Optimal control problem for unbounded bilinear systems and applications

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Optimal control problem for unbounded bilinear systems and applications

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Abstract

In this paper, we will investigate the optimal control problem for unbounded bilinear systems, with \((p,q)\)-admissible control operators. We will first study the case of finite time-horizon with unconstrained or constrained endpoint. This result is further applied to build the optimal control for infinite time horizon. Finally, we solve the bilinear optimal control for the transport equation. Then we consider the fractional diffusion equation, for which we prove the strong stabilisation by an optimal time-varying feedback control.

Keywords: Bilinear systems, quadratic cost, optimal control, feedback stabilization.

I. Introduction

In this paper, we consider a bilinear optimal control problem of the following form

\[
(P): \quad \min_{u \in U_{ad}} J(u) := \min_{u \in U_{ad}} a \|y(T) - y_d\|^{q'} + b \int_0^T \|y(t)\|^{q'} dt + c \int_0^T |u(t)|^p dt,
\]

where \(a, b\) and \(c\) are positive constants, \(0 < T \leq +\infty\), \(y_d \in X\), \(2 \leq p, q' < +\infty\) and \(u, y\) are such that

\[
\dot{y}(t) = Ay(t) + u(t)By(t), \quad t \in (0, T), \quad y(0) = y_0 \in X,
\]

\(1\)

- \(A : D(A) \subset X \mapsto X\) is the infinitesimal generator of a linear \(C_0\) semigroup \(S(t)\) on a real Hilbert space \(X\), whose inner product and corresponding norm are denoted respectively by \(\langle \cdot, \cdot \rangle\) and \(\|\cdot\|\),

- \(B : \quad X \mapsto X\) is a linear unbounded operator,

- \(u\) is a real valued control that belongs to the control space \(L^p(0,T;\mathbb{R})\) (we will use \(L^p(0,T)\) for simplicity) and \(y(t)\), when it exists, is the corresponding mild solution,

- \(U_{ad} = \{u \in L^p(0,T) / y(T) \in R\}\) is the set of admissible controls, where \(R\) is a subset of \(X\).

The case of bounded control operator is already treated by several authors (see [11] [16] [21] [26] [27] [29] [31] [32]). They provided necessary conditions on the optimal control via the solution of the adjoint equation. These works are based on two approaches: the Maximum Principle of
Pontryagin as in [11, 21, 26, 29] and the differentiation method as in [16, 31, 32]. Moreover, a few works have addressed the optimal control problem for particular systems with unbounded control operators, such as the Fokker Plank equation, in which case the control operator is skew adjoint (see [1, 5, 15, 18]).

Also in [12] the authors have studied the problem of optimal control with unconstrained endpoint for a class of abstract bilinear systems with a control operator \( B \in \mathcal{L}(V, X) \), where \( V \subset X \subset V^* \) is a Gelfand triple of real Hilbert spaces. This model is typically adapted to the Fokker plank equation and the fractional diffusion equation where the control operator \( B \) is \((-\Delta)^{\frac{1}{2}}\). However, the control operator \( B \) considered in this work is defined from \( X \) to some larger Banach space \( Y \supset X \) where \( B \) is \((p,q)\)-admissible. As an example, one can mention the fractional diffusion equation where the control operator \( B \) is \((-\Delta)^{\alpha}, 0 < \alpha < 1 \) (see [4, 7]). This type of unbounded control operator may also occur in many situations of PDEs when the control is applied on a part of the boundary or at a point of the geometrical domain. This also includes some situations of internal control action. The question of stabilisation of a system like (1) has been considered in several works (see [6, 7, 8, 10]).

