A State-scaling Design for Prescribed-time Stabilization of High-order Nonlinear Systems with Input Quantization

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Research Article

Keywords: High-order nonlinear systems (HONSs) , input quantization , state-scaling transformation (SST) , prescribed-time stabilization (PTS)

Posted Date: June 7th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-216085/v1

Noname manuscript No.
(will be inserted by the editor)

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the date of receipt and acceptance should be inserted later

Abstract This article considers global stabilization problem for a kind of uncertain high-order nonlinear systems (HONSs). Two distinct characteristics of this study are that the considered system possesses the input-quantized actuator, and the prescribed time convergence of the system states is wanted. To address these, a novel state-scaling transformation (SST) is firstly introduced to convert the aboriginal prescribed-time stabilization (PTS) to the asymptotic stabilization of the transformed one. Then, under the new framework of equivalent transformation, a quantized state feedback controller that achieves of the performance requirements is developed with the aid of the technique of adding a power integrator (API). Finally, simulation results of a liquid-level system are provided to confirm the efficacy of the proposed approach.

Keywords High-order nonlinear systems (HONSs) · input quantization · state-scaling transformation (SST) · prescribed-time stabilization (PTS)

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1 Introduction

As typical nonlinear dynamic systems, high-order nonlinear systems (HONSs) have received widespread attention during the last decades because of their significant values both in theory and practice [1, 2]. However, the distinctive feature of the non-existence and/or the lack of controllability/observability of the Jacobian linearization around the origin, makes the control of HONSs challenging. Thanks to the technique of adding a power integrator (API)[3], which develops the traditional backstepping technique by introducing the feedback domination mechanism and contributes to a technological breakthrough in coping with such intrinsic obstacle, a series of results have borne in the asymptotic stabilizing/tracking control of HONSs, for example, refer to [4–12] and references therein.

On the flip side, the research on finite-time control has proved to be popular recently because of the superior properties of finite-time stable system, such as fast response, good robustness and disturbance rejection. Especially, since the milestone work of the Lyapunov finite-time stability theorem was established in [13], many significant results have been obtained [14–20]. Note that the settling time functions achieved in the above-mentioned results critically rely on system initial conditions, which result in that the convergence time will rise unacceptably large along with the initial conditions growing. To handle this faultiness, Andrieu *et al.* in [21] put forward the concept of fixed-time stability, that requires that the connected settling time function is independent from initial system conditions. With this new framework, many works have appeared to study the fixed-time control designs of different linear or nonlinear systems [22–32]. Generally speaking, the existing methods on such control designs can be divided into two kinds: the bi-limit homogeneous method [21, 22] and the Lyapunov-based method [23–32]. However, it is important to note that both two methods have some inherent shortcomings. Namely in the former, the upper bound of the settling time (UBST) function exists but is unknown, and in the latter, the UBST is bounded and adjustable, but it is so hard to be prespecified discretionarily according to requirements on account that the derived settling time functions currently depend on a few design parameters, whose selections are laborious to meet the prespecified settling time requirements[33].

However, prespecifiable settling time is indeed expected by some practical applications, e.g., missile guidance [34]. This fact urges that the even stronger control objective of prescribed-time stability [35] (also named predefined-time stability [36]) where the UBST is a design parameter that can be selected (i.e., prescribed) by the control designer, has been introduced to address the stabilization problem of various types of systems by state or output feedback [37–40]. Especially, drawing support from scaling the system states by a function that grows unboundedly tending to the terminal time, a sate feedback controller which is computationally singular was given to the PTS of Brunovsky systems in [35]. This technique was further extended refined in [37], where a novel state-scaling transformation (SST) was proposed to solve the computationally singular problem and provided a solution to the problem of PTS for strict-feedback (switched) nonlinear systems. However, the powers of the considered systems are identically equal to 1 (i.e., $p_i = 1$) required in [35–40], which certainly limits its applications since quite a few practical systems are described by HONSs (or named p-normal systems), refer to the typical example of liquid-level system given in Section 4. Moreover, another common drawback of the aforementioned results is that the effect of input quantization is ignored.

As is known to all, most of control tasks of modern engineering application are achieved based on network information transfer, which means that the actual control signals in such systems must be quantized to overcome the communication constraints including limited data transmission rate of communication channels and the limited bandwidth. However, the application of quantizers inevitably introduce quantization errors, which seriously degrade the system's performance and prevent the implementation of quantizers [41–43]. Naturally, the folowing interesting question arises: *For a HONS with input quantization, is it possible to achieve its PTS? If yes, how can one design such quantized controller?*

This article focuses on addressing the problem of global PTS for a kind of HONSs with input quantization and giving a predicative answer to the above question. The main contributions are underlined as follows.

(i) Fully taking into consideration of practical system requirements, both quantized input and prescribed-time convergence are included firstly in this paper.
(ii) A novel SST is proposed to change the aboriginal PTS problem into the problem of asymptotic stabilization of the transformed one.
(iii) Under a new homogeneous-like restricted condition on system growth, a systematic design strategy ensuring the achievement of the performance requirements is proposed by elaborately using the API technique.

The rest of the article is organised as follows. In Section 2, the problem formulation and preliminaries of this article are introduced. In Section 3, details on the controller design are presented, followed by the rigorous stability analysis of the CLS. In Section 4, a practical example of the liquid-level system is provided with simulation studies to validate the efficiency of the proposed method. Finally, the article is concluded in Section 5.

2 Problem formulation and preliminaries

2.1 Problem formulation

Consider a kind of HONSs given by

$$\begin{cases} \dot{z}_1 = d_1(t)\lceil z_2 \rceil^{q_1} + f_1(z_1), \\ \dot{z}_2 = d_2(t)\lceil z_3 \rceil^{q_2} + f_2(\bar{z}_2), \\ \quad \vdots \\ \dot{z}_{m-1} = d_{m-1}(t)\lceil z_m \rceil^{q_{m-1}} + f_{m-1}(\bar{z}_{m-1}), \\ \dot{z}_m = d_m(t)Q(u) + f_m(\bar{z}_m), \end{cases} \tag{1}$$

where $\bar{z}_i = (z_1, \ldots, z_j)^T \in \mathbb{R}^j$ is the system state (vector), and $d_j \in \mathbb{R}$, $q_j \in \mathbb{R}^+$ (with $q_m = 1$), $j = 1, \ldots, m$ are the control coefficients and the power orders of the system, respectively. For $j = 1, \ldots, m-1$, $\lceil z_{j+1} \rceil^{q_j}$ are defined as $\lceil z_{j+1} \rceil^{q_j} = \mathrm{sign}(z_{j+1})|z_{j+1}|^{q_j}$ with the standard signum function $\mathrm{sign}(\cdot)$. $f_j \in \mathbb{R}$ $(j = 1, \ldots, m)$ are uncertain continuous functions satisfying $f_j(0) = 0$. $Q(u) \in \mathbb{R}$ denotes the quantized input described by

$$Q(u) = Q_1(t)u + Q_2(t), \tag{2}$$

where

$$Q_1(t) = \begin{cases} 1 + \vartheta_1\delta, & |u| \geq u_{min}, \\ 1, & |u| < u_{min}, \end{cases} \tag{3}$$

and

$$Q_2(t) = \begin{cases} 0, & |u| \geq u_{min}, \\ \vartheta_2 u_{min}, & |u| < u_{min}, \end{cases} \tag{4}$$

where $0 \le \delta < 1$ and u_{min} are known parameters and $-1 \le \vartheta_j \le 1$, $j = 1, 2$ are unknown parameters of the quantizer (2).

Remark 2.1. It is worth noting that many practical quantizers, such as logarithmic quantizer, hysteresis quantizer, and uniform quantizer, belong to the considered class described by (2). For instance, consider the logarithmic quantizer used in [43, 47], which is modeled as

$$Q(u) = \begin{cases} u_j, & \frac{u_j}{1+\delta} < u \le \frac{u_j}{1-\delta}, \\ 0, & 0 \le u < \frac{d}{1+\delta}, \\ -Q(-u), & u < 0, \end{cases} \quad (5)$$

where $u_j = \varsigma^{1-j} d$ $(j = 1, 2, \dots)$ with the parameters being selected to satisfy $d > 0$ and $0 < \varsigma < 1$. $\delta = \frac{1-\varsigma}{1+\varsigma}$ determine the quantization density of $Q(u)$. $u_0 = \frac{d}{1+\delta}$ determines the size of the dead zone for $Q(u)$. Clearly, this quantizer is in the shape of (2) with $\vartheta_1 = (Q(u) - u)/(\delta u)$ and $u_{min} = u_0$.

The aim of this article is to present a quantized state feedback control mechanism to stabilize the system (1) within prescribed finite time under the following wild assumptions.

