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Positive periodic solutions for a second-order singular differential equation with two indefinite weights *

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Abstract—This paper aims to show that the fixed point theorem of cone mapping can be applied to a second-order singular differential equation with two indefinite weights. Applying the positiveness of Green’s function of a second-order linear nonhomogeneous differential equation, we prove the existence of a positive periodic solution for a second-order singular differential equation with two indefinite weights.

Keyword—fixed point theorem of cone mapping; singularity; two indefinite weights; positive periodic solutions.

MSC—34B16; 34B18; 34C25.

1 Introduction

The main purpose of this paper is to consider the existence of a positive periodic solution for the following second-order singular differential equation with two indefinite weights

\[ u'' + a(t)u = \frac{b(t)}{u^\rho} - \frac{c(t)}{u^\mu} + e(t), \]  

(1.1)

where \( \rho \) and \( \mu \) are two positive real constants, \( a \in L^p(\mathbb{R}/T\mathbb{Z}) \) for some \( 1 \leq p < +\infty \) and \( e \in L^1(\mathbb{R}/T\mathbb{Z}) \) are positive, \( b, c \in L^1(\mathbb{R}/T\mathbb{Z}) \), \( b \) and \( c \) may change sign, even it may be zero, and \( T \) is a positive constant. According to the related literature [1, 2], the singular term \( \frac{b(t)}{u^\rho} - \frac{c(t)}{u^\mu} \) represents attractive-repulsive singularities if \( b(t) > 0 \) and \( c(t) > 0 \) for all \( t \in [0, T] \) (or \( b(t) < 0 \) and \( c(t) < 0 \) for all \( t \in [0, T] \)), an attractive singularity if \( b(t) < 0 \) and \( c(t) \equiv 0 \) for all \( t \in [0, T] \), a repulsive singularity if \( b(t) > 0 \) and \( c(t) \equiv 0 \) for all \( t \in [0, T] \), and indefinite singularity if \( b(t) \) and \( c(t) \) are allowed to change.

Attractive-repulsive singularities can be regarded as a generalized Lennard-Jones force [3, 4], and they are widely used in molecular dynamics to model the interaction between atomic particles [5]. By this reason, there have been some results published on differential equations with attractive-repulsive singularities [6, 7, 8, 9, 10, 11]. For examples, Hakl and Torres [9] in 2010 discussed the existence of a positive periodic solution for a special form of (1.1), where \( a(t) \equiv 0 \), i.e.,

\[ u'' = \frac{b(t)}{u^\rho} - \frac{c(t)}{u^\mu} + e(t). \]  

(1.2)

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Their proof was based on the method of lower and upper solutions. Afterward, Chu et al. [8] in 2016 improved the results [9], they studied the existence of twist periodic solutions for (1.2). Recently, by using Schauder’s fixed point theorem, Cheng and Ren [6] in 2018 proved that (1.1) had a positive periodic solution. We notice that in [6, 8, 9], $b, c \in L^1(\mathbb{R}/T\mathbb{Z})$ are positive, $\rho$ and $\mu$ are two positive constants, and $\rho \geq \mu$.

On the other hand, the study of the existence of positive periodic solutions for differential equations with indefinite singularity began in 2010, Bravo and Torres [12] investigated a special form of (1.1), where $a(t) \equiv c(t) \equiv e(t) \equiv 0$, i.e.,

$$u'' = \frac{b(t)}{u^\rho}. \quad (1.3)$$

They investigated that (1.3) had a positive periodic solution if $\int_0^T b(t) < 0$ and $\rho = 3$. After that, Hakl and Zamora [13] in 2017 improved the results of [12], they showed in their Theorem 1.2 that (1.3) had at least one positive periodic solution if $\rho \geq 1$ and $\int_0^T b(t) dt < 0$. Recently, Godoy and Zamora [14] in 2019 considered the existence of positive periodic solutions for (1.3) if $0 < \rho < 1$ and $\int_0^T b(t) dt < 0$. These two results in [13] and [14] are complementary, the proofs of Hakl [13] and Godoy [14] were based on the Leray-Schauder degree theory.

We are mainly motivated by the recent works [6, 8, 9, 12, 13, 14] and focus on the existence of a positive periodic solution of (1.1) in the cases that $\rho > \mu$, $\rho = \mu$, and $\rho < \mu$. Our proof is based on a fixed point theorem of cone mapping. Moreover, the results are applicable to weak as well as strong singularities.

## 2 Preliminaries and notations

Our proof is based on the following fixed point theorem of cone mapping, which can be found [15, Theorem 20.1]

**Lemma 2.1.** Let $X$ be a Banach space, and $K \subset X$ be a closed convex cone. Assume that $\Omega_1$ and $\Omega_2$ are bounded open subsets of $X$ with $0 \in \Omega_1$, $\Omega_1 \subset \Omega_2$. Let

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous mapping. If $T$ satisfies the following conditions

(i) $\nu \neq \tau T \nu$ for $\nu \in K \cap \partial \Omega_1$, $0 < \tau \leq 1$; and

(ii) there exists $\nu_0 \in K \setminus \{0\}$ such that $\nu - T \nu \neq \lambda \nu_0$ for $\nu \in K \cap \partial \Omega_2$ and $\lambda \geq 0$,

or the following conditions

(i) $\nu \neq \tau T \nu$ for $\nu \in K \cap \partial \Omega_2$, $0 < \tau \leq 1$; and

(ii) there exists $\nu_0 \in K \setminus \{0\}$ such that $\nu - T \nu \neq \lambda \nu_0$ for $\nu \in K \cap \partial \Omega_1$ and $\lambda \geq 0$.

