Tensor product of finite dimensional nilpotent evolution algebras

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TENSOR PRODUCT OF FINITE DIMENSIONAL NILPOTENT EVOLUTION ALGEBRAS

IZZAT QARALLEH

ABSTRACT. This paper investigates the tensor product of a finite-dimensional nilpotent evolution algebra. Some properties that translate from tensor products to factors and vice versa have been investigated, including the index of nilpotency and annihilator. The index of nilpotency of the tensor product of two nilpotent evolution algebras with different indexes of nilpotency is determined. Moreover, we investigate the tensorially decomposable of the 4-dimensional nilpotent evolution algebra. In addition, the decomposable nilpotent evolution algebra with the maximal nilindex of nilpotency has been carried out.

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Key words: evolution algebra; Tensor product; isomorphism; nilpotent;

1. INTRODUCTION

Several kinds of non-associative algebras, such as baric, evolution, Bernstein, train, stochastic, etc., have been linked to abstract algebra and biology [1–3]. Numerous substantial contributions to population genetics theory have been made by studying such algebras [3]. We note that the roots of population genetics problems may be traced to Bernstein’s work [4], where evolution operators were examined, which naturally define genetic algebras (see [3,5]). Evolution algebras are considered a type of genetic algebra, which are non-associative algebras with a dynamic nature. Such a type of algebra has been introduced in [6,7]. After that, in [8], the foundations of these algebras were established. Later on, evolution algebras are used to model non-Mendelian genetics laws [9–13]. Moreover, these algebras are tightly connected with group theory, the theory of knots, dynamic systems, Markov processes, and graph theory [14–18] and [43, 44, 46]. Evolution algebras introduced proper algebraic techniques and methods for investigating some digraphs because such algebras and weighted digraphs can be canonically identified [7,19]. In most investigations, evolution algebras are considered nilpotent [19–24]. A few papers are devoted to non-nilpotent evolution algebras [25–27]. In [28–30], a new class of evolution algebras called Lotka–Volterra evolution algebras has been introduced (see also [31]). An algebra is said to be nilpotent if there exists a natural integer m such that any product of m elements of the algebra is zero. If there is a non-zero product of m − 1 elements, then m is called the index of nilpotency of the algebra. Algebraic concepts such as nilpotency, right nilpotency, and solvability can be viewed as various sorts of vanishing (“deaths”) populations in biology. The matrix A is a quadratic matrix representing the structural constants of an evolution algebra (see Section 2). In [11], the equivalence between nil, right nilpotent evolution algebras and evolution algebras defined by upper triangular matrices A is demonstrated, as well as the classification of 2-dimensional complex evolution algebras. In [36], it was established a criterion for an n−dimensional nilpotent evolution algebra to be of maximal nilpotent index 2n−1 + 1. Furthermore, they have been constructed a wide classes of nilpotent evolution algebra whose the index of nilpotency of the form 2n−s + 1, where 2 ≤ s ≤ n − 1. Other properties of nilpotent evolution algebra such as derivation, local derivation, automorphism, and local automorphism have been studied in (see [23], [37]). In [24], the full invariant of n−dimensional nilpotent
evolution algebra with maximal nilindex have been found. These led to classified such kinds of algebras. Some other classes of evolution algebra, which is not nilpotent have been introduced (see [38], [39]). The relation of evolution algebra and other kinds of non-associative algebras has been established [17]. Recently, in [40] the tensor product of evolution algebras has been defined and studied for the inheritance of properties from the tensor product to the factor and conversely. For instance, non-degeneracy, irreducibility, perfection, and simplicity were investigated. In this paper, we study the tensor product of finite-dimensional nilpotent evolution algebras and investigate the properties of the tensor product to the factors, and conversely, then we discuss the index of nilpotency of the tensor product of two nilpotent evolution algebras with different indexes of nilpotency. Moreover, the decomposability of a nilpotent evolution algebra, whose index of nilpotency is maximal, has been considered.

The decomposable of some families of finite dimensional nilpotent evolution algebra in the sense $\mathcal{E} = I \oplus J$ where $I, J$ are indecomposable ideals, have been studied in [42, 45]. We note that such decomposable does not imply tensorally decomposable. Let us briefly describe the structure of this paper. Section 2 contains preliminary definitions of evolution algebra and tensor product of evolution algebras. In Section 3, we discuss the index of nilpotency of the tensor product of finite dimensional nilpotent evolution algebras. In Section 4, the decomposability of the tensor product of four dimensional nilpotent evolution algebra has been carried out.

2. Preliminaries

In this section, we recall the definitions of evolution algebra, and the tensor product and some definitions which are needed through the paper.

Definition 2.1. Let $\mathcal{E} := (\mathcal{E}, ,)$ be an algebra over a field $\mathbb{K}$. we say that $\mathcal{E}$ is an evolution algebra if it admits a basis $B := \{e_1, e_2, \ldots, e_n\}$, such that

$$e_i . e_j = \sum_{k=1}^{n} a_{ik} e_k, \quad \text{if } i = j,$$
$$0, \quad \text{if } i \neq j.$$  

(2.1)

The scalars $a_{ik}$ are called the structure constants of $\mathcal{E}$ relative to $B$. A basis $B$ satisfying (2.1) is called natural basis of $\mathcal{E}$.