In the present work, we study the optimal control problem for a class of unbounded abstract bilinear systems in finite and infinite time-horizon. We will also consider the case of endpoint constraint. We suppose that \( B \) is unbounded in the sense that it is bounded from \( X \) to some larger Banach space \( Y \supset X \) and we also assume that the control operator \( B \in \mathcal{L}(X, Y) \) is \((p,q)\)-admissible. Here we introduce a new technique of the proof, which is based on linear semigroup properties and the variation of constants formula. Furthermore, we shall suppose that \( A \) admits an extension \( A^Y \) generating a strongly continuous semi-group \( S^Y(t) \) on \( Y \), which extends the \( C_0 \)-semi-group \( S(t) \).

The paper is organized as follows: In Section 2, we will first study the well-posedness of a class of non-homogeneous unbounded bilinear systems, then we discuss the optimal control in a finite time horizon with an unconstrained endpoint. This result is further applied to express the optimal control with endpoint constraint and we will extend the results of finite time horizon to treat the case of infinite time horizon. Section 3 is devoted to solve the bilinear optimal control for the transport equation, and also the fractional diffusion equation, for which we further show the strong stability using a time-varying feedback control.

II. Optimal control problem

i. Preliminaries

Let us recall the notion of \((p,q)\)-admissible as given in [7]. For this, we consider the bilinear operator \( \phi_t \) defined by

\[
L^p_{loc}(0, +\infty) \times L^q_{loc}(0, +\infty; X) \mapsto Y
\]

\[
(u, y) \mapsto \Phi_t(u, y) = \int_0^t S^Y(t-s)u(s)By(s)ds.
\]

**Definition 1**

The operator \( B \) is \((p,q)\)-admissible with respect to the semi-group \( S(t) \) if for any \( t > 0 \) and \( 1 < p, q < +\infty \) such that \( \frac{1}{p} + \frac{1}{q} \leq 1 \), the operator \( \Phi_t \) is bounded from \( L^p(0, +\infty) \times L^q(0, +\infty; X) \) to \( X \).
Remark 2

- The \((p,q)\)-admissibility in the sense of Definition 1 is equivalent to the existence of a constant \(C_t > 0\) (for any fixed \(t > 0\)) such that
  \[
  \|\Phi_t(u,y)\| \leq C_t \|y\|_{L^p(0,t;X)} \|u\|_{L^p(0,t)} , \quad \forall (u,y) \in L^p(0,\infty) \times L^q(0,\infty;X)
  \]
- Note that for every fixed \((u,y) \in L^p(0,\infty) \times L^q(0,\infty;X)\), the mapping \(\Phi_t(u,y) : t \mapsto \Phi_t(u,y)\) is continuous.

The next lemma discusses the well-posedness of a perturbed bilinear system. The lemma is a generalization of Theorem 2.5 in [7] and is based on the fact that \(B\) is \((p,q)\)-admissible. For this end, we consider the following system

\[
\begin{aligned}
  y(t) &= Ay(t) + u(t)By(t) + f(t), \quad t \in (0,\infty) \\
  y(0) &= y_0
\end{aligned}
\]

(2)

where \(f : [0,\infty) \mapsto Y\) is such that the mapping \(g : [0,\infty) \mapsto X, \quad t \mapsto \int_0^t S^Y(t-s)f(s)ds\) is \(\alpha\)-Hölder continuous for some \(\alpha > 0\), (i.e. \(\|g(t_1) - g(t_2)\| \leq C|t_1 - t_2|^\alpha\) for \(t_1, t_2 \geq 0\) and \(C > 0\)).

Lemma 3

Assume that the operator \(B\) is \((p,q)\)-admissible. Then for any \(y_0 \in X\) and \(u \in L^p_{loc}(0,\infty)\), the system (2) admits a unique mild solution \(y \in C([0,\infty);X)\) given by the following variation of constants formula

\[
y(t) = S^Y(t)y_0 + \int_0^t S^Y(t-s)u(s)By(s)ds + \int_0^t S^Y(t-s)f(s)ds.
\]

(3)

Proof

Let \(0 < T < 1\) and \(a > 0\). Let us consider the set

\[
F = \{y \in C(0,T;X), \|y(t) - y_0\| \leq a, \forall t \in [0,T]\}.
\]