Then $T$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

A second important tool in our proofs is the concept of Green function. We consider the following nonhomogeneous linear differential equation

$$\begin{cases}
u'' + d(t)u = h(t), \\
u(0) = u(T), \quad u'(0) = u'(T),
\end{cases} \quad (2.1)$$

where $d$ and $h \in L^1(\mathbb{R}/T\mathbb{Z})$. As a consequence of Fredholm’s alternative, (2.1) has a unique $T$-periodic solution which can be written as

$$u(t) = \int_0^T G(t, s)h(s)ds,$$
where \( G(t,s) \) is the Green’s function of (2.1) (see [16, p. 645]). Torres [16] in 2003 discussed the sign of the Green’s function \( G(t,s) \), he obtained the following conclusion.

**Lemma 2.2.** (see [16, Corollary 2.3]) Define function

\[
A(\gamma) = \begin{cases}
\frac{2\pi}{\gamma^{\gamma+1}} \left( \frac{2}{\pi \gamma} \right)^{1-\frac{\gamma}{2}} \left( \frac{\Gamma(1+\frac{\gamma}{2})}{\Gamma(\gamma+1)} \right)^2, & \text{if } 1 \leq \gamma < \infty, \\
\frac{1}{\gamma}, & \text{if } \gamma = \infty,
\end{cases}
\]

where \( \Gamma \) is the Gamma function, i.e., \( \Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x} dx \). Assume that \( d(t) > 0 \) (i.e., \( d \geq 0 \) for almost every \( t \in [0,T] \)) and \( d \in L^p(\mathbb{R}/\omega \mathbb{Z}) \).

\[
||d||_p := \left( \int_0^T |d(t)|^p dt \right)^{\frac{1}{p}} < A(2p^*),
\]

where \( p^* = \frac{p}{p-1} \) if \( 1 \leq p < \infty \) and \( p^* = 1 \) if \( p = +\infty \), then \( G(t,s) > 0 \) for all \( (t,s) \in [0,T] \times [0,T] \).

**Remark 2.1.** (See [17, Lemma 2.3] and [18, Lemma 2.4]) In the special case \( d(t) \equiv \sigma^2 \) with \( \sigma > 0 \) and \( \sigma \neq \frac{2k\pi}{T} \) for any natural \( k \), the Green’s function has the following form

\[
G_1(t,s) = \begin{cases}
\cos \sigma(t - s - \frac{T}{2}), & 0 \leq s < t \leq T, \\
2\sigma \sin \frac{\sigma}{2} t, & 0 \leq t < s \leq T,
\end{cases}
\]

If \( \sigma < \frac{\pi}{2} \), then the Green’s function \( G_1(t,s) > 0 \) for all \( (t,s) \in [0,T] \times [0,T] \).

Denote

\[
m := \min_{0 \leq s,t \leq T} G(t,s), \quad M := \max_{0 \leq s,t \leq T} G(t,s), \quad \delta := \frac{m}{M + \max_{0 \leq s,t \leq T} \frac{|\partial G(t,s)|}{\partial t}}, \quad \eta := \frac{\max_{0 \leq s,t \leq T} |\partial G(t,s)|}{m},
\]

\[
m_1 := \min_{0 \leq s,t \leq T} G_1(t,s) = \frac{1}{2\sigma} \cot \frac{\sigma T}{2}, \quad M_1 := \max_{0 \leq s,t \leq T} G_1(t,s) = \frac{1}{2\sigma} \sin \frac{\sigma T}{2}, \quad \delta_1 := \frac{m_1}{M_1 + \max_{0 \leq s,t \leq T} \frac{|\partial G_1(t,s)|}{\partial t}}, \quad \eta_1 := \frac{\max_{0 \leq s,t \leq T} \frac{|\partial G_1(t,s)|}{\partial t}}{m_1} = \sigma \tan \frac{\sigma T}{2},
\]

(2.3)

It is clear that \( 0 < m \leq M \) and \( 0 < \delta \leq 1 \) from (2.2) of Lemma 2.2.

Let \( C_T := \{ u \in C(\mathbb{R}), \ u(t+T) \equiv u(t), \ \forall t \in \mathbb{R} \} \) with norm \( ||u||_{\infty} := \max_{t \in \mathbb{R}} |u(t)| \), \( C^1_T := \{ u \in C^1(\mathbb{R}) : u(t+T) \equiv u(t), \ u'(t+T) \equiv u'(t), \ \forall t \in \mathbb{R} \} \) with norm \( ||u|| := ||u||_{\infty} + ||u'||_{\infty} \). \( C_T \) and \( C^1_T \) are Banach space. Define

\[
\mathcal{K} := \{ u \in C^1_T : \min_{t \in \mathbb{R}} u(t) \geq \delta ||u||, \ |u'(t)| \leq \eta u(t), \ \forall t \in \mathbb{R} \}.
\]

It is easy to verify that \( \mathcal{K} \) is a cone in \( C^1_T \).

Finally, for a given periodic functions \( b(t) \) and \( e(t) \), we denote

\[
b^+(t) := \max\{0, b(t)\}, \quad b^-(t) := -\min\{0, b(t)\}, \quad \overline{b} := \frac{1}{T} \int_0^T b(t) dt, \quad e^+ := \max_{t \in [0,T]} e(t), \quad e^- := \min_{t \in [0,T]} e(t).
\]

3
3 Main results

In the section, we consider the existence of a positive periodic solution for (1.1) in the cases that \( \rho > \mu \), \( \rho = \mu \), and \( \rho < \mu \).