Assumption 2.1. For $j = 1, \dots, m$, there are smooth functions $\varphi_j \ge 0$ and a constant $\tau > 0$ such that

$$|f_j(\bar{z}_j)| \le \varphi_j(\bar{z}_j) \sum_{k=1}^{j} |z_k|^{\frac{\lambda_j - \tau}{\lambda_k}}, \quad (6)$$

where λ_j's are recursively defined by

$$\lambda_{m+1} = \tau, \ \ q_j \lambda_{j+1} = \lambda_j - \tau \ge 0, \ \ j = 1, \dots, m. \quad (7)$$

Assumption 2.2. There are positive constants $\underline{d}_j$ and $\overline{d}_j$, $j = 1, \dots, m$ such that $\underline{d}_j \le d_j(t) \le \overline{d}_j$.

Remark 2.2. Assumption 2.1 is a new type condition of homogeneous-growth-like because λ_j's given here are much different from the traditional ones employed in [4–12, 14–20] where they are recursively defined by $\lambda_1 = 1$, $q_j \lambda_{j+1} = \lambda_j - \tau \ge 0$, $j = 1, \dots, m$. The most important role of such assumption is that it can ensure the state-scaling-transformed system inheriting the homogeneous-like property of the original system (1) (see Proposition 3.1 below). In addition, it should be mentioned that, it is reasonable in engineering practice to impose the boundedness of the control coefficients in Assumption 2.2. Similar requirements can be found in the existing literature [14–18, 33, 47].

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2.2 Preliminary knowledge

Consider the nonlinear system

$$\dot{z} = f(t, z), \ \ z(0) = z_0 \in \mathbb{R}^n, \quad (8)$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to x and contents $f(t, 0) = 0$.

Definition 2.1[23]**.** The origin of system (8) is referred to be globally fixed-time stable if it is globally finite-time stable and there exists a bounded settling-time function $T(z_0)$ to make sure that the solution $z(t, z_0)$ of (8) satisfies $z(t, z_0) = 0$, $\forall t \ge T(z_0)$.

Definition 2.2. The origin of system (8) is referred to be globally prescribed-time stable if it is globally fixed-time stable and a tunable designing parameter $\vartheta \in \mathbb{R}$ exists to ensure $T(z_0) \le T_c$ for any prescribed finite time $T_c > 0$ and any $z_0 \in \mathbb{R}$.

Remark 2.3. The prescribed-time stability given in Definition 2.2 is further developed from the fixed-time stability, whose state converges to the origin before the time instant chosen by the users. Compared with the conventional fixed-time stability, the most significant difference is that the settling time of prescribed-time stability can be arbitrarily preassigned according to practical requirements.

Lemma 2.1[4]**.** For any x, $y \in \mathbb{R}$, and a constant $a \ge 1$, one has (i)$|x + y|^a \le 2^{a-1}|x^a + y^a|$; (ii)$(|x| + |y|)^{1/a} \le |x|^{1/q} + |y|^{1/a} \le 2^{(a-1)/a}(|x| + |y|)^{1/a}$.

Lemma 2.2[48]**.** If c, d are positive constants, then for any real-valued function one has $\delta(u, v) > 0$, $|u|^c|v|^d \le \frac{c}{c+d}\delta(u, v)|u|^{c+d} + \frac{d}{c+d}\delta^{-c/d}(u, v)|v|^{c+d}$.

Lemma 2.3[48]**.** Let $0 < p \le 1$ and $a > 0$ be constants. then for any $u, v \in \mathbb{R}$ there is $|\lceil u \rceil^{ap} - \lceil v \rceil^{ap}| \le 2^{1-p}|\lceil u \rceil^a - \lceil v \rceil^a|^p$.

3 Prescribed-Time Stabilization

In this section, we propose a constructive design mechanism of quantized state feedback controller which can stabilize system (1) within any prescribed finite time $T_c > 0$. The design of such stabilizing controller is specially given as two step. That is, when $t \in [0, T_c)$, a non-autonomous controller is firstly developed to force the system states to the origin within T_c regardless of initial conditions, thereafter an autonomous controller that maintain the system states staying at the origin is designed.

3.1 Controller design of $t \in [0, T_c)$

Firstly, to change the aboriginal PTS into the asymptotic stabilization framework, the following novel coordinate transformation of state-scaling is introduced.

$$\begin{aligned} &\zeta_j = \Gamma^{(1+c)\lambda_j} z_j,\ j=1,\ldots,m, \\ &Q(v) = \Gamma^{(1+c)\lambda_{m+1}} Q(u), \end{aligned} \tag{9}$$

where $c \geq (1/\tau)-1$ is a design constant and Γ is defined as

$$\Gamma = \frac{T_c}{T_c - t}. \tag{10}$$

Remark 3.1. It is obvious that $\Gamma(\cdot)$ has such important properties of monotonous growth on $[0, T_c)$ and satisfies $\Gamma(0) = 1$ and $\Gamma(T_c) = +\infty$.

From (9), system (1) is redescribed as

$$\begin{cases} \dot{\zeta}_1 = \Gamma^{(1+c)\tau}\left(d_1 \lceil \zeta_2 \rfloor^{q_1} + g_1(\zeta_1)\right), \\ \dot{\zeta}_2 = \Gamma^{(1+c)\tau}\left(d_2 \lceil \zeta_3 \rfloor^{q_2} + g_2(\bar{\zeta}_2)\right), \\ \qquad \vdots \\ \dot{\zeta}_{m-1} = \Gamma^{(1+c)\tau}\left(d_{m-1} \lceil \zeta_n \rfloor^{q_{m-1}} + g_{m-1}(\bar{\zeta}_{m-1})\right), \\ \dot{\zeta}_m = \Gamma^{(1+c)\tau}\left(d_m Q(v) + g_m(\bar{\zeta}_m)\right), \end{cases} \tag{11}$$

where

$$g_j(\bar{\zeta}_j) = \zeta_j \frac{(1+c)\lambda_j \dot{\Gamma}}{\Gamma^{1+(1+c)\tau}} + \Gamma^{(1+c)(\lambda_j - \tau)} f_j(\bar{z}_j), \quad j=1,\ldots,m, \tag{12}$$

Proposition 3.1. For $j = 1, \ldots, m$, nonnegative smooth functions $\bar{\varphi}_j$ exist such that

$$|g_j(\bar{\zeta}_j)| \leq \bar{\varphi}_j(\bar{\zeta}_j) \sum_{k=1}^{j} |\zeta_k|^{\frac{\lambda_j - \tau}{\lambda_k}}. \tag{13}$$

Proof. See the Appendix.

Next, a state feedback controller for asymptotic stabilization of system (11) is designed for the case of $t \in [0, T_c)$ by employing the API technique.

Step 1. Set $\rho \geq \max_{1 \leq j \leq m}\{\lambda_j\}$ be a constant and take the Lyapunov function V_1 as

$$V_1 = U_1 = \int_0^{\zeta_1} \left\lceil \lceil s \rfloor^{\frac{\rho}{\lambda_1}} - 0 \right\rfloor^{\frac{2\rho - \lambda_1}{\rho}} ds. \tag{14}$$

Applying Assumptions 2.1 and 2.2 and (13) produces

$$\begin{aligned} \dot{V}_1 &= \Gamma^{(1+c)\tau} \lceil \pi_1 \rfloor^{\frac{2\rho-\lambda_1}{\rho}} \left(d_1 \lceil \zeta_2 \rfloor^{q_1} + g_1\right) \\ &\leq \Gamma^{(1+c)\tau} \Big(\lceil \pi_1 \rfloor^{\frac{2\rho-\lambda_1}{\rho}} d_1 (\zeta_2 \rfloor^{q_1} - \lceil \zeta_2^* \rfloor^{q_1}) \\ &\quad + d_1 \lceil \pi_1 \rfloor^{\frac{2\rho-\lambda_1}{\rho}} \lceil \zeta_2^* \rfloor^{q_1} + |\pi_1|^{\frac{2\rho-\lambda_1}{\rho}} \bar{\varphi}_1 \Big), \end{aligned} \tag{15}$$

where $\pi_1 = \lceil \zeta_1 \rfloor^{\frac{\rho}{\lambda_1}}$ and ζ_2^* is the virtual controller of ζ_2 to be given.