For \(u \in L^p(0,T)\), we consider the mapping \(f_u : F \mapsto C(0,T;X)\) defined, for any \(y \in F\), by

\[
f_u(y)(t) = S^Y(t)y_0 + \int_0^t S^Y(t-s)u(s)By(s)ds + \int_0^t S^Y(t-s)f(s)ds.
\]

Let us show that \(f_u(y)(\cdot)\) is continuous. For this end, let \(0 < t < t_1 < T\) and let \(\epsilon = t_1 - t\), then

\[
\|f_u(y)(t_1) - f_u(y)(t)\| \leq \|S^Y(t_1)y_0 - S^Y(t)y_0\| + \|\Phi_{t_1-u,y} - \Phi_t(u,y)\| + \|\int_0^t S^Y(t-s)f(s)ds - \int_0^t S^Y(t-s)f(s)ds\|.
\]

Moreover, we have

\[
\Phi_{t_1-u,y} - \Phi_t(u,y) = S^Y(t) \int_0^\epsilon S^Y(\epsilon-s)u(s)By(s)ds + \int_0^t S^Y(t-s)(u(\epsilon+s)By(\epsilon+s) - u(s)By(s))ds.
\]

Then, using the admissibility of \(B\), we can conclude that for \(\epsilon > 0\) small enough and for some \(C_\epsilon > 0\) we obtain

\[
\|\Phi_{t_1-u,y} - \Phi_t(u,y)\| \leq C_\epsilon \left(\|u\|_{L^p(0,\epsilon)}\|y\|_{L^q(0,\epsilon;X)} + \|u(\cdot + \epsilon) - u(\cdot)\|_{L^p(0,1)}\|y(\cdot)\|_{L^q(0,1;X)} + \|u(\cdot)\|_{L^p(0,1)}\|y(\cdot + \epsilon) - y(\cdot)\|_{L^q(0,1;X)}\right).
\]
where $\tilde{C}_e = C_e \sup_{0 \leq t \leq T} \|S^y(t)\|_Y$. Thus according to the second point of Remark 2 we have that $t \mapsto \Phi_t(u, y)$ is continuous with respect to $t$ for any given $(u, y) \in L^p(0, +\infty) \times L^q(0, +\infty; X)$. Then, from the continuity of the mapping $t \mapsto \int_0^t S^y(t-s)f(s)ds$, we can conclude that $f_a(y)(.) \in C(0, T; X)$ and that for $T$ small enough we have $f_a(y) \in F$. Indeed, for the first point of Remark 2 and the $\kappa$-Hölder continuity of $f$, we deduce that
\[
\|f_a(y)(t) - y_0\| \leq \|S^y(t)\|_Y + C(\|u\|_{L^p(0, T)} \|y\|_{L^q(0, T; X)} + T^\kappa), \forall t \in [0, T],
\]
for some $C > 0$. In the sequel $C$ will denote a generic positive constant. Using the fact that $y \in F$, we get for some $C > 0$ that $\|y\|_{L^q(0, T; X)} \leq C(\|y_0\| + a)T^\frac{\kappa}{2}$.

Then, in order to prove that $f_a(y) \in F$, it suffices to have for some $C > 0$,
\[
\|S^y(t)\|_Y + C(\|u\|_{L^p(0, T)} \|y\|_X + a)T^\frac{\kappa}{2} + T^\kappa \leq a,
\]
which is verified for $T$ small enough.

Let us now show that $f_a$ is a contraction map from $F$ to $F$ for $T$ small enough. For $y_1, y_2 \in F$, we have for some $C > 0$
\[
\|f_a(y_1)(t) - f_a(y_2)(t)\| = \|\Phi_t(u, y_1) - \Phi_t(u, y_2)\| \leq CT^\frac{\kappa}{2} \|u\|_{L^p(0, T)} \|y_1 - y_2\|_{C(0, T; X)}.
\]

Then, by applying the contraction mapping principle, we conclude that for $T$ small enough, the system (2) admits a unique solution $y \in C(0, T; X)$. Let $[0, T_{\text{max}})$ be the maximal interval where the solution exists, we shall show that $T_{\text{max}} = +\infty$. By contradiction, we suppose that $T_{\text{max}} < +\infty$.