3.1 The case \( \rho > \mu \)

Observe the weight terms \( b \) and \( c \), (1.1) may possess both attractive, repulsive, and attractive-repulsive singularities depending on the variable \( t \), even the singularity can disappear if the weight terms \( b \) and \( c \) are equal to zero for some subintervals. To get around two indefinite singularities, the change of variable \( u = \nu^\alpha \), where \( \alpha = \frac{1}{1+\rho} \) and \( \alpha < 1 \) from \( \rho > 0 \), we transform and simplify (1.1) into the regular singular equation

\[
\frac{\nu''}{\alpha} + \frac{a(t)}{\alpha} \nu = \frac{e(t)}{\alpha} \nu^{1-\alpha} + (1-\alpha) \frac{\nu'^2}{\nu} + \frac{b(t)}{\alpha} \nu - \frac{c(t)}{\alpha} \nu^{\frac{\alpha}{1+\rho}}.
\]  

(3.1)

It is clear that the existence of a positive periodic solution to (1.1) reduces to the existence of a positive periodic solution to (3.1). Let \( d(t) = \frac{a(t)}{\alpha} \), by Lemmas 2.1 and 2.2, we obtain the following conclusion.

Theorem 3.1. Assume that \( \|a\|_\rho \leq \alpha \Lambda (2^\ast) \) and \( MT(1-\alpha)\eta^2 < \delta^2 \) hold. Furthermore, suppose the following inequality is satisfied

\[
\frac{b^\ast}{\bar{m}T\delta} > \frac{\alpha}{\bar{m}T\delta} \max \left\{ \left( \frac{2\|b^\ast\|_\infty}{e\ast} \right)^\frac{1}{\bar{m}}, \left( \frac{2\|c^\ast\|_\infty}{e\ast} \right)^\frac{1}{\bar{m}} \right\}, \tag{3.2}
\]

Then (1.1) admits at least one positive \( T \)-periodic solution if \( \rho > \mu \).

Proof. A \( T \)-periodic solution of (3.1) is just a fixed point of the map \( T \) defined by

\[
(T\nu)(t) := \int_0^T G(t,s) \left( \frac{e(s)}{\alpha} \nu^{1-\alpha}(s) + (1-\alpha) \frac{\nu'(s)^2}{\nu(s)} + \frac{b(s)}{\alpha} \nu - \frac{c(s)}{\alpha} \nu^{\frac{\alpha}{1+\rho}}(s) \right) ds,
\]

where we know \( G(t,s) > 0 \) for all \( (t,s) \in [0,T] \times [0,T] \) by Lemma 2.2.

Now we define two open sets

\[
\Omega_1 := \{ \nu \in C_T^1 : \|\nu\| < r_1 \} \quad \text{and} \quad \Omega_2 := \{ \nu \in C_T^1 : \|\nu\| < r_2 \}.
\]

Note that \( \Phi \) is well-defined in the set \( \mathcal{K} \cap (\Omega_2 \setminus \Omega_1) \), and it is a completely continuous operator by a standard application of Ascoli-Arzela Theorem. Our intention is to apply Lemma 2.1.

By (3.2), the positive constants \( r_1 \) and \( r_2 \) can be fixed such that

\[
r_2 > r_1 = \frac{mTb^\ast}{\alpha} > \frac{1}{\delta} \max \left\{ \left( \frac{2\|b^\ast\|_\infty}{e\ast} \right)^{\frac{1}{\bar{m}}}, \left( \frac{2\|c^\ast\|_\infty}{e\ast} \right)^{\frac{1}{\bar{m}}} \right\} > 0.
\]

Step 1. We claim that \( T(\mathcal{K} \cap (\Omega_2 \setminus \Omega_1)) \subset \mathcal{K} \). In fact, for any \( \nu \in \mathcal{K} \cap (\Omega_2 \setminus \Omega_1) \), we have

\[
\delta r_1 = \delta \|\nu\| \leq \|\nu(t)\| \leq \|\nu\|_\infty \leq r_2, \quad \text{for all} \quad t \in \mathbb{R}.
\]

Since \( r_1 > \frac{1}{\delta} \max \left\{ \left( \frac{2\|b^\ast\|_\infty}{e\ast} \right)^{\frac{1}{\bar{m}}}, \left( \frac{2\|c^\ast\|_\infty}{e\ast} \right)^{\frac{1}{\bar{m}}} \right\} \), we give

\[
\frac{e(t)}{2\alpha} \nu^{1-\alpha}(t) > \frac{e\ast}{2\alpha} (\delta r_1)^{1-\alpha} \geq \frac{\|b^\ast\|_\infty}{\alpha} \geq \frac{b^\ast(t)}{\alpha} > 0,
\]

for all \( t \in \mathbb{R} \), and

\[
\frac{e(t)}{2\alpha} \nu^{1-\alpha}(t) \nu^{-\frac{\alpha}{1+\rho}}(t) = \frac{e(t)}{2\alpha} \nu^{1-\alpha}(t) \nu^{-\frac{\alpha}{1+\rho}}(t) > \frac{e\ast}{2\alpha} (\delta r_1)^{1-\alpha} \geq \frac{\|c^\ast\|_\infty}{\alpha} \geq \frac{c^\ast(t)}{\alpha} > 0,
\]

for all \( t \in \mathbb{R} \), and
for all $t \in \mathbb{R}$. Combining (3.3) and (3.4), we get

$$
\frac{e(t)}{\alpha} t^{\nu - (t)} + (1 - \alpha) \frac{\nu^{y}}{\nu} + b(t) - \frac{c(t)}{\alpha} \nu^{d}(t) = \frac{e(t)}{\alpha} t^{\nu - (t)} + (1 - \alpha) \frac{\nu^{y}}{\nu} + b^{+}(t) - b(t) - \frac{c^{(t)} - c(t)}{\alpha} \nu^{d}(t)
$$