Take the virtual controller ζ_2^* as

$$\zeta_2^* = -\lceil \pi_1 \rfloor^{\frac{\lambda_2}{\rho}} \beta_1^{\frac{\lambda_2}{\rho}}(\zeta_1), \tag{16}$$

where

$$\beta_1(\zeta_1) \geq \left(\frac{m + \bar{\varphi}_1}{\underline{d}_1}\right)^{\frac{\rho}{q_1 \lambda_2}}, \tag{17}$$

is a smooth function. Then, putting (16), (17) and (15) together produces

$$\begin{aligned} \dot{V}_1 &\leq -m\Gamma^{(1+c)\tau} |\pi_1|^{\frac{2\rho-\tau}{\rho}} \\ &\quad + \Gamma^{(1+c)\tau} d_1 \lceil \pi_1 \rfloor^{\frac{2\rho-\lambda_1}{\rho}} \left(\lceil \zeta_2 \rfloor^{q_1} - \lceil \zeta_2^* \rfloor^{q_1}\right). \end{aligned} \tag{18}$$

putting together

Step 2. Define $\pi_2 = \lceil \zeta_2 \rfloor^{\frac{\rho}{\lambda_2}} - \lceil \zeta_2^* \rfloor^{\frac{\rho}{\lambda_2}}$ and take the Lyapunov function $V_2 = V_1 + U_2$ with

$$U_2 = \int_{\zeta_2^*}^{\zeta_2} \left\lceil \lceil s \rfloor^{\frac{\rho}{\lambda_2}} - \lceil \zeta_2^* \rfloor^{\frac{\rho}{\lambda_2}} \right\rfloor^{\frac{2\rho-\lambda_2}{\rho}} ds. \tag{19}$$

From

$$\begin{cases} \dfrac{\partial U_2}{\partial \zeta_2} = \lceil \pi_2 \rfloor^{\frac{2\rho-\lambda_2}{\rho}}, \\ \dfrac{\partial U_2}{\partial \zeta_1} = -\dfrac{2\rho - \lambda_2}{\rho} \dfrac{\partial\left(\lceil \zeta_2^* \rfloor^{\frac{\rho}{\lambda_2}}\right)}{\partial \zeta_1} \\ \qquad \times \displaystyle\int_{\zeta_2^*}^{\zeta_2} \left| \lceil s \rfloor^{\frac{\rho}{\lambda_2}} - \lceil \zeta_2^* \rfloor^{\frac{\rho}{\lambda_2}} \right|^{\frac{\rho-\lambda_2}{\rho}} ds, \end{cases} \tag{20}$$

a direct calculation gives

$$\begin{aligned} \dot{V}_2 &\leq -m\Gamma^{(1+c)\tau} |\pi_1|^{\frac{2\rho-\tau}{\rho}} \\ &\quad + \Gamma^{(1+c)\tau} d_1 \lceil \pi_1 \rfloor^{\frac{2\rho-\lambda_1}{\rho}} \left(\lceil \zeta_2 \rfloor^{q_1} - \lceil \zeta_2^* \rfloor^{q_1}\right) \\ &\quad + \frac{\partial U_2}{\partial \zeta_1} \Gamma^{(1+c)\tau} (d_1 \lceil \zeta_2 \rfloor^{q_1} + g_1) \\ &\quad + \frac{\partial U_2}{\partial \zeta_2} \Gamma^{(1+c)\tau} (d_2 \lceil \zeta_3 \rfloor^{q_2} + g_2) \\ &\leq -m\Gamma^{(1+c)\tau} |\pi_1|^{\frac{2\rho-\tau}{\rho}} \\ &\quad + \Gamma^{(1+c)\tau} d_1 \lceil \pi_1 \rfloor^{\frac{2\rho-\lambda_1}{\rho}} \left(\lceil \zeta_2 \rfloor^{q_1} - \lceil \zeta_2^* \rfloor^{q_1}\right) \\ &\quad + \Gamma^{(1+c)\tau} \Big(\frac{\partial U_2}{\partial \zeta_1} (d_1 \lceil \zeta_2 \rfloor^{q_1} + g_1) \\ &\quad + d_2 \lceil \pi_2 \rfloor^{\frac{2\rho-\lambda_2}{\rho}} \left(\lceil \zeta_3 \rfloor^{q_2} - \lceil \zeta_3^* \rfloor^{q_2}\right) \\ &\quad + d_2 \lceil \pi_2 \rfloor^{\frac{2\rho-\lambda_2}{\rho}} \lceil \zeta_3^* \rfloor^{q_2} + \lceil \pi_2 \rfloor^{\frac{2\rho-\lambda_2}{\rho}} g_2 \Big), \end{aligned} \tag{21}$$

where ζ_3^* is the virtual controller of ζ_3 to be specified later. To continue, the following estimates are needed.

Firstly, with the definitions of π_j and ζ_j^* $(j = 1, 2)$ and Lemma 2.3, one has

$$\begin{aligned} \left| \lceil \zeta_2 \rfloor^{q_1} - \lceil \zeta_2^* \rfloor^{q_1} \right| &= \left| \left(\lceil \zeta_2 \rfloor^{\frac{\rho}{\lambda_2}}\right)^{\frac{\lambda_2 q_1}{\rho}} - \left(\lceil \zeta_2^* \rfloor^{\frac{\rho}{\lambda_2}}\right)^{\frac{\lambda_2 q_1}{\rho}} \right| \\ &\leq 2^{1-\frac{\lambda_2 q_1}{\rho}} \left| \lceil \zeta_2 \rfloor^{\frac{\rho}{\lambda_2}} - \lceil \zeta_2^* \rfloor^{\frac{\rho}{\lambda_2}} \right|^{\frac{\lambda_2 q_1}{\rho}} \\ &= 2^{1-\frac{\lambda_2 q_1}{\rho}} |\pi_2|^{\frac{\lambda_2 q_1}{\rho}}. \end{aligned} \tag{22}$$

Thus, from (22), Assumptions 2.2 and Lemma 2.2, it is obtained that

$$\begin{aligned} & d_1\lceil\pi_1\rceil^{\frac{2\rho-\lambda_1}{\rho}}\left(\lceil\zeta_2\rceil^{q_1}-\lceil\zeta_2^*\rceil^{q_1}\right) \\ & \leq 2^{1-\frac{\lambda_2 q_1}{\rho}}\bar{d}_1|\pi_1|^{\frac{2\rho-\lambda_1}{\rho}}|\pi_2|^{\frac{\lambda_2 q_1}{\rho}} \\ & \leq \frac{1}{3}|\pi_1|^{\frac{2\rho-\tau}{\rho}}+|\pi_2|^{\frac{2\rho-\tau}{\rho}}\varrho_{21}, \end{aligned} \tag{23}$$

for a nonnegative smooth function ϱ_{21}.

Secondly, by Proposition 3.1 and Lemma 2.1, one gets

$$\begin{aligned} |g_2| & \leq \bar{\varphi}_2\left(|\zeta_1|^{\frac{\lambda_2-\tau}{\lambda_1}}+|\zeta_2|^{\frac{\lambda_2-\tau}{\lambda_2}}\right) \\ & \leq \bar{\varphi}_2\left(|\pi_1|^{\frac{\lambda_2-\tau}{\rho}}+|\pi_2|^{\frac{\lambda_2-\tau}{\rho}}+\beta_1^{\frac{\lambda_2-\tau}{\rho}}|\pi_1|^{\frac{\lambda_2-\tau}{\rho}}\right) \\ & \leq \tilde{\varphi}_2\left(|\pi_1|^{\frac{\lambda_2-\tau}{\rho}}+|\pi_2|^{\frac{\lambda_2-\tau}{\rho}}\right), \end{aligned} \tag{24}$$

where $\tilde{\varphi}_2 \geq \left(1+\beta_1^{\frac{\lambda_2-\tau}{\rho}}\right)\bar{\varphi}_2$ is a smooth function.

Using (24) and Lemma 2.2 yields

$$\begin{aligned} \lceil\pi_2\rceil^{\frac{2\rho-\lambda_2}{\rho}}g_2 & \leq \lceil\pi_2\rceil^{\frac{2\rho-\lambda_2}{\rho}}\tilde{\varphi}_2\left(|\pi_1|^{\frac{\lambda_2-\tau}{\rho}}+|\pi_2|^{\frac{\lambda_2-\tau}{\rho}}\right) \\ & \leq \frac{1}{3}|\pi_1|^{\frac{2\rho-\tau}{\rho}}+|\pi_2|^{\frac{2\rho-\tau}{\rho}}\varrho_{22}, \end{aligned} \tag{25}$$

for a nonnegative smooth function ϱ_{22}.