Now, Let us recall the definition of tensor product of vector space The tensor product of two vector spaces $V$ and $W$ over a field $\mathbb{K}$ is a vector space defined as the set

$$V \otimes W = \left\{ \sum_{i=1}^{n} \lambda_i v_i \otimes w_i : \lambda_i \in \mathbb{K}, v_i \in V, w_i \in W, n \in \mathbb{N}^* \right\}$$

where $v \otimes w$ is a bilinear map

$$v \otimes w : V^* \times W^* \to \mathbb{K}$$
$$\forall v \in V, \forall w \in W \text{ and } V^* \text{ and } W^* \text{ denote the dual vector space of } V \text{ and } W \text{ respectively. Moreover, if } \{e_i\}_{i \in \Lambda} \text{ is a basis of } V \text{ and } \{f_j\}_{j \in \Gamma} \text{ is a basis of } W, \text{ then } \{e_i \otimes f_j\}_{i \in \Lambda, j \in \Gamma} \text{ is a basis of the tensor product } V \otimes W.$$

Definition 2.2. [40] Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two $\mathbb{K}$-algebras with bases $\{e_i\}_{i \in \Lambda}$ and $\{f_j\}_{j \in \Gamma}$ respectively. We define a product on $\mathcal{E}_1 \otimes \mathcal{E}_2$ as follows:

$$(e_i \otimes f_j) \cdot (e_k \otimes f_r) = e_i e_k \otimes f_j f_r.$$
Definition 2.3. [40] We say that a \( \mathbb{K} \)-algebra \( E \) is tensorially decomposable if it is isomorphic to \( E_1 \otimes E_2 \) where \( E_1 \) and \( E_2 \) are \( \mathbb{K} \)-algebras with \( \dim(E_1), \dim(E_2) > 1 \). Otherwise we say that \( E \) is tensorially indecomposable.

Proposition 2.4. [40] If \( E_1 \) and \( E_2 \) are evolution \( \mathbb{K} \)-algebras, then \( E_1 \otimes E_2 \) is also an evolution \( \mathbb{K} \)-algebra. Furthermore, if \( B_1 = \{ e_i \}_{i \in \Lambda} \) be a natural basis of \( E_1 \) and let \( B_2 = \{ f_j \}_{j \in \Gamma} \) be a natural basis of \( E_2 \), then \( \{ e_i \otimes f_j \}_{i \in \Lambda, j \in \Gamma} \) is a natural basis of \( E_1 \otimes E_2 \).

Recall that if \( E \) is evolution algebra then \( \text{ann}(E) := \text{span}\{ e_i : e_i^2 = 0 \} \).

For the evolution algebra \( E \) we have the following for \( m > 1 \):

\[
E^m = \sum_{j=1}^{m-1} E^j E^{m-j}
\]

Due to commutative of \( E \), we have

\[
E^m = \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} E^j E^{m-j}
\]

Definition 2.5. An evolution algebra \( E \) is called nilpotent if there exists some \( n \in \mathbb{N} \) such that \( E^n = 0 \). The smallest \( n \) such that \( E^n = 0 \) is called the index of nilpotency.

In [11], it is proved that the notions of nilpotent and right nilpotent are equivalent.

Theorem 2.6 ([11]). The following statements are equivalent for an \( n \)-dimensional evolution algebra \( E \):

(a) The matrix corresponding to \( E \) can be written as

\[
\hat{A} = \begin{pmatrix}
0 & a_{12} & a_{13} & \ldots & a_{1n} \\
0 & 0 & a_{23} & \ldots & a_{2n} \\
0 & 0 & 0 & \ldots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix};
\]

(b) \( E \) is a right nilpotent algebra;
(c) \( E \) is a nil algebra.

Remark 2.7. It will know that there is index of nilpotency which is not in form \( 2^m + 1 \) see remark (3.14) in [36]. In this paper, we consider only the finite nilpotent evolution algebra when \( a_{ii+1} \neq 0 \) for any non zero row of structural matrix of \( E \).

Definition 2.8. An evolution algebra \( E \) is tensorially decomposable if it is isomorphic to \( E_1 \otimes E_2 \) where \( E_1 \) and \( E_2 \) are evolution algebras with \( \dim(E_1), \dim(E_2) > 1 \). Otherwise we say that \( E \) is tensorially indecomposable.

Remark 2.9. It is worth to mentioned that in [40], it was found an example of 4 dimensional algebra which is decomposable for two algebra which are not evolution algebras. In this paper, we are focus on the nilpotent evolution algebras which are decomposable to two evolution algebras.
3. Index of Nilptency of the Tensor Product of Finite Dimensional Evolution Algebras

**Proposition 3.1.** Let $E_1$ and $E_2$ be two finite dimensional nilpotent evolution algebras with natural basis $B_1 = \{e_i\}_{i=1}^N$, $B_2 = \{f_j\}_{j=1}^M$ respectively, then the following statements hold true:

(i) $(e_i \otimes f_j)^2 = 0$ for any $i$, and any $j$ such that $f_j \in \text{ann}(E_2)$,
(ii) $(e_i \otimes f_j)^2 = 0$, for any $j$, and any $i$ such that $e_i \in \text{ann}(E_1)$.