(3.5)

for all $t \in \mathbb{R}$. It follows from (2.3) and (3.5) that

$$
\text{min}_{t \in \mathbb{R}} \int_{t}^{T} G(t, s) \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds \geq m \int_{t}^{T} \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds
$$

$$
= \delta M \int_{t}^{T} \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds + \delta \max_{0 \leq s, t \leq T} \left| \frac{\partial G(t, s)}{\partial t} \right| \int_{0}^{T} \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds
$$

$$
= \delta \| T \nu \|_{\infty} + \delta \| (T \nu) \|_{\infty}
$$

Besides, we deduce

$$
\left| \frac{d(T \nu)}{dt} \right| = \left| \int_{0}^{T} \frac{\partial G(t, s)}{\partial t} \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds \right|
$$

$$
\leq \int_{0}^{T} \left| \frac{\partial G(t, s)}{\partial t} \right| \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds
$$

$$
\leq \max_{t \in \mathbb{R}} \int_{0}^{T} \left| \frac{\partial G(t, s)}{\partial t} \right| \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds
$$

$$
= m \int_{0}^{T} \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds
$$

$$
\leq \eta \int_{0}^{T} G(t, s) \left( \frac{e(s)}{\alpha} t^{\nu - (s)} + (1 - \alpha) \frac{\nu^{y}(s)}{\nu} + b(s) - \frac{c(s)}{\alpha} \nu^{d}(s) \right) ds
$$

$$
= \eta(T \nu)(t).
$$

This shows that $T(\mathcal{K} \cap (\Omega_{2} \setminus \Omega_{1})) \subset \mathcal{K}$.

**Step 2.** We prove that

$$
\nu \neq \tau T \nu, \quad \text{for all } \nu \in \mathcal{K} \cap \partial \Omega_{2}. \quad (3.6)
$$

In fact, if (3.6) does not hold, there exist $\nu_{1} \in \mathcal{K} \cap \partial \Omega_{2} \cap 0 < \tau \leq 1$ such that

$$
\nu_{1} = \tau T \nu_{1}.
$$

Because $\nu_{1} \in \mathcal{K} \cap \partial \Omega_{2}$, we arrive at

$$
\delta r_{2} \leq \nu_{1}(t) \leq r_{2}, \quad \text{for all } t \in \mathbb{R}.
$$
Because \( \nu \) is a fixed point of (1.1) in \( \Omega \), then we have
\[
\nu_0 = \int_0^T G(t, s) \left( \frac{c(s)}{\alpha} \nu_1^1(s) + (1 - \alpha) \left( \frac{\nu_0^1(s)}{\nu_1(s)} - \frac{b(s)}{\alpha} \nu_2^1(s) \right) + \frac{c(s)}{\alpha} \frac{\nu_0^2(s)}{\nu_2^1(s)} \right) ds \leq \frac{MT}{\alpha} \left( \frac{e^2}{\alpha} r^2_2 + \frac{b^2}{\alpha} + \frac{c^2}{\alpha} \nu_2^1 \right),
\]
where the inequality holds because \( |\nu(t)| \leq \eta \nu(t) \) for all \( t \in \mathbb{R} \). Further, we claim that
\[
MT \left( \frac{e^2}{\alpha} r^2_2 + \frac{b^2}{\alpha} + \frac{c^2}{\alpha} \nu_2^1 \right) \leq \delta r_2.
\]
In fact, from \( MT(1 - \alpha)\eta^2 < \delta^2 \), (3.7) is rewritten as
\[
MT \left( \frac{e^2}{\alpha} r^2_2 + \frac{b^2}{\alpha} + \frac{c^2}{\alpha} \nu_2^1 \right) \leq \delta \delta r_2 - MT(1 - \alpha)\eta^2 < r_2.
\]
Since \( 1 - \alpha < 1 \) and \( \frac{e^2}{\alpha} < 1 \), we can choose \( r_2 > 0 \) large appropriate such that (3.8) holds. It is a contradiction to \( \nu_1 \geq \delta r_2 \). Therefore, (3.6) is satisfied.

**Step 3.** Let \( \nu_0 = 1 \), then we have \( \nu_0 \in K \setminus \{0\} \). Now we prove that
\[
\nu \neq T \nu + \lambda \nu_0, \quad \text{for all } \nu \in K \cap \partial \Omega_1, \quad \lambda \geq 0.
\]
In fact, if not, there exist \( \nu_2 \in K \cap \partial \Omega_1 \) and \( \lambda_0 \geq 0 \) such that
\[
\nu_2 = T \nu_2 + \lambda_0 \nu_0.
\]
Because \( \nu_2 \in K \cap \partial \Omega_1 \), we see that
\[
\delta r_1 \leq \nu_2(t) \leq r_1, \quad \text{for all } t \in \mathbb{R}.
\]
It follows from (2.3) and (3.5) that
\[
\nu_2 = T \nu_2 + \lambda_0 \nu_0
\]

\[
= \int_0^T G(t, s) \left( \frac{c(s)}{\alpha} \nu_1^1(s) + (1 - \alpha) \frac{\nu_0^1(s)}{\nu_1(s)} + \frac{b(s)}{\alpha} \nu_2^1(s) - \frac{c(s)}{\alpha} \frac{\nu_0^2(s)}{\nu_2^1(s)} \right) ds + \lambda_0
\]

\[
\geq \int_0^T G(t, s) \left( \frac{c(s)}{\alpha} \nu_1^1(s) + \frac{b(s)}{\alpha} \nu_2^1(s) - \frac{c(s)}{\alpha} \frac{\nu_0^2(s)}{\nu_2^1(s)} \right) ds + \lambda_0
\]

\[
> \int_0^T G(t, s) \frac{b^2(s)}{\alpha} ds
\]

\[
\geq \frac{MTb^2}{\alpha} = r_1
\]

since \( r_1 = \frac{MTb^2}{\alpha} \) from definition of \( r_1 \). It is a contradiction with \( \nu_2 \leq r_1 \). Hence, (3.9) holds.