Let $\|\Phi_t\|$ denote the norm of the continuous bilinear operator $\Phi_t$, which satisfies $\|\Phi_t\| \leq \|\Phi_t\|$ for $t' \leq t$. Then we have for $t \in [0, T_{\text{max}})$ and for some $C = C(T_{\text{max}})$
\[
\|y(t)\| \leq (\|S^y(t)\|_Y + \int_0^t S(t-s)f(s)ds + \|\Phi_t(u, y)\|)^q \leq C(1 + \|y\|_{L^q(0, T; X)}),
\]
Using Gronwall’s lemma we get, for some $M_{T_{\text{max}}} > 0$, that $\|y(t)\| \leq M_{T_{\text{max}}}$.

Then, applying Theorem 1.4 in [25], we conclude that $T_{\text{max}} = +\infty$, which achieves the proof.

ii. Optimal control in finite time-horizon

ii.1 Unconstrained optimal control

This part is devoted to the problem (P) without constraint on the state, that is $U_{\text{ad}} = \{u \in L^p(0, T) / y(T) \in X\}$. This is the subject of Theorem 4 below. For the sake of simplicity we can take, without loss of generality, $q' = q$ in the problem (P). Note that for bounded control operator, results like Theorem 4 are quite well known. The case of unbounded control operator, however, needs special attention. It should also be pointed out that the proof is not based on partial differential equation techniques, but on rather linear semigroup properties and the properties of adjoint equation.

**Theorem 4**

For all $T > 0$ the problem (P) possesses a solution $u^* \in L^p(0, T)$. Moreover, there exists an adjoint state $\phi \in C(0, +\infty; X)$ such that :
\[
\begin{align*}
\dot{\phi}(t) &= -A^*\phi(t) - u^*(t)B^*\phi(t) - q\|y^*(t)\|^{q-2}y^*(t), \quad t \in (0, T) \\
\Phi(T) &= qa[y(T) - y_d]\|y^*(T) - y_d\|^{q-2} \\
u^*(t) &= \left(\frac{1}{pc}\right)^{\frac{1}{p-1}} \text{sign} \left(-\langle y^*(t), B^*\phi(t)\rangle\right) \|\langle y^*(t), B^*\phi(t)\rangle\|^{\frac{1}{p-1}}
\end{align*}
\]
where $y^*$ is the solution of $[I]$ corresponding $u = u^*$.

Indeed, we start by proving the existence of a solution for the problem $(P)$ with unconstrained endpoint (i.e. $R = X$). For this end we first prove the following result.

**Lemma 5**

Let $T > 0$. Then for any $y_0 \in X$, the problem $(P)$ with $R = X$ has a solution $u^* \in L^p(0, T)$.

**Proof of the lemma.**

Let $J^* = \inf_{u \in U_{ad}} J(u)$, and let us consider a minimizing sequence $(u_n)$ such that $J(u_n) \to J^*$. Then the sequence $(u_n)$ is bounded and admits a sub-sequence which weakly converges to $u^* \in L^p(0, T)$. Let $y_n$ be the mild solution of $[I]$ corresponding to $u_n$. Then from the definition of $J(u_n)$, we deduce that the sequence $(y_n)$ is bounded, so $(y_n)$ admits a sub-sequence, also denoted by $(y_n)$, which weakly converges to $y^* \in L^q(0, T; X)$.