**Proof.** Assume that $f_j \in \text{ann}(E_2)$, then $f_j^2 = 0$. Therefore,

$$(e_i \otimes f_j)^2 = (e_i \otimes f_j)(e_i \otimes f_j) = (e_i^2 \otimes f_j^2) = (e_i^2 \otimes 0) = 0.$$ 

Similarly, if $e_i \in \text{ann}(E_1)$, then $e_i^2 = 0$, we have

$$(e_i \otimes f_j)^2 = (e_i \otimes f_j)(e_i \otimes f_j) = (e_i^2 \otimes f_j^2) = (0 \otimes f_j^2) = 0.$$ 

$\square$

**Proposition 3.2.** Let $E_1$ and $E_2$ be two finite dimensional nilpotent evolution algebras with natural basis $B_1 = \{e_i\}_{i=1}^N$, $B_2 = \{f_j\}_{j=1}^M$ respectively, then the following statements hold true:

(i) $(e_i \otimes f_j) \notin (E_1 \otimes E_2)^2$ for any $j$,
(ii) $(e_i \otimes f_1) \notin (E_1 \otimes E_2)^2$ for any $i$.

**Proof.** Since $E_1$ is nilpotent then $e_i \notin E_1^2$ this means that $(e_i \otimes f_j) \notin E_1^2 \otimes E_2^2 = (E_1 \otimes E_2)^2$ for any $1 \leq j \leq M$. Similarly, since $E_2$ is nilpotent then $f_1 \notin E_2^2$ this means that $(e_i \otimes f_1) \notin E_1^2 \otimes E_2^2 = (E_1 \otimes E_2)^2$ for any $1 \leq i \leq N$. $\square$

**Theorem 3.3.** Let $E_1$ and $E_2$ be two finite dimensional nilpotent evolution algebras, with natural basis $B_1 = \{e_i\}_{i=1}^N$ and $B_2 = \{f_j\}_{j=1}^M$ respectively, then $E_1 \otimes E_2$ is nilpotent evolution algebra.

**Proof.** We know from proposition (2.4) that $E_1 \otimes E_2$ is also evolution algebra and $E_1 \otimes E_2 = \langle e_i \otimes f_j \rangle$ , where $1 \leq i \leq N$, $1 \leq j \leq M$. Now, we have to show that the structural matrix of $E_1 \otimes E_2$ which is denoted by $M_{E_1 \otimes E_2}$ can be written as the form of theorem (2.6). Consider

$$(e_i \otimes f_j)^2 = (e_i \otimes f_j)(e_i \otimes f_j) = (\sum_{k=1}^N a_{ik} e_k \otimes \sum_{m=1}^M b_{jm} f_m)$$

But $E_1$ and $E_2$ are nilpotent, then $a_{ik} = 0$, $b_{jm} = 0$ for any $1 \leq k \leq i$, $1 \leq m \leq j$. Hence,

$$(e_i \otimes f_j)^2 = \left( \sum_{k=i+1}^N a_{ik} e_k \otimes \sum_{m=j+1}^M b_{jm} f_m \right)$$

$$= \sum_{k=i+1}^N a_{ik} e_k \otimes \sum_{m=j+1}^M b_{jm} f_m$$

$$= \sum_{k=i+1}^N \sum_{m=j+1}^M a_{ik} b_{jm} (e_k \otimes f_m).$$

Moreover, since $(e_k \otimes f_m) \notin (E_1 \otimes E_2)^2$ and $e_k \otimes f_m \in \text{ann}(E_1 \otimes E_2)$ if and only if $e_k \in \text{ann}(E_1)$ or $f_m \in \text{ann}(E_2)$. Hence, $M_{E_1 \otimes E_2}$ has the form as in (2.6). Thus, $E_1 \otimes E_2$ is nilpotent evolution algebra. $\square$
Remark 3.4. [40] (Structure matrix of tensor product of finite dimensional evolution algebras) Suppose \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are two finite dimensional evolution \( \mathbb{K} \)-algebras with natural basis \( B_1 = \{e_1, \ldots, e_N\} \) and \( B_2 = \{f_1, \ldots, f_M\} \) respectively. Let \( M_{B_1} = (a_{ij}) \) and \( M_{B_2} = (b_{km}) \) be the structure matrices associated to \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) respectively. Then, the structure matrix of the evolution algebra \( \mathcal{E}_1 \otimes \mathcal{E}_2 \) relative to the basis \( B_1 \otimes B_2 = \{e_1 \otimes f_1, \ldots, e_1 \otimes f_M, \ldots, e_N \otimes f_1, \ldots, e_N \otimes f_N\} \) is the Kronecker product of \( M_{B_1} \) and \( M_{B_2} \), i.e., \( M_{B_1 \otimes B_2} = M_{B_1} \otimes M_{B_2} \).

Next corollary is about the dimension of annihilator of the tensor product to two finite dimensional nilpotent evolution algebras with maximal nil-index.