By Lemma 2.1, we obtain that \( \mathcal{T} \) has a fixed point \( \nu \in K \cap (\Omega_2 \setminus \Omega_1) \). Obviously, the fixed point is a positive \( T \)-periodic solution of (1.1).

It is clear that the calculation of \( m \) and \( M \) is very difficult. In order to get around this problem, in what follows, we consider the existence of a positive \( T \)-periodic solution for (1.1) in the case that \( a(t) \equiv a^2 \) for all \( t \in [0, T] \).
Corollary 3.1. Assume that \( \sigma < \frac{\pi \sqrt{3}}{2} \) and \( M_1 T (1 - \alpha) \eta T^2 < \delta_1^2 \) hold. Furthermore, suppose the following inequality is satisfied
\[
\frac{b^+}{\alpha} > \frac{\alpha}{m_1 T \delta_1} \max \left\{ \left( \frac{2 \| b^- \|_\infty}{e_*} \right)^{\frac{1}{m}}, \left( \frac{2 \| c^+ \|_\infty}{e_*} \right)^{\frac{1}{m}} \right\}.
\]
(3.10)

Then (1.1) admits at least one positive \( T \)-periodic solution if \( \rho > \mu \).

This Corollary can be proved by Theorem 3.1 because \( a(t) \equiv \sigma^2 \).

Remark 3.1. In view of (2.2), it is important to note that \( m_1, M_1, \delta_1 \) and \( \eta_1 \) are functions of \( \sigma \) and \( T \). In fact, the sufficient condition (3.10) has an explicit (but rather cumbersome) expression as
\[
\frac{\sigma T (1 - \alpha) \tan \frac{\sigma t}{2}}{2 \cos \frac{\sigma t}{2}} < \left( \frac{\cos \frac{\sigma t}{2}}{\sin \frac{\sigma t}{2} + 1} \right)^2 \quad \text{and} \quad b^+ > \frac{2 \sigma a (\sigma \sin \frac{\sigma t}{2} + 1)}{T \csc \frac{\sigma t}{2} \cos \frac{\sigma t}{2}} \max \left\{ \left( \frac{2 \| b^- \|_\infty}{e_*} \right)^{\frac{1}{m}}, \left( \frac{2 \| c^+ \|_\infty}{e_*} \right)^{\frac{1}{m}} \right\}.
\]
(3.11)

From here it is easy to construct explicit examples. For instance, taking \( b(t) = 4 \cos 4 t, \ c(t) = \frac{1}{2} (\sin 4 t - 1) \), \( e(t) = 60 - 40 \sin 4 t \), \( \rho = \frac{1}{5} > \frac{\delta_1^2}{\eta_1} = \mu \), \( \alpha = \frac{1}{4 + \sqrt{2}} \approx \frac{5}{6}, \ T = \frac{\pi}{2}, \ \sigma = 1 < \frac{\pi \sqrt{3}}{2} = \frac{2 \sqrt{3 \pi}}{6} \). Besides, we have \( \| b \|_\infty = 4, \ |c| \|_\infty = 1, \ e_* = 20 \), and
\[
\int_0^T b^+ (t) dt = 4 \left( \int_0^{\frac{\pi}{2}} \cos 4 t dt + \int_{\frac{\pi}{2}}^{\pi} \cos 4 t dt \right) = 2.
\]

Furthermore, we have
\[
\frac{\sigma T (1 - \alpha) \tan \frac{\sigma t}{2}}{2 \cos \frac{\sigma t}{2}} = \frac{1}{12} < \left( \frac{\cos \frac{\sigma t}{2}}{\sin \frac{\sigma t}{2} + 1} \right)^2 \approx 0.172,
\]
and
\[
\frac{2 \sigma a (\sigma \sin \frac{\sigma t}{2} + 1)}{T \csc \frac{\sigma t}{2} \cos \frac{\sigma t}{2}} \max \left\{ \left( \frac{2 \| b^- \|_\infty}{e_*} \right)^{\frac{1}{m}}, \left( \frac{2 \| c^+ \|_\infty}{e_*} \right)^{\frac{1}{m}} \right\}
\]
\[
= \frac{\frac{5}{6}}{2} \times \frac{2}{(\sqrt{2} - 1)} \max \left\{ \left( \frac{2}{5} \right)^6, \left( \frac{1}{10} \right)^{\frac{36}{7}} \right\}
\]
\[
\approx 0.017 < 2 = T b^+.
\]

The above computations show that (3.11) holds and the (1.1) admits at least one \( \frac{\pi}{2} \)-periodic solution.

3.2 The case \( \rho = \mu \)

If \( \rho = \mu \), (3.1) can be presented in the form
\[
\nu'' + \frac{a(t)}{\alpha} \nu = \frac{e(t)}{\alpha} \nu^{1 - \alpha} + (1 - \alpha) \frac{\nu^2}{\nu} + \frac{b(t) - c(t)}{\alpha}.
\]
(3.12)

Let \( d(t) = \frac{a(t)}{\alpha} \), by Lemma 2.1 and Theorem 3.1, we obtain the following conclusion.