From the continuity of the mapping $L^p(0, T) \times L^q(0, T; X) \to L^q(0, T; X); (u, y) \mapsto \Phi(t, y)$, we can see that $S^Y(.)y_0 + \Phi(u_n, y_n)$ weakly converges to $S^Y(.)y_0 + \Phi(u^*, y^*) \in L^q(0, T; X)$.

By the variation of constants formula (3), we conclude, for all $t \in [0, T]$, that

$$y^*(t) = S^Y(t)y_0 + \Phi(t, u^*, y^*)$$

Then $y^*$ is the mild solution of the system $[I]$ corresponding to $u^*$. Moreover, we have

$$y_n(t) - y^*(t) = \Phi(t, y_n(t), y^*)$$

Then, using the continuity of the mapping $L^p(0, T) \times L^q(0, T; X) \to X; (u, y) \mapsto \Phi(t, y)$, we deduce that $\Phi(t, y_n(t), y^*)$ weakly converges to $\Phi(t, u^*, y^*) \in X$. So for all $t \in [0, T]$, we can conclude that $y_n(t)$ weakly converges to $y^*(t)$.

Now, using the lower semi-continuity of the norm we obtain

$$f(u^*) = a\|y^*(T) - y_d\|^q + b \int_0^T \|y^*(t)\|^q dt + c \int_0^T |y^*(t)|^p dt$$

$$\leq a \liminf_{n \to +\infty} \|y_n(T) - y_d\|^q + b \liminf_{n \to +\infty} \int_0^T \|y_n(t)\|^q dt + c \liminf_{n \to +\infty} \int_0^T |y_n(t)|^p dt$$

$$\leq \liminf_{n \to +\infty} J(u_n)$$

$$\leq J^*.$$  

This gives $f(u^*) = J^*$, and hence $u^*$ is a solution of the problem $(P)$.

Before giving the optimality conditions for the problem $(P)$, we state and prove the differentiability of the function $u \mapsto y_u(t)$, where $y_u$ is the solution of the system $[I]$ corresponding to $u$. This is the aim of the following lemma.

**Lemma 6**

The mapping $\Lambda : u \mapsto y_u(t)$ is Fréchet differentiable and its derivative at $u \in L^p(0, T)$ is given, for all $h \in L^p(0, T)$, by $\Lambda'(u)h = z_h$, where $z_h$ is the mild solution of the following system

$$\begin{cases}
  z_h(t) = A^Yz_h(t) + u(t) Bz_h(t) + h(t)By_u(t) \\
  z_h(0) = 0
\end{cases}$$

(5)
Proof
Let \( u, h \in L^p(0, T) \) and let \( y_u, y_{u+h} \) be the solutions of the system (1) corresponding respectively to \( u \) and \( u + h \). Letting \( z = y_{u+h} - y_u - z_h \), we can see that \( z \) is the mild solution of the following system

\[
\begin{aligned}
\dot{z}(t) &= A^\top z(t) + u(t)Bz(t) + h(t)Bw(t) \\
z(0) &= 0
\end{aligned}
\]

where \( w = y_{u+h} - y_u \).

By the variation of constants formula, we have, for all \( t \in [0, T] \)

\[
z(t) = \int_0^t S^\top (t-s)u(s)Bz(s)ds + \int_0^t S^\top (t-s)h(s)Bw(s)ds.
\]

Then, by the \((p, q)\)-admissibility of the operator \( B \), we obtain

\[
\|z(t)\| \leq \|\Phi_t\| \|u\|_{L^p(0, T)} \|z\|_{L^q(0, T; X)} + \|\Phi_t\| \|h\|_{L^p(0, T)} \|w\|_{L^q(0, T; X)}
\]

Using the convexity of the function \( x \mapsto x^q \) and the Gronwall lemma, we conclude that for some \( M_1 = M_1(u, y_0, T) > 0 \), we have

\[
\|z(t)\|^q \leq 2h^{-1}(\|\Phi_T\| \|h\|_{L^p(0, T)} \|w\|_{C(0, T; X)})^q \exp(M_1).
\]  \( \text{(6)} \)