Corollary 3.5. Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be two finite dimensional nilpotent evolution algebras with maximal nilindex with natural basis \( B_1 = \{e_i\}_{i=1}^N \), \( B_2 = \{f_j\}_{j=1}^M \) respectively, then

\[
\dim(\text{ann}(\mathcal{E}_1 \otimes \mathcal{E}_2)) = N + M - 1.
\]

Proof. Let us define the following sets:

\[
A_1 = \text{span}\{e_i \otimes f_M : 1 \leq i \leq N\}, \quad A_2 = \text{span}\{e_N \otimes f_j : 1 \leq j \leq M - 1\}
\]

It is not hard to show that \( A_1 \) and \( A_2 \) are subalgebras. Clearly,

\[
\text{ann}(\mathcal{E}_1 \otimes \mathcal{E}_2) = A_1 \cup A_2, \quad A_1 \cap A_2 = \{0\}.
\]

Moreover, \( \dim(A_1) = N \), \( \dim(A_2) = M - 1 \)

Then,

\[
\dim(\text{ann}(\mathcal{E}_1 \otimes \mathcal{E}_2)) = \dim(A_1) + \dim(A_2) = N + M - 1.
\]

\(\square\)

Proposition 3.6. Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be two finite dimensional nilpotent evolution algebras with natural basis \( B_1 = \{e_i\}_{i=1}^N \), \( B_2 = \{f_j\}_{j=1}^M \) respectively, then

\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k+s} = \text{span}\{e_i \otimes f_j : k + 2 \leq i \leq N, \ k + 2 \leq j \leq M\}
\]

\[
s = 1, 2, \ldots, 2^k, \ k = 0, 1, 2, \ldots
\]

Proof. The proof process by induction. For \( k = 0 \), using Lemma (3.2), we have \( (e_1 \otimes f_j) \notin (\mathcal{E}_1 \otimes \mathcal{E}_2)^2 \) and \( (e_i \otimes f_1) \notin (\mathcal{E}_1 \otimes \mathcal{E}_2)^2 \) for any \( i, j \). Since \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are nilpotent, then \( a_{ii+1} \neq 0 \) and \( b_{jj+1} \neq 0 \) for any \( 1 \leq i \leq N - \dim(\text{ann}(\mathcal{E}_1)), \text{ and } 1 \leq j \leq M - \dim(\text{ann}(\mathcal{E}_2)) \) and thus,

\[
e_i^2 \otimes f_j^2 = \left( \sum_{k=i+1}^N a_{ik} e_k \otimes \sum_{m=j+1}^M b_{jm} f_m \right)
\]

Then, \( (e_i \otimes f_j) \notin (\mathcal{E}_1 \otimes \mathcal{E}_2)^2 \) for any \( i, j \neq 1 \). Hence,

\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^2 = \text{span}\{e_i \otimes f_j : 2 \leq i \leq N, \ 2 \leq j \leq M\}.
\]

For \( k = 1 \), we have

\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^3 = (\mathcal{E}_1 \otimes \mathcal{E}_2)^4 = (\mathcal{E}_1 \otimes \mathcal{E}_2)^2 (\mathcal{E}_1 \otimes \mathcal{E}_2)^2.
\]

Using step when \( k = 0 \), one has

\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^2 (\mathcal{E}_1 \otimes \mathcal{E}_2)^2 = \text{span}\{e_i^2 \otimes f_j^2 : 2 \leq i \leq N, \ 2 \leq j \leq M\}.
\]

Since \( e_2 \notin \mathcal{E}_1^2 \) and \( f_2 \notin \mathcal{E}_2^2 \) then \( e_2 \otimes f_j \notin (\mathcal{E}_1 \otimes \mathcal{E}_2)^3 \) and \( e_i \otimes f_2 \notin (\mathcal{E}_1 \otimes \mathcal{E}_2)^3 \) for any \( i, j \geq 2 \). Then

\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^3 = (\mathcal{E}_1 \otimes \mathcal{E}_2)^4 = \text{span}\{e_i \otimes f_j : 3 \leq i \leq N, \ 3 \leq j \leq M\}.
\]

Assume the equality is true for \( k - 1 \), then

\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k-1+s} = \text{span}\{e_i \otimes f_j : k + 1 \leq i \leq N, \ k + 1 \leq j \leq M\}.
\]
Taking the maximum value of \( s = 2^{k-1} \) yields,
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k} = \text{span}\{e_i \otimes f_j : \ k + 1 \leq i \leq N, \ k + 1 \leq j \leq M\}.
\]
Now, for \( k \), we have
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k+i} = \sum_{m=1}^{2^{k-1+\lfloor \frac{i}{2} \rfloor}} (\mathcal{E}_1 \otimes \mathcal{E}_2)^m (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{k-s-m}+s-m} \]
\[
= \sum_{m=1}^{2^{k-1+\lfloor \frac{i}{2} \rfloor}} (\mathcal{E}_1 \otimes \mathcal{E}_2)^m (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{k-1+s}} \]
\[
= \sum_{m=1}^{2^{k-1+\lfloor \frac{i}{2} \rfloor}} (\mathcal{E}_1 \otimes \mathcal{E}_2)^m (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k} \]
\[
= (\mathcal{E}_1 \otimes \mathcal{E}_2) (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k} = (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k} (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k}.
\]
Hence,
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{k+s}} = (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k} (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2k} = \text{span}\{e_i^2 \otimes f_j^2 : \ k + 1 \leq i \leq N, \ k + 1 \leq j \leq M\}.
\]
This implies that
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{k+s}} = \text{span}\{e_i \otimes f_j : \ k + 2 \leq i \leq N, \ k + 2 \leq j \leq M\}.
\]
This completes the proof.