Theorem 3.2. Assume that \( |a|_p \leq \alpha \Lambda (2p^*) \) and \( M T (1 - \alpha) \eta T^2 < \delta^2 \) hold. Furthermore, suppose the following inequality is satisfied
\[
\frac{b^+}{\alpha} > \frac{\alpha}{m T \delta} \left( \frac{\| c^+ \|_\infty}{e_*} \right)^{\frac{1}{m}}.
\]
(3.13)

Then (1.1) admits at least one positive \( T \)-periodic solution if \( \rho = \mu \).
By (3.13), the positive constants $r$ define two open sets

$$\Omega_3 := \{ \nu \in C^1_T : \| \nu \| < r_3 \} \quad \text{and} \quad \Omega_4 := \{ \nu \in C^1_T : \| \nu \| < r_4 \}.$$ 

By (3.13), the positive constants $r_3$ and $r_4$ can be fixed such that

$$r_4 > r_3 = \frac{MT \bar{b}}{\alpha} > \frac{1}{\delta} \left( \frac{\| b^- + c^+ \|_{\infty}}{\epsilon_*} \right)^{\frac{1}{1-\alpha}} > 0.$$

First, we claim that $\Phi(\mathcal{K} \cap (\Omega_4 \setminus \Omega_3)) \subset \mathcal{K}$. In fact, for any $\nu \in \mathcal{K} \cap (\Omega_4 \setminus \Omega_3)$, we have

$$\delta r_3 = \delta \| \nu \| \leq \nu(t) \leq \| \nu \|_{\infty} \leq r_4, \quad \text{for all} \quad t \in \mathbb{R}.$$

Since $r_3 > \frac{1}{\delta} \left( \frac{\| b^- + c^+ \|_{\infty}}{\epsilon_*} \right)^{\frac{1}{1-\alpha}}$, we get

$$\frac{e(t) \nu^{1-\alpha}(t) + (1 - \alpha) \nu^2}{\alpha} + \frac{b(t) - c(t)}{\alpha}$$

$$= \frac{e(t) \nu^{1-\alpha}(t) + (1 - \alpha) \nu^2}{\alpha} + \frac{b^+(t) - b^-(t) - c^+(t) + c^-(t)}{\alpha}$$

$$> \frac{e(t) \nu^{1-\alpha}(t) - b^-(t) + c^+(t)}{\alpha}$$

$$> 0,$$

for all $t \in \mathbb{R}$. Similar to the Step 1 of Theorem 3.1, we get that $\Phi(\mathcal{K} \cap (\Omega_4 \setminus \Omega_3)) \subset \mathcal{K}$.

The left of proof is the same as Theorem 3.1.

In the following, we study (1.1) in the case that $a(t) \equiv \sigma^2$ for all $t \in [0, T]$.

**Corollary 3.2.** Assume that $\sigma < \frac{\alpha \sqrt{\pi}}{\sqrt{\epsilon_\ast}}$ and $M_1 T (1 - \alpha) \eta_1^2 < \delta_1^2$ hold. Furthermore, suppose the following inequality is satisfied

$$\bar{b}^+ > \frac{\alpha}{m_1 T \delta_1} \left( \frac{\| b^- + c^+ \|_{\infty}}{\epsilon_*} \right)^{\frac{1}{1-\alpha}}. \quad (3.16)$$

Then (1.1) admits at least one positive $T$-periodic solution if $\rho = \mu$.

This Corollary can be proved by Theorem 3.2 because $a(t) \equiv \sigma^2$.

**Remark 3.2.** In view of (2.2), it is important to note that $m_1$, $M_1$, $\delta_1$ and $\eta_1$ are functions of $\sigma$ and $T$. In fact, the sufficient condition (3.16) has an explicit (but rather cumbersome) expression as

$$\frac{\sigma T (1 - \alpha) \tan \frac{\pi}{2}}{2 \cos \frac{\pi}{2}} < \left( \frac{\cos \frac{\pi}{2}}{\sigma \sin \frac{\pi}{2} + 1} \right)^2 \quad \text{and} \quad \bar{b}^+ > \frac{2 \sigma \alpha (\sigma \sin \frac{\pi}{2} + 1)}{T \cos \frac{\pi}{2} \cos \frac{\pi}{2}} \left( \frac{\| b^- + c^+ \|_{\infty}}{\epsilon_*} \right)^{\frac{1}{1-\alpha}}.$$

The proof of Theorem 3.2 relies mainly on $\bar{b}^+ > \frac{\alpha}{m T \delta} \left( \frac{\| b^- + c^+ \|_{\infty}}{\epsilon_*} \right)^{\frac{1}{1-\alpha}}$. In the following, we will show several other different results.

**Theorem 3.3.** Assume that $\| a \|_p \leq \alpha \Lambda (2p^*)$ and $M T (1 - \alpha) \eta_1^2 < \delta_1^2$ hold. Furthermore, suppose the following inequality is satisfied

$$\bar{c}^- > \frac{\alpha}{m T \delta} \left( \frac{\| b^- + c^+ \|_{\infty}}{\epsilon_*} \right)^{\frac{1}{1-\alpha}}. \quad (3.17)$$

Then (1.1) admits at least one positive $T$-periodic solution if $\rho = \mu$. 

---

**Proof.** A $T$-periodic solution of (3.12) is just a fixed point of the map $\Phi$ defined by

$$\Phi(\nu)(t) := \int_0^T G(t, s) \left( \frac{e(s)}{\alpha} \nu^{1-\alpha}(s) + (1 - \alpha) \frac{|\nu'(s)|^2}{\nu(s)} + \frac{b(s) - c(s)}{\alpha} \right) ds. \quad (3.14)$$

Define two open sets

$$\Omega_3 := \{ \nu \in C^1_T : \| \nu \| < r_3 \} \quad \text{and} \quad \Omega_4 := \{ \nu \in C^1_T : \| \nu \| < r_4 \}.$$ 

The left of proof is the same as Theorem 3.1.

In the following, we will show (1.1) in the case that $a(t) \equiv \sigma^2$ for all $t \in [0, T]$. 