Observing that \( w \) is the mild solution of the following system

\[
\begin{aligned}
\dot{w}(t) &= A^\top w(t) + (u(t) + h(t))Bw(t) + h(t)By_u(t), \quad t \in [0, T] \\
w(0) &= 0
\end{aligned}
\]

Then we can write

\[
w(t) = \int_0^t S^\top (t-s)(u(s) + h(s))Bw(s)ds + \int_0^t S^\top (t-s)h(s)By_u(s)ds,
\]

and hence

\[
\|w(t)\| \leq \|\Phi_t\| \|u + h\|_{L^p(0, T)} \|w\|_{L^q(0, T; X)} + \|\Phi_t\| \|h\|_{L^p(0, T)} \|y_u\|_{L^q(0, T; X)}
\]

Similarly to (5), we conclude that for some \( M_2 = M_2(u, y_0, T) > 0 \) and for \( \|h\|_{L^p(0, T)} \leq 1 \), we have

\[
\|w(t)\|^q \leq 2h^{-1}(\|\Phi_T\| \|h\|_{L^p(0, T)} \|y_u\|_{L^q(0, T; X)})^q \exp(M_2).
\]  \( \text{(7)} \)

Then from (6) and (7), we conclude that for some \( M_3 = M_3(u, y_0, T) > 0 \) we have

\[
\|z(t)\|^q \leq M_3 \|h\|_{L^p(0, T)}^2, \quad \forall t \in [0, T].
\]

We conclude that the mapping \( \Lambda : u \mapsto y_u(t) \) is Frechet differentiable at any \( u \in L^p(0, T) \), and that for all \( h \in L^p(0, T) \), the derivative \( z_h := \Lambda'(u)h \) is given by (5), and the lemma is proved.

We can finally prove the Theorem 4

Proof of Theorem 4
Since the mappings \( u \mapsto y(t) \), \( y \mapsto \|y\|_{L^p(0, T; X)}^q \) and \( u \mapsto \|u\|_{L^p(0, T)}^p \) are Frechet differentiable, the cost function \( J(u) \) is Frechet differentiable as well, and we have

\[
D_u J(h) = (f'(u), h)_{L^p(0, T) \times L^p(0, T)}
\]

\[
= qa \|y(T) - y_u\|^{q-2}(y(T) - y_u, z_h(T)) + qb \int_0^T \|y(t)\|^{q-2}(y(t), z_h(t))dt
\]