**Theorem 3.7.** Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be two finite dimensional nilpotent evolution algebras with maximal nilindex with natural basis \( B_1 = \{e_i\}^N_{i=1}, \ B_2 = \{f_j\}^M_{j=1} \) respectively, then the index of nilpotency of \( \mathcal{E}_1 \otimes \mathcal{E}_2 \) is \( 2^{\min\{N,M\}} - 1 + 1 \).

**Proof.** Using Proposition (3.6) we have
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{k+s}} = \text{span}\{e_i \otimes f_j : \ k + 2 \leq i \leq N, \ k + 2 \leq j \leq M\},
\]
\[
s = 1, 2, \ldots, 2^k, \ k = 0, 1, 2, \ldots
\]
Without loss of generality, we may assume that \( \min\{N, M\} = M \).
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{M-2+s}} = \{e_i \otimes f_M\}
\]
Take the maximum value of \( S = 2^{M-1} \), one has
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{M-1}} = \{e_i \otimes f_M\}.
\]
Using the fact \( (\mathcal{E}_1 \otimes \mathcal{E}_2)^{k+1} = (\mathcal{E}_1 \otimes \mathcal{E}_2)^k (\mathcal{E}_1 \otimes \mathcal{E}_2)^k \), we have
\[
(\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{M-1}+s} = (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{M-1}} (\mathcal{E}_1 \otimes \mathcal{E}_2)^{2^{M-1}} = \{e_i^2 \otimes f_M^2\} = 0.
\]
Thus, then the index of nilpotency of \( \mathcal{E}_1 \otimes \mathcal{E}_2 \) is \( 2^{M-1} + 1 \) where \( M = \min\{N, M\} \). This completes the proof.

Next theorem is about if we have two finite dimensional nilpotent evolution algebra whose the index of nilpotency is necessary maximal nilindex.Its known that the class of finite dimensional evolution algebra is huge and we can not come over all cases. In what follows, we will consider the classes of nilpotent evolution algebras that have the index of nilpotency \( 2^{\dim(\mathcal{E}) - \dim(\text{ann}(\mathcal{E}))} + 1 \).

We notice that if \( \dim(\text{ann}(\mathcal{E})) = 1 \), then \( \mathcal{E} \) has maximal nil-index.
Proposition 3.8. Let $E$ be a finite dimensional evolution algebra with natural basis $B = \{e_i\}_{i=1}^N$, then, up to isomorphism, $e_{N-\dim(\text{ann}(E))+i} \in \text{ann}(E)$, for all $1 \leq i \leq \dim(\text{ann}(E))$.

Theorem 3.9. Let $E_1$ and $E_2$ be two finite dimensional nilpotent evolution algebras with natural basis $B_1 = \{e_i\}_{i=1}^N$, $B_2 = \{f_j\}_{j=1}^M$ respectively, if the index of nilpotency of $E_1$ is $2^{N-\dim(\text{ann}(E_1))} + 1$ and the index of nilpotency of $E_2$ is $2^{M-\dim(\text{ann}(E_2))} + 1$, then the index of nilpotency of $E_1 \otimes E_2$ is $2^r + 1$ where $r = \min\{N - \dim(\text{ann}(E_1)), M - \dim(\text{ann}(E_2))\}$.

Proof. Without loss of generality, we may assume that $r = M - \dim(\text{ann}(E_2))$. From Proposition (3.6), we have

\begin{equation}
(E_1 \otimes E_2)^{r+1} = \text{span}\{e_i \otimes f_j : k + 2 \leq i \leq N, k + 2 \leq J \leq M\}.
\end{equation}

Let $k = r - 1$ in equation (3.4), then one has that

\begin{equation}
(E_1 \otimes E_2)^r = \text{span}\{e_i \otimes f_j : r + 1 \leq i \leq N, r + 1 \leq j \leq M\}.
\end{equation}

By proposition (3.8), one has that $f_{r+s} \in \text{ann}(E_2)$. Hence, $f_{r+s}^2 = 0$ for all $1 \leq s \leq \dim(\text{ann}(E_2))$. Therefore,

\begin{equation}
(E_1 \otimes E_2)^{2r+1} = (E_1 \otimes E_2)^r (E_1 \otimes E_2)^r = \text{span}\{e_i^2 \otimes f_j^2 : r + 1 \leq i \leq N, r + 1 \leq j \leq M\}.
\end{equation}

But

\begin{equation}
\text{span}\{e_i^2 \otimes f_j^2\} = \text{span}\{e_i^2 \otimes 0\} = 0.
\end{equation}

for all $r + 1 \leq i \leq N$, $r + 1 \leq j \leq M$. Thus, the index of nilpotency of $2^r + 1$. This completes the proof.