At last, notice that

$$\begin{aligned} & \frac{2\rho-\lambda_2}{\rho}\int_{\zeta_2^*}^{\zeta_2}\left|\lceil s\rceil^{\frac{\rho}{\lambda_2}}-\lceil\zeta_2^*\rceil^{\frac{\rho}{\lambda_2}}\right|^{\frac{\rho-\lambda_2}{\rho}}ds \\ & \leq \frac{2\rho-\lambda_2}{\rho}|\pi_2|^{\frac{\rho-\lambda_2}{\rho}}|\zeta_2-\zeta_2^*| \\ & \leq \frac{2\rho-\lambda_2}{\rho}2^{1-\frac{\lambda_2}{\rho}}|\pi_2|, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \left|\frac{\partial\left(\lceil\zeta_2^*\rceil^{\frac{\rho}{\lambda_2}}\right)}{\partial\zeta_1}\right| & = \left|\frac{\partial(\beta_1\lceil\pi_1\rceil)}{\partial\zeta_1}\right| \\ & \leq \left|\frac{\partial\beta_1}{\partial\zeta_1}\right||\pi_1|+\frac{\rho}{\lambda_1}\beta_1|\pi_1|^{\frac{\rho-\lambda_1}{\rho}} \\ & \leq |\pi_1|^{\frac{\rho-\lambda_1}{\rho}}\gamma_2, \end{aligned} \tag{27}$$

where $\gamma_2 \geq 0$ is a smooth function.

Therefore, on the basis of (24), (26), (27) and Lemma 2.2, one has

$$\begin{aligned} & \frac{\partial U_2}{\partial\zeta_1}(d_1\lceil\zeta_2\rceil^{q_1}+g_1) \\ & \leq \frac{2\rho-\lambda_2}{\rho}\int_{\zeta_2^*}^{\zeta_2}\left|\lceil s\rceil^{\frac{\rho}{\lambda_2}}-\lceil\zeta_2^*\rceil^{\frac{\rho}{\lambda_2}}\right|^{\frac{\rho-\lambda_2}{\rho}}ds \\ & \times\left|\frac{\partial\left(\lceil\zeta_2^*\rceil^{\frac{\rho}{\lambda_2}}\right)}{\partial\zeta_1}\right|(d_1\lceil\zeta_2\rceil^{q_1}+g_1) \\ & \leq \frac{1}{3}|\pi_1|^{\frac{2\rho-\tau}{\rho}}+|\pi_2|^{\frac{2\rho-\tau}{\rho}}\varrho_{23}, \end{aligned} \tag{28}$$

with $\varrho_{23} \geq 0$ being a smooth function.

Bringing (23), (25) and (28) to (22) gives

$$\begin{aligned} \dot{V}_2 \leq & -(m-1)\Gamma^{(1+c)\tau}|\pi_1|^{\frac{2\rho-\tau}{\rho}} \\ & +\Gamma^{(1+c)\tau}d_2\lceil\pi_2\rceil^{\frac{2\rho-r_2}{\rho}}\left(\lceil\zeta_3\rceil^{q_2}-\lceil\zeta_3^*\rceil^{q_2}\right) \\ & +\Gamma^{(1+c)\tau}\Big(d_2\lceil\pi_2\rceil^{\frac{2\rho-r_2}{\rho}}\lceil\zeta_3^*\rceil^{q_2} \\ & +(\varrho_{21}+\varrho_{22}+\varrho_{23})|\pi_2|^{\frac{2\rho-\tau}{\rho}}\Big). \end{aligned} \tag{29}$$

Then, one can design the virtual controller

$$\zeta_3^* = -\lceil\pi_2\rceil^{\frac{\lambda_3}{\rho}}\beta_2^{\frac{\lambda_3}{\rho}}(\bar{\zeta}_2), \tag{30}$$

where the smooth function β_2 satisfies

$$\beta_2(\bar{\zeta}_2) \geq \left(\frac{m-1+\varrho_{21}+\varrho_{22}+\varrho_{23}}{\underline{d}_2}\right)^{\frac{\rho}{q_2\lambda_3}}, \tag{31}$$

such that

$$\begin{aligned} \dot{V}_2 \leq & -(m-1)\Gamma^{(1+c)\tau}\left(|\pi_1|^{\frac{2\rho-\tau}{\rho}}+|\pi_2|^{\frac{2\rho-\tau}{\rho}}\right) \\ & +\Gamma^{(1+c)\tau}d_2\lceil\pi_2\rceil^{\frac{2\rho-r_2}{\rho}}\left(\lceil\zeta_3\rceil^{q_2}-\lceil\zeta_3^*\rceil^{q_2}\right). \end{aligned} \tag{32}$$

Step j ($j = 3, \ldots, m-1$)**.** The following proposition is obtained in this step.

Proposition 3.2. Assume that at step $j-1$, there exists a C^1 Lyapunov function V_{j-1} that is positive definite and proper, and a series of C^0 virtual controllers $\zeta_1^*, \ldots, \zeta_j^*$ defined by

$$\begin{aligned} & \zeta_1^* = 0, && \pi_1 = \lceil\zeta_1\rceil^{\frac{\rho}{\lambda_1}}-\lceil\zeta_1^*\rceil^{\frac{\rho}{\lambda_1}}, \\ & \zeta_2^* = -\lceil\pi_1\rceil^{\frac{\lambda_2}{\rho}}\beta_1^{\frac{\lambda_2}{\rho}}(\zeta_1), && \pi_2 = \lceil\zeta_2\rceil^{\frac{\rho}{\lambda_2}}-\lceil\zeta_2^*\rceil^{\frac{\rho}{\lambda_2}}, \\ & \quad\vdots && \quad\vdots \\ & \zeta_j^* = -\lceil\pi_{j-1}\rceil^{\frac{\lambda_j}{\rho}}\beta_{j-1}^{\frac{\lambda_j}{\rho}}(\bar{\zeta}_{j-1}), && \pi_j = \lceil\zeta_j\rceil^{\frac{\rho}{\lambda_j}}-\lceil\zeta_j^*\rceil^{\frac{\rho}{\lambda_j}}, \end{aligned} \tag{33}$$

with $\beta_j > 0$, $j = 1, \ldots, m-1$ being smooth, to render

$$\begin{aligned} \dot{V}_{j-1} \leq & -(m-j+2)\Gamma^{(1+c)\tau}\sum_{k=1}^{j-1}|\pi_k|^{\frac{2\rho-\tau}{\rho}} \\ & +\Gamma^{(1+c)\tau}d_{j-1}\lceil\pi_{j-1}\rceil^{\frac{2\rho-\lambda_{j-1}}{\rho}}\left(\lceil\zeta_j\rceil^{q_{j-1}}-\lceil\zeta_j^*\rceil^{q_{j-1}}\right). \end{aligned} \tag{34}$$

Then the jth Lyapunov function $V_j = V_{j-1} + U_j$ with

$$U_j = \int_{\zeta_j^*}^{\zeta_j}\left\lceil\lceil s\rceil^{\frac{\rho}{\lambda_j}}-\lceil\zeta_j^*\rceil^{\frac{\rho}{\lambda_j}}\right\rceil^{\frac{2\rho-\lambda_j}{\rho}}ds, \tag{35}$$

is C^1, positive definite and proper, and there is a C^0 state feedback controller

$$\zeta_{j+1}^* = -\beta_j^{\frac{\lambda_{j+1}}{\rho}}(\bar{\zeta}_j)\lceil\pi_j\rceil^{\frac{\lambda_{j+1}}{\rho}}, \tag{36}$$

such that

$$\dot{V}_j \le -(m-j+1)\Gamma^{(1+c)\tau}\sum_{k=1}^{j}|\pi_k|^{\frac{2\rho-\tau}{\rho}} +\Gamma^{(1+c)\tau}d_j\lceil\pi_j\rceil^{\frac{2\rho-r_j}{\rho}}(\lceil\zeta_{j+1}\rceil^{q_j}-\lceil\zeta^*_{j+1}\rceil^{q_j}). \tag{37}$$

Proof. See the Appendix.

Step m. Selecting

$$V_m=\sum_{j=1}^{m}U_j=\sum_{j=1}^{m}\int_{\zeta_j^*}^{\zeta_j}\left\lceil\lceil s\rceil^{\frac{\rho}{\lambda_j}}-\lceil\zeta_j^*\rceil^{\frac{\rho}{\lambda_j}}\right\rceil^{\frac{2\rho-\lambda_j}{\rho}}ds, \tag{38}$$

the above inductive step indicates that there exists a desired quantized output

$$\zeta^*_{m+1}=-\lceil\pi_m\rceil^{\frac{\lambda_{m+1}}{\rho}}\beta_m^{\frac{\lambda_{m+1}}{\rho}}(\bar{\zeta}_m), \tag{39}$$

such that

$$\begin{aligned}\dot{V}_m &\le -\Gamma^{(1+c)\tau}\sum_{k=1}^{m}|\pi_k|^{\frac{2\rho-\tau}{\rho}} +\Gamma^{(1+c)\tau}\lceil\pi_m\rceil^{\frac{2\rho-\lambda_m}{\rho}}\left(Q(v)-\zeta^*_{m+1}\right)\\ &\le -\Gamma^{(1+c)\tau}\sum_{k=1}^{m}|\pi_k|^{\frac{2\rho-\tau}{\rho}} +\Gamma^{2(1+c)\tau}\lceil\pi_m\rceil^{\frac{2\rho-\lambda_m}{\rho}}\left(Q(u)-\Gamma^{-(1+c)\tau}\zeta^*_{m+1}\right).\end{aligned} \tag{40}$$

Therefore, the state feedback control u is designed as

$$u=\begin{cases}\left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_{m+1}}{1-\delta}+u_{min}\right), & \zeta^*_{m+1}>0,\\ 0, & \zeta^*_{m+1}=0,\\ \left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_{m+1}}{1-\delta}-u_{min}\right), & \zeta^*_{m+1}<0,\end{cases} \tag{41}$$

which renders the inequality (42) of next page holds.