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Proof. Define two open sets
\[ \Omega_5 := \{ \nu \in C_I^3 : \| \nu \| < r_5 \} \quad \text{and} \quad \Omega_6 := \{ \nu \in C_I^3 : \| \nu \| < r_6 \}. \]
By (3.17), the positive constants \( r_5 \) and \( r_6 \) can be fixed such that
\[ r_6 > r_5 = \frac{mTc}{\alpha} > \frac{1}{\delta} \left( \frac{\| b^- + c^+ \|_\infty}{e_*} \right)^{\frac{1}{1-n}} > 0. \]

Similar to the Step 1 of Theorem 3.1, we get that \( \Phi(K \cap (\Omega_6 \setminus \Omega_5)) \subset K \), here \( \Phi \) is defined in (3.14).
Next, let \( \nu_0 = 1 \), then we have \( \nu_0 \in K \setminus \{0\} \). Now we prove that
\[ \nu \neq \Phi \nu + \lambda \nu_0, \quad \text{for all} \quad \nu \in K \cap \partial \Omega_5, \quad \lambda \geq 0. \]
(3.18)

In fact, if not, there exist \( \nu_3 \in K \cap \partial \Omega_5 \) and \( \lambda_1 \geq 0 \) such that
\[ \nu_3 = \Phi \nu_3 + \lambda_1 \nu_0. \]
Because \( \nu_3 \in K \cap \partial \Omega_5 \), we see that
\[ \delta r_5 \leq \nu_3(t) \leq r_5, \quad \text{for all} \quad t \in \mathbb{R}. \]
From (2.3) and (3.15) that
\[
\nu_3 = \Phi \nu_3 + \lambda_1 \nu_0 \\
= \int_0^T G(t, s) \left( \frac{e(s)}{\alpha} \nu_3^{1-\alpha}(s) + (1 - \alpha) \frac{\| \nu_3'(s) \|^2}{\nu_3(s)} + \frac{b(s) - c(s)}{\alpha} \right) ds + \lambda_1 \\
\geq \int_0^T G(t, s) \left( \frac{e(s)}{\alpha} \nu_3^{1-\beta}(s) + \frac{b^+(s) - b^-(s) - c^+(s) + c^-(s)}{\alpha} \right) ds + \lambda_1 \\
\geq \int_0^T G(t, s) \frac{e^{-}(s)}{\alpha} ds \\
\geq \frac{mTc}{\alpha} = r_5
\]
since \( r_5 = \frac{mTc}{\alpha} \) from definition of \( r_5 \). It is a contradiction with \( \nu_3 \leq r_5 \). Hence, (3.18) holds.

The left of proof is the same as Theorem 3.1. \( \square \)

In what follows, we consider (1.1) in the case that \( a(t) \equiv \sigma^2 \) for all \( t \in [0, T] \).

Corollary 3.3. Assume that \( \sigma < \frac{\sqrt{\pi}}{2} \) and \( M_1 T (1-\alpha) \eta^2_1 < \delta^2_1 \) hold. Furthermore, suppose the following inequality is satisfied
\[
\frac{\sigma}{c} > \frac{\alpha}{m_1 T \delta_1} \left( \frac{\| b^- + c^+ \|_\infty}{e_*} \right)^{\frac{1}{1-n}}. \quad (3.19)
\]
Then (1.1) admits at least one positive \( T \)-periodic solution if \( \rho = \mu \).

This Corollary can be proved by Theorem 3.3 because \( a(t) \equiv \sigma^2 \).

Remark 3.3. In view of (2.2), it is important to note that \( m_1, M_1, \delta_1 \) and \( \eta_1 \) are functions of \( \sigma \) and \( T \). In fact, the sufficient condition (3.19) has an explicit (but rather cumbersome) expression as
\[
\frac{\sigma T (1-\alpha) \tan \frac{\alpha T}{2}}{2 \cos \frac{\alpha T}{2}} < \left( \frac{\cos \frac{\alpha T}{2}}{\sigma \sin \frac{\alpha T}{2} + 1} \right)^{2} \quad \text{and} \quad \frac{\sigma}{c} > \frac{2 \sigma \alpha (\sigma \sin \frac{\alpha T}{2} + 1)}{T \cot \frac{\alpha T}{2} \cos \frac{\alpha T}{2}} \left( \frac{\| b^- + c^+ \|_\infty}{e_*} \right)^{\frac{1}{1-n}}.
\]
3.3 The case \( \rho < \mu \)

Take \( u = \nu^\beta \), where \( \beta = \frac{1}{1+\mu} < 1 \) from \( \mu > 0 \), (1.1) is rewritten in the following form

\[
\nu'' + \frac{a(t)}{\beta} \nu = \frac{c(t)}{\beta} \nu^{1-\beta} + (1-\beta) \frac{\nu'^2}{\nu} + \frac{b(t)}{\beta} \nu^{\mu-\nu} - c(t). 
\tag{3.20}
\]

Let \( d(t) = \frac{a(t)}{\beta} \), by Lemma 2.1 and Theorem 3.1, we obtain the following conclusion.

**Theorem 3.4.** Assume that \( \|a\|_p \leq \beta \Lambda (2p^*) \) and \( MT(1-\beta)\eta^2 < \delta^2 \) hold. Furthermore, suppose the following inequality is satisfied

\[
\frac{c}{\| \|} > \frac{\beta}{MT} \max \left\{ \left( \frac{2\|c^+\|_{\infty}}{e_*} \right)^{\frac{1}{\delta}}, \left( \frac{2\|b^-\|_{\infty}}{e_*} \right)^{\frac{1}{\delta}} \right\}. 
\tag{3.21}
\]

Then (1.1) admits at least one positive \( T \)-periodic solution if \( \rho < \mu \).