Example 3.10. Let $E_1$ and $E_2$ be two dimensional nilpotent evolution algebras with natural basis $B_1 = \{e_i\}_{i=1}^4$, $B_2 = \{f_j\}_{j=1}^3$ respectively, if $\dim(\text{ann}(E_1)) = 1$ and $\dim(\text{ann}(E_2)) = 2$, then the index of nilpotency of $E_1 \otimes E_2$ is

\begin{equation}
2^r + 1 = 2^{3-2} + 1 = 3.
\end{equation}

Remark 3.11. The advantage of discussing such properties of tensor product for finite dimensional evolution algebra is that if we have nilpotent evolution algebra $E$ whose dimension is $N \times M$ and if $E \cong E_1 \otimes E_2$ which means $E$ is decomposable for two nilpotent evolution algebra, then one can find the index of nilpotency of $E$ through the minimum index of nilpotency of $E_1$ and $E_2$.

Now, the natural question arising here is that what is the result of tensor product of two evolution algebras if one of them at least is not nilpotent. Next, theorem will answer the question when one of them is nilpotent and other one is not.

Theorem 3.12. Let $E_1$ and $E_2$ be finite dimensional evolution algebras relative to the natural basis $B_1 = \{e_i\}_{i=1}^N$, $B_2 = \{f_j\}_{j=1}^M$ if $E_1$ or $E_2$ is nilpotent then $E_1 \otimes E_2$ is nilpotent. Moreover, then index of the nilpotency of $E_1 \otimes E_2$ is the same index of nilpotency of $E_1$ or $E_2$.

Proof. Assume that $E_1$ is nilpotent evolution algebra and $E_2$ is evolution algebra which is not nilpotent. As we know $E_1 \otimes E_2 = \text{span}\{e_i \otimes f_j : 1 \leq i \leq N, 1 \leq j \leq M\}$. Since $E_1$ is nilpotent then $e_i \notin \langle f_j^2 \rangle$, for all $1 \leq i \leq N$. Consequently, $e_i \otimes f_j \notin \langle (e_i \otimes f_j)^2 \rangle$, for all $1 \leq i \leq N, 1 \leq j \leq M$. Then, $MB_1 \otimes B_2$ can be written in the form (2.4). Thus, $E_1 \otimes E_2$ is nilpotent.

Now, $E_1$ is nilpotent then the index of nilpotency of $E_1$ is $2^{N-\dim(\text{ann}(E_1))} + 1$. In (3.1), put $k = N - \dim(\text{ann}(E_1)) - 1$, then one finds that

\begin{equation}
(E_1 \otimes E_2)^{2^{N-\dim(\text{ann}(E_1))}} = \text{span}\{e_i \otimes f_j : N - \dim(\text{ann}(E_1)) + 1 \leq i \leq N, 1 \leq j \leq M\}.
\end{equation}
Then
\[(E_1 \otimes E_2)^{2^N - \dim(\text{ann}(E_1)) + 1} = \text{span}\{e_i^2 \otimes f_j^2 : N - \dim(\text{ann}(E_1)) + 1 \leq i \leq N, 1 \leq j \leq M\} .\]

But \(e_{N - \dim(\text{ann}(E_1)) + s} \in \text{ann}(E_1)\), then \(e_{N - \dim(\text{ann}(E_1)) + s}^2 = 0\), for all \(1 \leq s \leq \dim(\text{ann}(E_1))\).

Hence,
\[
\text{span}\{e_i^2 \otimes f_j^2\} = \{0\}, N - \dim(\text{ann}(E_1)) + 1 \leq i \leq N, 1 \leq j \leq M .
\]

This implies that the index of nilpotency of \(E_1 \otimes E_2\) is \(2^{N - \dim(\text{ann}(E_1)) + 1}\). □

4. Decomposability of the tensor product of nilpotent evolution algebras

In this section, we are going to study the decomposability of four dimensional nilpotent evolution algebras. In [41] it has been classified the four dimensional nilpotent evolution algebras to 10 no isomorphic classes including trivial one.

**Theorem 4.1.** Let \(E_1\) and \(E_2\) be finite dimensional evolution algebras with natural basis \(B_1 = \{e_i\}_{i=1}^N, B_2 = \{f_j\}_{j=1}^M\) respectively, if \(E_1 \otimes E_2\) is nilpotent evolution algebra then either \(E_1\) or \(E_2\) is nilpotent evolution algebra.

**Proof.** Suppose that neither \(E_1\) nor \(E_2\) is nilpotent. Since \(E_1 \otimes E_2\) is nilpotent, then there is \(r \in \mathbb{N}^+\) such that
\[
(E_1 \otimes E_2)^{2r - \dim(\text{ann}(E_1), \text{ann}(E_2))} = \text{span}\{e_i \otimes f_j : r - \dim(\text{ann}(E_1)) + 1 \leq i \leq N, r - \dim(\text{ann}(E_2)) + 1 \leq j \leq M\}.
\]

And
\[
(E_1 \otimes E_2)^{2r - \dim(\text{ann}(E_1), \text{ann}(E_2)) + 1} = \text{span}\{e_i^2 \otimes f_j^2\} = 0.
\]

\[
r - \dim(\text{ann}(E_1)) + 1 \leq i \leq N, r - \dim(\text{ann}(E_2)) + 1 \leq j \leq M .
\]

Therefore, \(e_i^2 \otimes f_j^2 = 0\) for all
\[
r - \dim(\text{ann}(E_1)) + 1 \leq i \leq N, r - \dim(\text{ann}(E_2)) + 1 \leq j \leq M .
\]