By noticing that $-\lceil\pi_m\rceil^{\frac{2\rho-\lambda_m}{\rho}}\zeta^*_{n+1}\ge 0$, one gets

$$\dot{V}_m\le-\Gamma^{(1+c)\tau}\sum_{k=1}^{m}|\pi_l|^{\frac{2\rho-\tau}{\rho}}\le-\sum_{k=1}^{m}|\pi_k|^{\frac{2\rho-\tau}{\rho}}. \tag{43}$$

Consequently, the following theorem is drew out.

Theorem 3.1. For the system (1) under Assumptions 2.1 and 2.2, the quantized state feedback controller (41) consisting of (33) and (39) renders the states of the closed-loop system convergent to zero within prescribed finite time $T_c>0$.

Proof. represents that property that positive definite and proper of V_n given in Proposition 3.2 together with (43) and Lemma 4.3 in [49] reveal that, there exist class $\mathcal{K}_\infty$ functions κ_1, κ_2 and κ_3 such that

$$\kappa_1(|\zeta|)\le V_m(\zeta)\le\kappa_2(|\zeta|), \tag{44}$$

$$\dot{V}_m\le-\kappa_3(|\zeta|), \tag{45}$$

which indicate that $\zeta(t)$ is asymptotically convergent and is bounded on $[0,T_c)$.

On the other hand, the SST (9) gives

$$\begin{aligned}z_j(t)&=\Gamma^{-(1+c)\lambda_j}\zeta_j(t)\\ &=\left(\frac{T_c-t}{T_c}\right)^{(1+c)\lambda_j}\zeta_i(t),\quad j=1,\dots,m.\end{aligned} \tag{46}$$

Consequently, it further is obtained that

$$\begin{aligned}\lim_{t\to T_c}z_j(t)&=\lim_{t\to T_c}\left(\frac{T_c-t}{T_c}\right)^{(1+c)\lambda_j}\zeta_j(t)\\ &=0,\quad j=1,\dots,m.\end{aligned} \tag{47}$$

Therefore, the proof is completed.

3.2 Controller design for $t\in[T_c,+\infty)$ and main result

Notice that the quantized controller that drives system states to zero within prescribed time $T_c>0$ has been designed in the above subsection. As a result, in this subsection we need only consider how to design a controller such that the states reach and stay at the origin for all $t\in[T_c,+\infty)$.

On basis of the solution properties of existence and continuation, it is obtained that $z(T_c)=0$. Therefore, the control u can be simply selected as $u=0$, which together with $f_j(0)=0$ guarantees $z(t)=0$ for any $t\in[T_c,+\infty)$ [37]. However, this choice will render that the CLS is sensitive to uncertainties/disturbances. To avoid this, we here give an alternative solution for $t\in[T_c,+\infty)$. Notice the fact that the aboriginal system (1) and the transformed system (11) possess the similar structure except the control coefficient $\Gamma^{(1+c)\tau}$. This means that, by simply setting $\Gamma=1$, a new controller u of from (41) can be designed to keep the states at the origin for all $t\ge T_c$.

Unill now, the control design of PTS for the system (1) is finished. Thereby the main results of this article is summed up as follows.

Theorem 3.2. For the system (1) under Assumptions 2.1 and 2.2, if the quantized state feedback controller

$$u=\begin{cases}\left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_{m+1}}{1-\delta}+u_{min}\right), & \zeta^*_{m+1}>0,\\ 0, & \zeta^*_{m+1}=0,\\ \left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_{m+1}}{1-\delta}-u_{min}\right), & \zeta^*_{m+1}<0,\end{cases} \tag{48}$$

with

$$\Gamma_1=\begin{cases}\dfrac{T_c}{T_c-t}, & t\in[0,T_c),\\ 1, & t\in[T_c,+\infty),\end{cases} \tag{49}$$

$$\left(Q(u) - \Gamma^{-(1+c)\tau}\zeta^*_{m+1}\right) = \begin{cases} Q_1(t)\left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_{m+1}}{1-\delta} + u_{min}\right) - \Gamma^{-(1+c)\tau}\zeta^*_{m+1} > 0, & \zeta^*_{m+1} > 0, \\ 0, & \zeta^*_{m+1} = 0, \\ Q_1(t)\left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_{m+1}}{1-\delta} - u_{min}\right) - \Gamma^{-(1+c)\tau}\zeta^*_{m+1} < 0, & \zeta^*_{m+1} < 0. \end{cases} \tag{42}$$

$$\zeta^*_{m+1} = -\lceil \pi_m \rfloor^{\frac{\lambda_{m+1}}{\rho}} \beta_m^{\frac{\lambda_{m+1}}{\rho}} (\bar{\zeta}_m), \tag{50}$$

is introduced, then the origin of the closed-loop system is globally prescribed-time stable.

Proof. The properties that, on $[0, T_c)$, $\Gamma(t) = T_c/(T_c - t)$ monotonously grows and $\zeta(t)$ is asymptotically convergent, give

$$|z(t)| \le |\zeta(t)| \le |\zeta(0)| = |z(0)|. \tag{51}$$

Putting this and $z(t) = 0$ for any $t \in [T_c, +\infty)$ together lead to

$$|z(t)| \le |z(0)|, \quad t \ge 0. \tag{52}$$

That is to say, the origin of the closed-loop system is globally Lyapunov stable. Furthermore, with the global prescribed-time convergent of the closed-loop system in mind, this theorem is straightforwardly concluded from Definition 2.2.

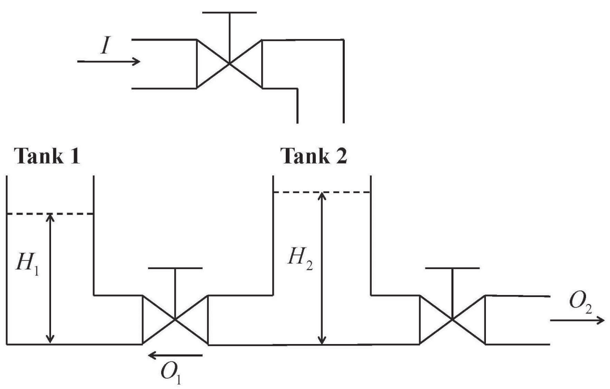


Fig. 1 Schematic diagram of the liquid-level system.

4 Simulation example

To example the utilization of the proposed control scheme, this section considers a liquid-level system exhibited in Fig.1, the dynamics of which are represented by

$$\begin{aligned} C_1\dot{H}_1 &= O_1 \\ C_2\dot{H}_2 &= I - O_1 - O_2 \\ O_1 &= \begin{cases} k_1\sqrt{2g|H_2 - H_1|}, & H_2 \ge H_1, \\ -k_1\sqrt{2g|H_2 - H_1|}, & H_2 < H_1, \end{cases} \\ O_2 &= k_2\sqrt{2gH_2}, \end{aligned} \tag{53}$$

with the same physical meanings of system parameters given in [29].

By drawing into the variable changes

$$z_1 = H_1 - H,\ z_2 = H_2 - H_1,\ u = \frac{I}{C_2} - \frac{k_2\sqrt{2gH}}{C_2}, \tag{54}$$

and taking the quantized input nonlinearity into account, the dynamics of (53) can be further modelled as

$$\begin{aligned} \dot{z}_1 &= d_1\lceil z_2 \rfloor^{\frac{1}{2}}, \\ \dot{z}_2 &= Q(u) + f_2(\bar{z}_2), \end{aligned} \tag{55}$$

where $d_1 = \frac{k_1\sqrt{2g}}{C_1}$ and $f_2(\bar{z}_2) = -\frac{C_1}{C_2}d_1\lceil z_2 \rfloor^{\frac{1}{2}} - \frac{k_2\sqrt{2g}}{C_2}\lceil z_1 + z_2 + H \rfloor^{\frac{1}{2}} + \frac{k_2\sqrt{2g}}{C_2}\lceil H \rfloor^{\frac{1}{2}}$, Q denotes the quantized input nonlinearity described by (5). With Lemma 2.1, it is easy to check that Assumptions 2.1 and 2.2 hold with $\lambda_3 = \tau = 1$, $\lambda_1 = \lambda_2 = 2$, $\varphi_2 = \frac{\sqrt{2g}}{C_2}(k_1 + k_2)$.

