistics; \( c = 0 \) and \( d = q \). The equivalence classes \((c,d)\) are intrinsic properties of the entire statistical space, where \((c,d) = (1,1)\) characterize Boltzmann statistics and \((c,d) = (0,q)\) represents the Tsallis type. The thermodynamic quantities either intensive or extensive, such as temperature, entropy, pressure, and energy density, should remain identical in the entire statistical space, i.e., they should not be changing when moving from BG to Tsallis or to any other type of nonextensive statistics. The interpretations of \( q \) and of \((c,d)\) are not just dummy fluctuations but rather nonextensive statistical characteristics. Their nature is related to the properties of the relevant statistical ensemble.

**B Inclusion of very high-mass resonances and missing states in the hadron resonance gas model**

The authors of ref.\(^{57}\) introduced two ingredients to the extensive additive equilibrium hadron resonance gas model. The first one is very high-mass resonances (\( > 2 \) GeV). The second one is the \( \sigma \) meson or \( f_0(600) \). The various decay channels are also taken into consideration.

The \( \sigma \) meson is characterized by energy-dependong mass and varying decay constants. Its dominant decay channel produces pion pair. This might be the motivation of ref.\(^{57}\). Accordingly, it has been estimated that \( \sigma \) meson comes up with about 3.5% enhancement in the pion yields. The inclusion of very high-mass resonances also increase the pion yields, this time by about 13%. Enhancing pion yields against Kaon yields seems to reproduce the nonmonotonic \( K^+ / \pi^+ \) ratio, on one hand. On the other, the standard of evidence prerequisites that the Kaon yields should be enhanced, similarly! Alternatively, the ensemble of the hadron resonances remains unbalanced.

Fig. 4 presents unbiased balanced reproduction of \( K^+ / \pi^+ \) ratio (solid curve), where all hadrons and resonances reported in recent PDG, including \( \sigma \) meson and its heavier resonances\(^{68}\), as well as 56 missing states are included in. Most if not all decay channels are also taken into consideration, so that the final number of any particle is summed over that particle and all other decay channels producing it weighted with the corresponding decay width. The masses of the resonances can be as heavier as 12 GeV. We conclude that the inclusion of the very heavy-mass resonances would slightly enhance the respective particle yields, on one hand. On the other hand, such an enhancement is faded away in the relative production ratio. The \( K^+ / \pi^+ \) ratio is not well reproduced either with or without the inclusion of missing states.

The contributions of the missing states to the various thermodynamic quantities\(^{60}\) are depicted in Fig. 5. At 7.7, 19.6, 39 and 200 GeV, the number density, energy density, and pressure calculated in the extensive additive equilibrium hadron resonance gas model, are compared with (symbols) and without (curves) missing stated and when including (symbols) and excluding (curves) very-heavy mass resonances. We conclude that the inclusion of the missing states and very-heavy mass resonances minimally contributes to the particle yields. These contributions are apparently balanced in the relative particle production ratios, Fig. 4. In this regard, we recall that the authors of ref.\(^{60}\) also have drawn the same conclusion that “susceptibilities of conserved charges, […] can be attributed to the missing resonances in the strange baryon sector.” In other words, the missing
states become first dominant in the higher-order moments of conserved charges, such as susceptibilities and cumulants. But they slightly contribute to lower-order moments, such as the number density. Even this slight contribution is faded away in the particle ratios.

C Contour plots of the quality of fits for $\gamma_{q,s}$

A suppression in $\gamma_{q}$ was assumed as a signature of onset of the color deconfinement\textsuperscript{26}. The nonequilibrium quark occupation factors, $\gamma_{q,s} \neq 1$, are conjectured to dictate the numbers of light- and strange-quarks, respectively, which can be accommodated in the phase-space volume and accordingly the production rates of light and strange particle yields\textsuperscript{8,21,22}. In all these studies, extensive additive equilibrium BG statistics was utilized.

When $\gamma_{q}$ is determined from the statistical fit of one particle ratio, the other particle ratios are not well reproduced. Although various other particle ratios can simultaneously be reproduced for nonequilibrium $\gamma_{q,s}$, the resulting values of $\gamma_{q,s}$ are scattered, Table 1 of ref.\textsuperscript{8}. Values of $\gamma_{q,s}$ much greater than unity are tough to incorporate with a nonequilibrium approach.

Figure ?? shows the contour plots of the quality of fits for $\gamma_{q}$ (top left panel), $\gamma_{s}$ (top right panel), and $\gamma_{q,s}$ (bottom panel). The resulting of $\gamma_{q,s}$ are parameterized in Eq. (7). When comparing with Table 1 of ref.\textsuperscript{8}, we conclude that the present estimations can be well incorporated with a nonequilibrium approach.

With the same sets of equivalence classes given by Eqs. (8)-(9) and nonequilibrium quark-occupation factors $\gamma_{q,s}$ given by Eq. (7), not only $K^{+}/\pi^{+}$ ratio is well reproduced, but other particle ratios, as well, for example, $P/\pi^{+}$ and $\Lambda/\pi^{-}$, Fig. 7. We like to highlight that at $> 4$ TeV, the ensemble of the experimental results available so-far is not large enough to deduce the equivalence classes. Thus, we have assumed that their results, at lower LHC energies, remain unchanged. There is a little overestimation, which can be improved with the release of LHC results in the near future.

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Declaration

- The authors declare that there are no conflicts of interest regarding the publication of this published article!
Figure 5. Temperature dependence of number density, energy density and pressure calculated in the extensive additive equilibrium hadron resonance gas model, at 7.7, 19.6, 39, and 200 GeV, respectively. The curves show the results with truncated hadron mass spectrum, $< 2$ GeV. With the symbols, we show the results, where 56 missing states are included in and the hadron mass can be as heavy as 12 GeV.

- The authors declare that the present script is in compliance with ethical standards regarding its content!
- The funding agencies have no role in the design of the study; in the collection, analysis or interpretation of the data; in the writing of the manuscript; or in the decision to publish the results!

Dataset Availability

All data generated or analyzed during this study are included in this published article. The data used to support the findings of this study are included within the published article and properly cited! The row data and the thermal model calculations that support the findings of this study are also available from the corresponding author upon reasonable request.

Author contributions statement

AT proposed the conception of the present study, designed and managed the research, interpreted the results, derived the various statistical approaches, and prepared the manuscript. ERA and HY collected the empirical data, generated the thermal models calculations, and drawn the figures. All authors contributed to the analysis, reviewed the results, and approved the final version of the manuscript.

References

Figure 6. Top panel: contour plot of the quality of fit for $\gamma_q$ at $\gamma_s = 1$ (left) and $\gamma_s$ at $\gamma_q = 1$ (right), over different collision energies. Bottom panel: the same as in the top panel but for varying $\gamma_q$ and $\gamma_s$.


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This work, $\chi_{q=1}, \chi_{s=1}$

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Figure 7. Left panel: Energy dependence of $P/\pi^+$, where the same sets of the equivalence classes are given by Eqs. (8)-(9) and $\gamma_{q,s}$ by Eq. (7), as the ones for $K^+/\pi^+$ are utilized. Right panel: The same as in the left panel but here the energy dependence of $\Lambda/\pi^-$. 