This implies that either \(e_i^2 = 0\) or \(f_j^2 = 0\) for all
\[
r - \dim(\text{ann}(E_1)) + 1 \leq i \leq N, r - \dim(\text{ann}(E_2)) + 1 \leq j \leq M .
\]

Without loss of generality, we may assume that \(e_i^2 = 0\) then \(E_1^{2r - \dim(\text{ann}(E_1)) + 1} = 0\). But this contradicts our assumption. Therefore, either \(E_1\) or \(E_2\) is nilpotent. This completes the proof. □

**Corollary 4.2.** Let \(E_1\) and \(E_2\) be finite dimensional evolution algebras relative to the natural basis \(B_1 = \{e_i\}_{i=1}^N, B_2 = \{f_j\}_{j=1}^M\), then
\[
\dim(\text{ann}(E_1 \otimes E_2)) = \dim(\text{ann}(E_1))M + N\dim(\text{ann}(E_2)) - \dim(\text{ann}(E_1))\dim(\text{ann}(E_2)) .
\]

This corollary leads to the following result:

**Lemma 4.3.** If \(E\) is a finite dimensional nilpotent evolution algebras with maximal nilindex, then \(E\) is not decomposable.

**Proof.** Assume that \(E_1\) and \(E_2\) are two finite dimensional evolution algebras. Suppose that \(E \cong E_1 \otimes E_2\), then
\[
\dim(\text{ann}(E)) = \dim(\text{ann}(E_1 \otimes E_2)) .
\]

Since \(E\) is nilpotent with maximal nilindex, then \(\dim(\text{ann}(E)) = 1\), By corollary (4.2), the equation \(\dim(\text{ann}(E_1))M + \dim(\text{ann}(E_2))N - \dim(\text{ann}(E_1))\dim(\text{ann}(E_2)) = 1\). Assume that \(\dim(\text{ann}(E_1)) = k_1\) and \(\dim(\text{ann}(E_2)) = k_2\). By theorem (4.1), we have that \((k_1, k_2) \neq (0, 0)\).
then, we have to find solution of the following equation \( k_1M + k_2N - k_1k_2 = 1 \) in \( \mathbb{N}^* \). it easy to see that
\[
k_1M + k_2N - k_1k_2 > k_1^2 + k_2^2 - 2k_1k_2 = (k_1 + k_1)^2 \geq 1
\]
Then the equation \( k_1M + k_2N - k_1k_2 = 1 \) has no solution in \( \mathbb{N}^* \).

Thus, \( E \not\cong E_1 \otimes E_2 \). This implies that \( E \) is not decomposable. \( \square \)

**Theorem 4.4.** Any 4-dimensional nilpotent evolution algebra over an algebraically closed field \( \mathbb{F} \) is isomorphic to one of the following pairwise non-isomorphic algebras:

<table>
<thead>
<tr>
<th>( E )</th>
<th>Multiplication Table</th>
<th>( \text{ann} (E) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{4,1} )</td>
<td>( e_1^2 = e_2 )</td>
<td>( \langle e_2, e_3, e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,2} )</td>
<td>( e_1^2 = e_3 ) ( e_2^2 = e_3 )</td>
<td>( \langle e_3, e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,3} )</td>
<td>( e_1^2 = e_2 ) ( e_3^2 = e_4 )</td>
<td>( \langle e_3, e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,4} )</td>
<td>( e_1^2 = e_2 ) ( e_3^2 = e_2 ) ( e_4^2 = e_3 )</td>
<td>( \langle e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,5} )</td>
<td>( e_1^2 = e_3 ) ( e_4^2 = e_3 ) ( e_2^2 = e_4 )</td>
<td>( \langle e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,6} )</td>
<td>( e_1^2 = e_3 ) ( e_2^2 = e_3 ) ( e_4^2 = e_4 )</td>
<td>( \langle e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,7} )</td>
<td>( e_1^2 = e_4 ) ( e_2^2 = e_4 ) ( e_3^2 = e_4 )</td>
<td>( \langle e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,8} )</td>
<td>( e_1^2 = e_4 ) ( e_2^2 = e_3 ) ( e_3^2 = e_4 )</td>
<td>( \langle e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,9} )</td>
<td>( e_1^2 = e_4 ) ( e_2^2 = e_3 ) ( e_4^2 = e_4 )</td>
<td>( \langle e_4 \rangle )</td>
</tr>
<tr>
<td>( E_{4,10} )</td>
<td>( e_1^2 = e_3 ) ( e_2^2 = e_4 )</td>
<td>( \langle e_3, e_4 \rangle )</td>
</tr>
</tbody>
</table>

Table 1. Nilpotent evolution algebras of dimension four.

**Theorem 4.5.** Let \( E \) is 4-dimensional nilpotent evolution algebra then \( E \) is decomposable if it isomorphic to one of \( E_{4,1}, E_{4,2}, E_{4,3} \) or \( E_{4,10} \), which are given in Theorem (4.4).

**Proof.** Assume that \( E \cong E_{4,1} \), one can easily find \( E_{4,1} \cong E_1 \otimes E_2 \) where \( E_1 \) and \( E_2 \) are 2 dimensional trivial evolution algebras. Hence, \( E \) is decomposable.