Introducing $\zeta_i = \Gamma_1^{(1+c)\lambda_i} z_i$, $i = 1, 2$ with

$$\Gamma_1 = \begin{cases} \dfrac{T_c}{T_c - t}, & t \in [0, T_c), \\ 1, & t \in [T_c, +\infty), \end{cases} \tag{56}$$

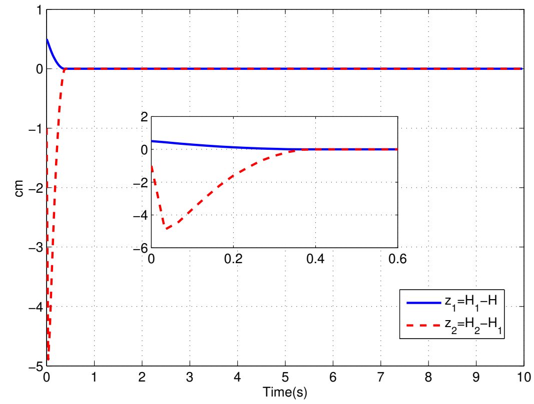
and taking $\rho = 2$ and $c = 0$, according to Theorem 3.2 one can design a quantized state feedback controller

$$u = \begin{cases} \left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_3}{1-\delta} + u_{min}\right), & \zeta^*_3 > 0, \\ 0, & \zeta^*_3 = 0, \\ \left(\dfrac{\Gamma^{-(1+c)\tau}\zeta^*_3}{1-\delta} - u_{min}\right), & \zeta^*_3 < 0, \end{cases} \tag{57}$$

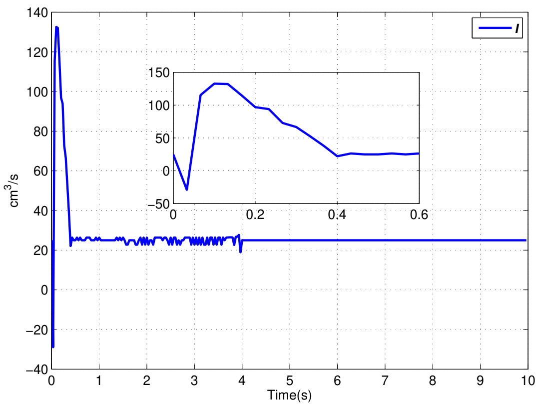
$$\zeta^*_3 = -\left(0.1 + \varrho_{21} + \varrho_{22} + \varrho_{23}\right)\lceil \pi_2 \rfloor^{\frac{1}{2}}, \tag{58}$$

with $\beta_1 = (1.1 + \frac{2}{T_c}(1 + \zeta_1^2)^{\frac{1}{2}})/d_1$ if $t \in [0, T_c)$ and $\beta_1 = 1.1/d_1$ if $t \in [T_c, +\infty)$, $\pi_2 = \zeta_2 - \zeta_2^*$, $\zeta_2^* = -\beta_1\zeta_1^2$, $\tilde{\varphi}_2 = (1 + \beta_1^{\frac{1}{2}})(1+c)\lambda_2|\zeta_2|^{\tau/\lambda_2}/T_c + \varphi_i$, $\varrho_{21} = 3.7712 d_1^{\frac{3}{2}}$, $\varrho_{22} = 0.6667\tilde{\varphi}_2^{\frac{3}{2}} + \tilde{\varphi}_2$, $\varrho_{23} = |\frac{\partial\zeta_2^*}{\partial\zeta_1}d_1| + 0.6667|\frac{\partial\zeta_2^*}{\partial\zeta_1}|^3(d_1\beta_1^{\frac{1}{2}} + \frac{2}{T_c}(1 + \zeta_1^2)^{\frac{1}{2}})^3$, which renders the system (55) globally prescribed-time stable.

In the simulation, we select the system parameters as $H = 100\text{cm}$, $g = 9.8\text{m/s}^2$, $C_1 = C_2 = \sqrt{2g} = 4.427\text{cm}^2$, $k_1 = 1\text{cm}^2$, $k_2 = 0.25\text{cm}^2$, $d = 0.05$, $\delta = 0.2$ and the prescribed time as $T_c = 4s$. With the different

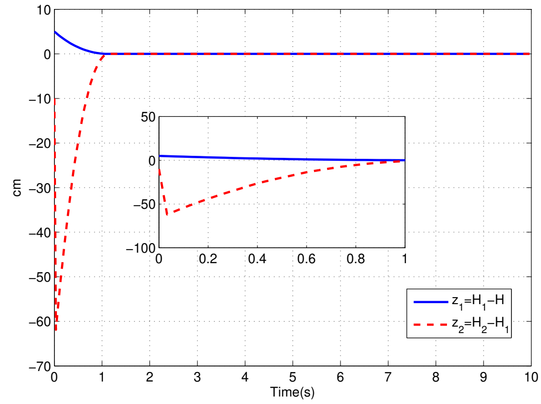


(a) System states

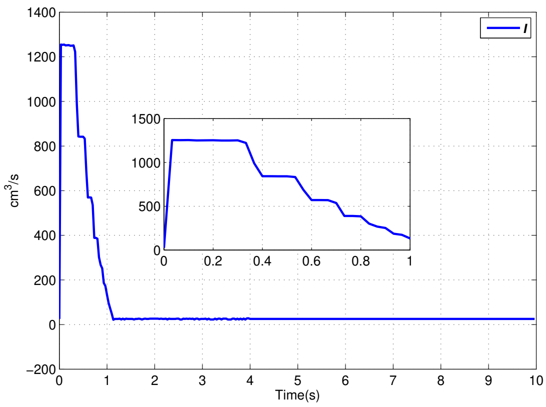


(b) Actual input I

Fig. 2 The responses of the CLS with initial condition $(z_1(0), z_2(0)) = (0.5, -1)$.



(a) System states



(b) Actual input I

Fig. 3 The responses of the CLS with initial condition $(z_1(0), z_2(0)) = (5, -10)$.

initial conditions, Figs. 2 and 3 are given to exhibit the responses of the CLS. It is clearly see that the CLS is stable and the convergence time maintains below the prescribed finite time $4s$ despite the initial value growing, which confirms the validity of the control scheme.

Remark 3.3. In this remark, our contributions is stressed by contrasting these main features of the proposed approach with the existing approaches on prescribed/fixed-time controllers in Table 1.

5 Conclusion

In this article, the problem of prescribed-time state feedback stabilization has addressed a kind of HON-Ss with quantized input nonlinearity. Based on a novel SST to translate the aboriginal problem of PTS into the asymptotic stabilization of the transformed one, a constructive quantized control design procedure of state feedback is established with the aid of the API technique. A significant advantage of the presented scheme is that the settling time can be preset and easy to adjust discretionarily in line with practical requirements. The efficacy is confirmed by the practical application of a liquid-level system.

Disclosure statement

The authors declare that they have no conflict of interest.

Appendix

Proof of Proposition 3.1. From the definition of Γ in (10), one has $\dot{\Gamma} = \Gamma^2/T_c$, which together with (9)

Table 1 Qualitative comparison with state-of-the-art approaches.

Approaches and features	Priori setting time	Singular controller	For HONSs	Involving quantized input nonlinearity
Bi-limit homogeneous approach [21, 22]	No	No	No	No
Lyapunov-based approach [23–32]	No	No	No	No
Scaling transformation approach [35]	Yes	Yes	No	No
The proposed SST approach	Yes	No	Yes	Yes

and Assumption 2.1 implies

$$\begin{aligned}|g_j(\bar{\zeta}_j)| &= \left|\zeta_j \frac{(1+c)\lambda_j \dot{\Gamma}}{\Gamma^{1+(1+c)\tau}} + \Gamma^{(1+c)(\lambda_j-\tau)} f_j(\bar{z}_j)\right| \\ &\le \frac{(1+c)\lambda_j}{T_c}\Gamma^{1-(1+c)\tau}|\zeta_j| \\ &\quad + \left|\Gamma^{(1+c)(\lambda_j-\tau)}\varphi_j \sum_{k=1}^{j}|z_k|^{\frac{\lambda_j-\tau}{\lambda_k}}\right| \\ &\le \frac{(1+c)\lambda_j}{T_c}\Gamma^{1-(1+c)\tau}|\zeta_j| + \varphi_j \sum_{k=1}^{j}|\zeta_k|^{\frac{\lambda_j-\tau}{\lambda_k}}.\end{aligned} \tag{59}$$

By noting that $c \ge (1/\tau) - 1$ and $\Gamma \ge 1$ for all $t \in [0, T_c)$, the smooth functions $\bar{\varphi}_j(\bar{\zeta}_j) \ge (1+c)\lambda_j|\zeta_j|^{\tau/\lambda_j}/T_c$ $+\varphi_j$ exists to ensure that Proposition 3.1 is true.