**Proof.** A \( T \)-periodic solution of (3.20) is just a fixed point of the map \( \Psi \) defined by

\[
(\Psi \nu)(t) = \int_0^T G(t, s) \left( \frac{c(s)}{\beta} \nu^{1-\beta}(s) + (1-\beta) \frac{\nu'(s)^2}{\nu(s)} + \frac{b(s)}{\beta} \nu^{\mu-\nu}(s) - \frac{c(s)}{\beta} \right) ds.
\]

Define two open sets

\[
\Omega_7 := \{ \nu \in C_T^1 : \| \nu \| < r_7 \} \quad \text{and} \quad \Omega_8 := \{ \nu \in C_T^1 : \| \nu \| < r_8 \}.
\]

By (3.21), the positive constants \( r_7 \) and \( r_8 \) can be fixed such that

\[
r_8 > r_7 = \frac{MT\frac{c}{\| \|}}{\beta} > \frac{1}{\delta} \max \left\{ \left( \frac{2\|c^+\|_{\infty}}{e_*} \right)^{\frac{1}{\delta}}, \left( \frac{2\|b^-\|_{\infty}}{e_*} \right)^{\frac{1}{\delta}} \right\} > 0.
\]

Similar to the Step 1 of Theorem 3.1, we get that \( \Psi(K \cap (\Omega_8 \setminus \Omega_7)) \subset K \).

Next, we prove that

\[
\nu \neq \tau \Psi \nu, \quad \text{for all} \quad \nu \in K \cap \partial \Omega_8.
\tag{3.22}
\]

In fact, if not, there exist \( \nu_4 \in K \cap \partial \Omega_8 \) and \( 0 < \tau_2 \leq 1 \) such that

\[
\nu_4 = \tau_2 \Psi \nu_4.
\]

In the following, for any \( \nu \in K \cap \partial \Omega_8 \), we have

\[
\delta r_8 \leq \nu(t) \leq r_8, \quad \text{for all} \quad t \in \mathbb{R}.
\]

It follows from (2.3) that

\[
\nu_4 = \tau_2 \Psi \nu_4
\]

\[
= \tau_2 \int_0^T G(t, s) \left( \frac{c(s)}{\beta} \nu_4^{1-\beta}(s) + (1-\beta) \frac{\nu_4'(s)^2}{\nu_4(s)} + \frac{b(s)}{\beta} \nu_4^{\mu-\nu}(s) - \frac{c(s)}{\beta} \right) ds
\leq \int_0^T G(t, s) \left( \frac{c(s)}{\beta} \nu_4^{1-\beta}(s) + (1-\beta) \frac{\nu_4'(s)^2}{\nu_4(s)} + \frac{b(s)}{\beta} \nu_4^{\mu-\nu}(s) - \frac{c(s)}{\beta} \right) ds
\leq MT \left( \frac{c^*}{\beta} \frac{r_8^{1-\beta}}{\delta} + (1-\beta) \left( \frac{\eta r_8}{\delta} \right) + \frac{b^*}{\beta} \frac{r_8^{\mu-\nu}}{\delta} + \frac{c^-}{\beta} \right).
\]

Further, we claim that

\[
MT \left( \frac{c^*}{\beta} \frac{r_8^{1-\beta}}{\delta} + (1-\beta) \left( \frac{\eta r_8}{\delta} \right) + \frac{b^*}{\beta} \frac{r_8^{\mu-\nu}}{\delta} + \frac{c^-}{\beta} \right) < \delta r_8.
\tag{3.23}
\]
In fact, because $MT(1-\beta)\eta^2 < \delta^2$, write (3.23) as
\[ MT \left( \frac{e^+}{\beta} r_8^{1-\beta} + \frac{b^+}{\beta} r_8^{1+\beta} + \frac{c^-}{\beta} \right) \delta^2 - MT(1-\beta)\eta^2 < r_8. \] (3.24)
Since $1-\beta < 1$ and $\frac{c^-}{\beta r_8} < 1$ from $\beta = \frac{1}{1+\mu}$ and $\mu > \rho > 0$, we can choose $r_8 > 0$ large appropriate such that (3.24) holds. It is a contradictory to $\nu_4 \geq \delta r_8$. Therefore, (3.22) is satisfied.

The left of the proof is the same as Theorem 3.1.

In what follows, we consider (1.1) in the case that $a(t) \equiv \sigma^2$ for all $t \in [0, T]$.

**Corollary 3.4.** Assume that $\sigma < \frac{r_8^2}{\pi^2}$ and $M_1T(1-\beta)\eta_1^2 < \delta_1^2$ hold. Furthermore, suppose the following inequality is satisfied
\[ c^- > \frac{\beta}{m_1 T \delta_1} \max \left\{ \left( \frac{2\|c^+\|_\infty}{e_*} \right)^{\frac{1}{1+\mu}}, \left( \frac{2\|b^-\|_\infty}{e_*} \right)^{\frac{1}{1-\beta}} \right\}. \] (3.25)
Then (1.1) admits at least one positive $T$-periodic solution if $\rho < \mu$.

This Corollary can be proved by Theorem 3.4 because $a(t) \equiv \sigma^2$.

**Remark 3.4.** In view of (2.2), it is important to note that $m_1$, $M_1$, $\delta_1$ and $\eta_1$ are functions of $\sigma$ and $T$. In fact, the sufficient condition (3.25) has an explicit (but rather cumbersome) expression as
\[ \frac{\sigma T (1-\beta) \tan \frac{\sigma T}{2}}{2 \cos \frac{\sigma T}{2}} < \left( \frac{\cos \frac{\sigma T}{2}}{\sigma \sin \frac{\sigma T}{2} + 1} \right)^2 \quad \text{and} \quad c^- > \frac{2\sigma\beta(\sigma \sin \frac{\sigma T}{2} + 1)}{T \cot \frac{\sigma T}{2} \cos \frac{\sigma T}{2}} \max \left\{ \left( \frac{2\|c^+\|_\infty}{e_*} \right)^{\frac{1}{1+\mu}}, \left( \frac{2\|b^-\|_\infty}{e_*} \right)^{\frac{1}{1-\beta}} \right\}. \]

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**Declarations**

**Conflict of interest:** The authors declare that they have no competing interests.

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