If \( E \cong E_{4,2} \), The nilpotent evolution algebra \( E_{4,2} \cong E_1 \otimes E_2 \), where the table of multiplication of \( E_1 \) is \( e_1^2 = e_1 \), \( e_2^2 = 0 \) and the table of multiplication of \( E_2 \) is \( f_1^2 = f_2 \), \( f_2^2 = 0 \). Thus, \( E \) is decomposable.

If \( E \cong E_{4,3} \), The nilpotent evolution algebra \( E_{4,3} \cong E_1 \otimes E_2 \), where the table of multiplication of \( E_1 \) is \( e_1^2 = e_2 \), \( e_2^2 = 0 \) and the table of multiplication of \( E_2 \) is \( f_1^2 = f_1 \), \( f_2^2 = f_1 \). Thus, \( E \) is decomposable.

If \( E \cong E_{4,10} \), The nilpotent evolution algebra \( E_{4,10} \cong E_1 \otimes E_2 \), where the table of multiplication of \( E_1 \) is \( e_1^2 = e_2 \), \( e_2^2 = 0 \) and the table of multiplication of \( E_2 \) is \( f_1^2 = f_1 \), \( f_2^2 = f_2 \). Thus, \( E \) is decomposable. This completes the proof. \( \square \)

**Theorem 4.6.** Let \( E \) is 4-dimensional nilpotent evolution algebra then \( E \) is not decomposable if it isomorphic to one of \( E_{4,4}, E_{4,5}, E_{4,6}, E_{4,7}, E_{4,8} \) or \( E_{4,9} \), which are given in Theorem (4.4).

**Proof.** If \( E \cong E_{4,4} \), Suppose that \( E_{4,4} \) is decomposable then \( E_{4,4} \cong E_1 \otimes E_2 \). By Theorem (4.1) we have either \( E_1 \) is nilpotent or \( E_2 \) is nilpotent, then the index of nilpotency of \( E_1 \otimes E_2 \) is 3, but the index of nilpotency of \( E \) is 5. Hence, \( E \not\cong E_{4,4} \), which contradicts our condition. Thus, \( E_{4,4} \) is not decomposable.

If \( E \cong E_{4,j} \), \( 5 \leq j \leq 9 \). Clearly, \( \dim(\text{ann}(E_{4,j})) = 1 \). Suppose that \( E_{4,j} \cong E_1 \otimes E_2 \). By Corollary (4.2), we have \( \dim(\text{ann}(E_1 \otimes E_2)) = 2k_1 + 2k_2 - k_1k_2 \), where \( k_1 = \dim(\text{ann}(E_1)) \), and \( k_2 = \dim(\text{ann}(E_2)) \). But we know that either \( E_1 \) is nilpotent or \( E_2 \) is nilpotent. Without loss of
generality, we may assume that $E_1$ is nilpotent then $k_1 = 2$. Hence, the equation $2k_1 + 2k_2 - k_1k_2 = 1$ has no solution in $\mathbb{N}^*$. Thus, $E_{4,j} \not\cong E_1 \otimes E_2$. Therefore, $E$ is not decomposable.

By combining Theorems (4.5) and (4.6), we have the following table:

**Theorem 4.7.** Let $E$ be a 4-dimensional nilpotent evolution algebra over an algebraically closed field $\mathbb{F}$. Then we have the following table:

<table>
<thead>
<tr>
<th>$E$</th>
<th>Multiplication Table</th>
<th>$\text{ann}(E)$</th>
<th>Decomposability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{4,1}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,2}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,3}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,4}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,5}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,6}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,7}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,8}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,9}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$E_{4,10}$</td>
<td>$e_1 = e_2, e_1^2 = e_3, e_2^2 = e_3$</td>
<td>$(e_1, e_2)$</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 2. Decomposability of nilpotent evolution algebras of dimension four.

**Theorem 4.8.** Let $E$ be a 6-dimensional decomposable nilpotent evolution algebra then the index of nilpotency of $E$ is 2, 3 or 5.

**Proof.** Since $E$ is decomposable, then $E \cong E_1 \otimes E_2$ with $\dim(E_1), \dim(E_2) > 1$. By Theorem (4.1), we have either $E_1$ or $E_2$ is nilpotent. Let us now, consider the following cases:

**Case 1.** If $E_1$ is nilpotent and $E_2$ is not nilpotent, then the index of nilpotency of $E_1 \otimes E_2$ is the same index of nilpotency of $E_1$. If $\dim(E_1) = 2$ then the index of nilpotency of $E_1$ is 3. Therefore, the index of nilpotency of $E_1 \otimes E_2$ is 3. If $\dim(E_1) = 3$ then the possible indexes of nilpotency are 3 or 5.

**Case 2.** If $E_1$ and $E_2$ is nilpotent, then by Theorem (3.9) the index of nilpotency $E_1 \otimes E_2$ is the minimum index of nilpotency between $E_1$ and $E_2$ which is 3.

**Case 3.** If $E_1$ or $E_2$ is trivial evolution algebra, then again by Theorem (3.9) one has that the index of nilpotency of $E_1 \otimes E_2$ is 2.

So, the possibilities index of nilpotency of $E$ are 2, 3, and 5.

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**Data Availability.** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**References**


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