Proof of Proposition 3.2. First of all, some simple calculations lead to

$$\begin{cases} \dfrac{\partial U_j}{\partial \zeta_j} = \lceil \pi_j \rfloor^{\frac{2\rho-\lambda_j}{\rho}}, \\ \dfrac{\partial U_j}{\partial \zeta_k} = -\dfrac{2\rho-\lambda_j}{\rho} \dfrac{\partial \left(\lceil \zeta_j^* \rfloor^{\frac{\rho}{\lambda_j}}\right)}{\partial \zeta_k} \\ \qquad \times \displaystyle\int_{\zeta_j^*}^{\zeta_j} \left|\lceil s \rfloor^{\frac{\rho}{\lambda_j}} - \lceil \zeta_j^* \rfloor^{\frac{\rho}{\lambda_j}}\right|^{\frac{\rho-\lambda_j}{\rho}} ds, \end{cases} \tag{60}$$

for $k = 1, \dots, j-1$. By $\rho \ge \max_{1\le j\le m}\{\lambda_j\}$ and $\beta_j(\cdot)$ being smooth, it is clear that U_j, and also V_j is C^1.

Second, by employing the idea of classified discussion [4], one can prove that

$$U_k \ge C_k|\zeta_k - \zeta_k^*|^{\frac{\rho-\lambda_k}{\rho}}, \tag{61}$$

for some constant $C_k > 0$.

Furthermore one has

$$V_j = V_{j-1} + U_j \ge V_{j-1} + U_j|\zeta_j - \zeta_j^*|^{\frac{\rho-\lambda_j}{\rho}}, \tag{62}$$

and thus V_j is positive definite and proper.

At last, we prove the inequality (37) is true. From (34) and (60), one has

$$\begin{aligned}\dot{V}_j \le &-(m-j+2)\Gamma^{(1+c)\tau}\sum_{k=1}^{j-1}|\pi_k|^{\frac{2\rho-\tau}{\rho}} \\ &+\Gamma^{(1+c)\tau}\Bigg(d_{j-1}\lceil \pi_{j-1}\rfloor^{\frac{2\rho-\lambda_{j-1}}{\rho}}(\lceil \zeta_j \rfloor^{q_{j-1}} - \lceil \zeta_j^* \rfloor^{q_{j-1}}) \\ &+d_j\lceil \pi_j \rfloor^{\frac{2\rho-\lambda_j}{\rho}}\lceil \zeta_{j+1}\rfloor^{q_j} + \lceil \pi_j \rfloor^{\frac{2\rho-\lambda_j}{\rho}} g_j \\ &+\sum_{k=1}^{j-1}\frac{\partial U_j}{\partial \zeta_k}(d_k\lceil \zeta_{k+1}\rfloor^{q_k} + g_k)\Bigg).\end{aligned} \tag{63}$$

Similar as those in Step 2, the estimates of some terms of (63) on the basis of Lemma 2.1–2.3 can be provided as

$$\begin{aligned}&d_{j-1}\lceil \pi_{j-1}\rfloor^{\frac{2\rho-\lambda_{j-1}}{\rho}}(\lceil \zeta_j \rfloor^{q_{j-1}} - \lceil \zeta_j^* \rfloor^{q_{j-1}}) \\ &\le \frac{1}{3}|\pi_{j-1}|^{\frac{2\rho-\tau}{\rho}} + |\pi_j|^{\frac{2\rho-\tau}{\rho}}\varrho_{j1},\end{aligned} \tag{64}$$

$$\begin{aligned}&\lceil \pi_j \rfloor^{\frac{2\rho-\lambda_j}{\rho}} g_j \\ &\le \frac{1}{3}\sum_{k=1}^{j-1}|\pi_j|^{\frac{2\rho-\tau}{\rho}} + |\pi_j|^{\frac{2\rho-\tau}{\rho}}\varrho_{j2},\end{aligned} \tag{65}$$

$$\begin{aligned}&\sum_{j=1}^{k-1}\frac{\partial U_j}{\partial \zeta_k}(d_k\lceil \zeta_{k+1}\rfloor^{q_k} + g_k) \\ &\le \frac{1}{3}\sum_{k=1}^{j-1}|\pi_k|^{\frac{2\rho-\tau}{\rho}} + |\pi_j|^{\frac{2\rho-\tau}{\rho}}\varrho_{j3},\end{aligned} \tag{66}$$

where ϱ_{ij}, $j = 1, 2, 3$ are nonnegative smooth functions.

Substituting (64)–(66) into (63) results in

$$\begin{aligned}\dot{V}_j \le &-(m-j+1)\Gamma^{(1+c)\tau}\sum_{k=1}^{j-1}|\pi_k|^{\frac{2\rho-\tau}{\rho}} \\ &+\Gamma^{(1+c)\tau}\Bigg(d_j\lceil \pi_j \rfloor^{\frac{2\rho-\lambda_j}{\rho}}(\lceil \zeta_{j+1}\rfloor^{q_j} - \lceil \zeta_{j+1}^*\rfloor^{q_j}) \\ &+d_j\lceil \pi_j \rfloor^{\frac{2\rho-\lambda_j}{\rho}}\lceil \zeta_{j+1}^*\rfloor^{q_j} + |\pi_j|^{\frac{2\rho-\tau}{\rho}}(\varrho_{j1}+\varrho_{j2}+\varrho_{j3})\Bigg).\end{aligned} \tag{67}$$

Then, the virtual (actual) controller

$$\zeta_{j+1}^{*} = -\lceil \pi_j \rfloor^{\frac{\lambda_{j+1}}{\rho}} \beta_j^{\frac{\lambda_{j+1}}{\rho}}(\bar{\zeta}_j), \tag{68}$$

where $\beta_j(\cdot)$ is smooth and satisfies

$$\beta_j(\bar{\zeta}_j) \geq \left(\frac{m-j+1+\varrho_{j1}+\varrho_{j2}+\varrho_{j3}}{\underline{d}_j} \right)^{\frac{\rho}{q_j \lambda_{j+1}}}, \tag{69}$$

renders

$$\begin{aligned} \dot{V}_j \leq & -(m-j+1)\Gamma^{(1+c)\tau} \sum_{k=1}^{j} |\pi_k|^{\frac{2\rho-\tau}{\rho}} \\ & +\Gamma^{(1+c)\tau} d_j \lceil \pi_j \rfloor^{\frac{2\rho-\lambda_j}{\rho}} (\lceil \zeta_{j+1} \rfloor^{q_j} - \lceil \zeta_{j+1}^{*} \rfloor^{q_j}). \end{aligned} \tag{70}$$

This completes the proof.

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Figures

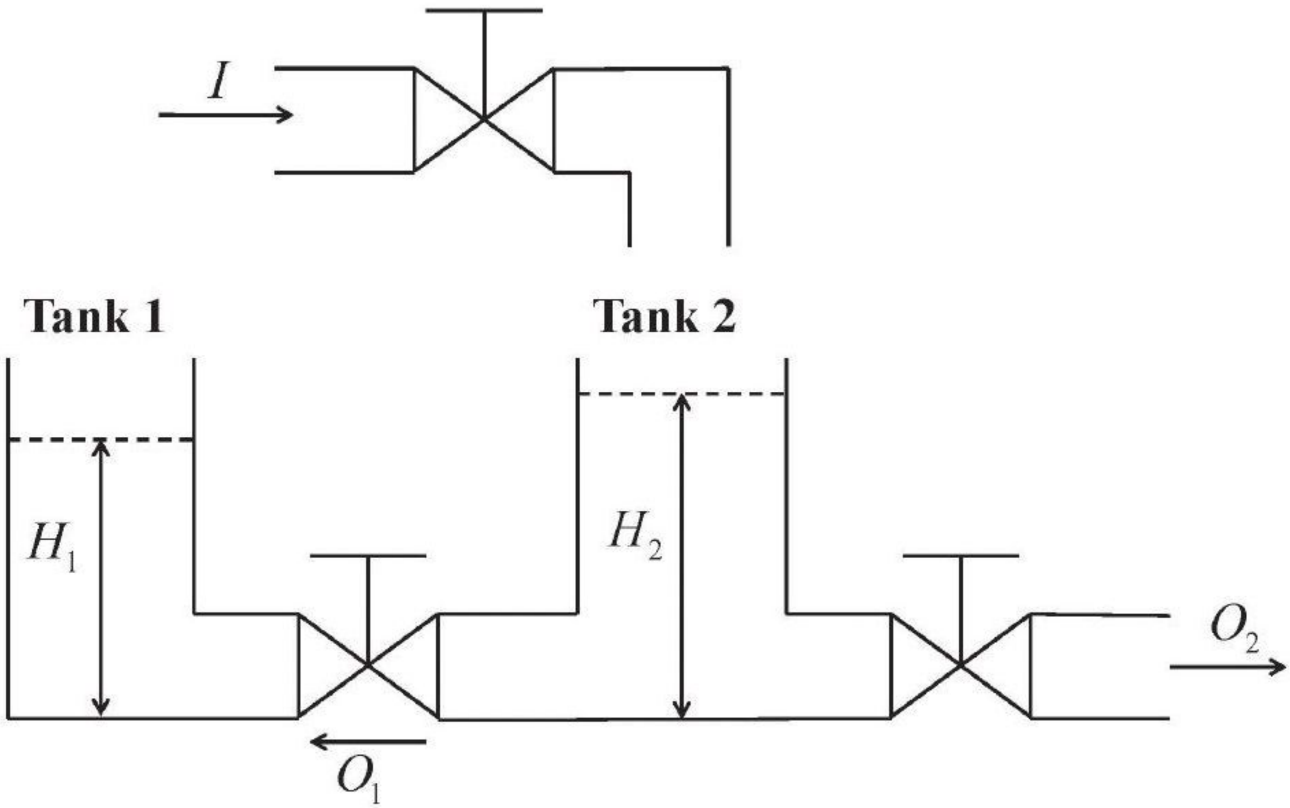
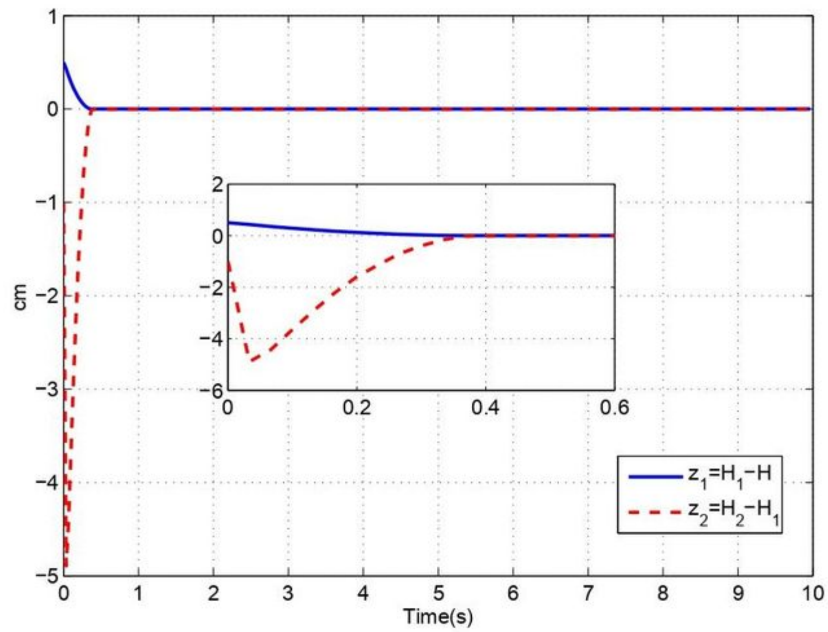
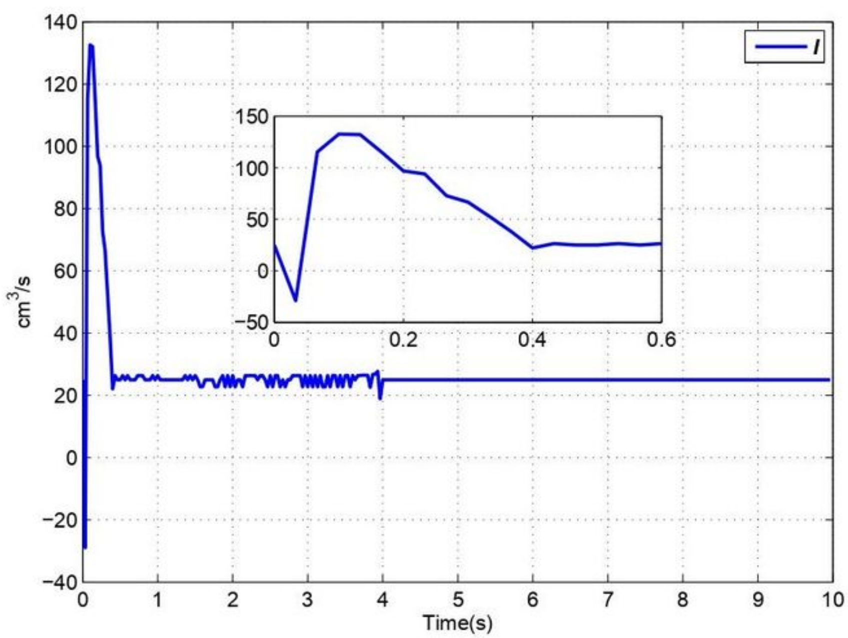


Figure 1

Schematic diagram of the liquid-level system.



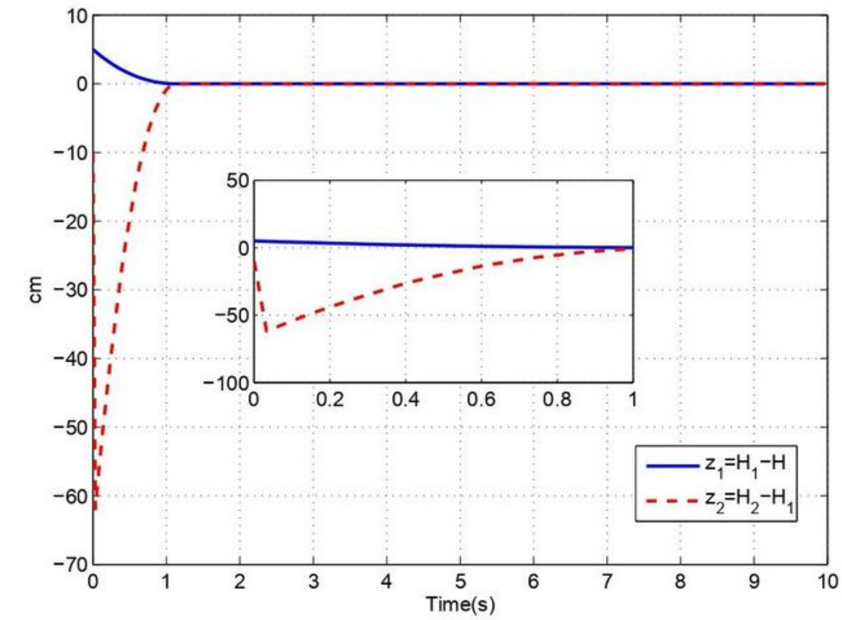
(a) System states



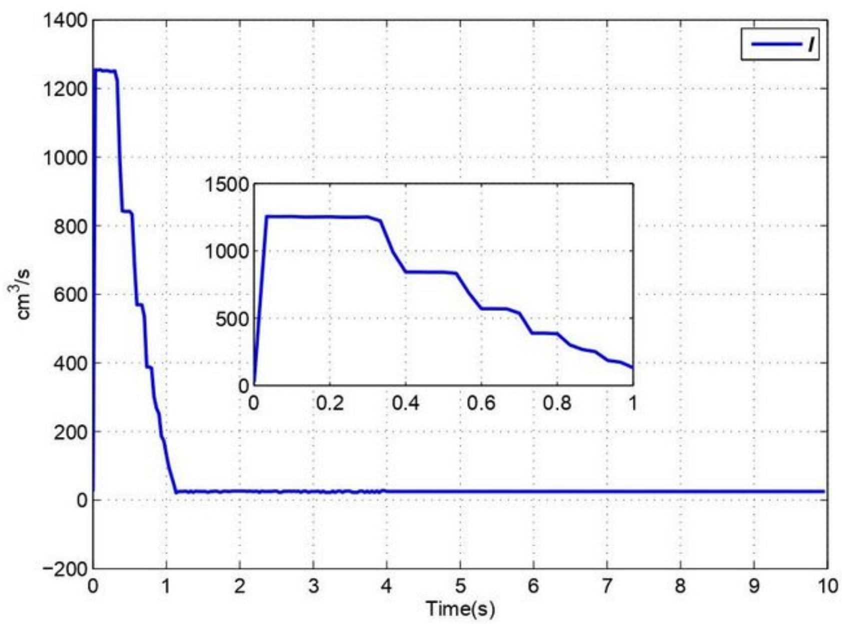
(b) Actual input I

Figure 2

The responses of the CLS with initial condition (z1(0), z2(0)) = (0.5, −1).



(a) System states



(b) Actual input I

Figure 3

The responses of the CLS with initial condition (z1(0), z2(0)) = (5, −10).

Supplementary Files

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