\[
+ pc \int_0^T \|u(t)\|^{p-2}(u(t), h(t))dt.
\]  \( \text{(8)} \)
Let $\phi$ be the mild solution of the following adjoint system
\[
\begin{aligned}
\phi(t) &= -A^*\phi(t) - u^*(t)B^*\phi(t) - qb\|y^*(t)\|^q-2y^*(t), \quad t \in [0, T] \\
\phi(T) &= qa\|y(T) - y_d\|^q-2(y^*(T) - y_d)
\end{aligned}
\] (9)
whose existence and uniqueness is guaranteed by Lemma via the following change of variables
\[
\begin{aligned}
q(t) &= \phi(T - t) \\
g(t) &= qb\|y^*(T - t)\|^q-2y^*(T - t) \\
v(t) &= u^*(T - t) \\
q(0) &= \phi(T) = qa\|y(T) - y_d\|^q-2(y^*(T) - y_d)
\end{aligned}
\] (10)
Let $A_n = nAR(n, A^Y) = nA(nI - A^Y)^{-1}$ be the Yosida approximation of the operator $A$ and let $B_n = nR(n, A^Y)B$. Let $u \in L^2(0, T)$ and let $y_{m,n}$ and $\phi_{m,n}$ be the respective solutions to (1) and (9) with $A_m$ instead of $A$ and $B_n$ instead of $B$. The introduction of the adjoint equation allows us to clarify the derivative of the state according to the control $u$. Then observing that for all $y \in X$ we have $B_n y \in X$, we deduce that
\[
\int_0^T \langle qb\|y_{m,n}(t)\|^q-2y_{m,n}(t), z_{m,n}(t) \rangle dt = \int_0^T (\langle -\phi_{m,n}(t) - A_m^*\phi_{m,n}(t) - u(t)B_n^*\phi_{m,n}(t), z_{m,n}(t) \rangle dt
\]
\[
= -\int_0^T \langle \phi_{m,n}(t), z_{m,n}(t) \rangle + \langle \phi_{m,n}(t), A_mz_{m,n}(t) + u(t)B_nz_{m,n}(t) \rangle dt
\]
\[
= -\int_0^T \langle \phi_{m,n}(t), z_{m,n}(t) \rangle + \langle \phi_{m,n}(t), z_{m,n}(t) - h(t)B_ny_{m,n}(t) \rangle dt
\]
\[
= -\int_0^T \langle \phi_{m,n}(t), z_{m,n}(t) \rangle + \langle \phi_{m,n}(t), z_{m,n}(t) \rangle dt
\]
\[
+ \int_0^T \langle \phi_{m,n}(t), h(t)B_ny_{m,n}(t) \rangle dt
\]
\[
= -\langle \phi_{m,n}(T), z_{m,n}(T) - \phi_{m,n}(0), z_{m,n}(0) \rangle
\]
\[
+ \int_0^T \langle \phi_{m,n}(t), h(t)B_ny_{m,n}(t) \rangle dt.
\]
Moreover, since $\phi_{m,n}(T) = qa\|y_{m,n}(T) - y_d\|^q-2(y_{m,n}(T) - y_d)$ and $z_{m,n}(0) = 0$, we conclude that
\[
\int_0^T \langle qb\|y_{m,n}(t)\|^q-2y_{m,n}(t), z_{m,n}(t) \rangle dt = -qa\|y_{m,n}(T) - y_d\|^q-2(y_{m,n}(T) - y_d, z_{m,n}(T))
\]
\[
+ \int_0^T \langle \phi_{m,n}(t), h(t)B_ny_{m,n}(t) \rangle dt
\]
\[
= -qa\|y_{m,n}(T) - y_d\|^q-2(y(T) - y_d, z_{m,n}(T))
\]
\[
+ \int_0^T ((B_ny_{m,n}(t))^*\phi_{m,n}(t), h(t)) dt.
\]
where $(B_ny_{m,n}(t))^*$ is the adjoint operator of $(B_ny_{m,n}(t)) : R \rightarrow X$.
Using the boundedness of the operator $B_n$ and the fact that for all $x \in D(A)$ we have $A_mx \rightarrow Ax$, we then get
\[
\lim_{m \rightarrow +\infty} \phi_{m,n} = \phi_n \quad \text{and} \quad \lim_{m \rightarrow +\infty} y_{m,n} = y_n
\]
where $y_n$ and $\varphi_n$ are respectively the solutions to (1) and (2) with $B_n$ instead of $B$.

Now, letting $m \to +\infty$

\[
\int_0^T \langle q b \| y_n(t) \|^q - 2 y_n(t), z_n(t) \rangle dt = - q a \| y_n(T) - y_d \|^q - 2 (y_n(T) - y_d, z_n(T)) \]

\[
+ \int_0^T \langle (B_n y_n(t))^* \varphi_n(t), h(t) \rangle dt
\]

\[
= - q a \| y_n(T) - y_d \|^q - 2 (y_n(T) - y_d, z_n(T))
\]

\[
+ \int_0^T \langle \langle \varphi_n(t), B_n y_n(t) \rangle, h(t) \rangle dt
\]

\[
= - q a \| y_n(T) - y_d \|^q - 2 (y_n(T) - y_d, z_n(T))
\]

\[
+ \int_0^T \langle \langle R^*(n, A^Y) \varphi_n(t), By_n(t) \rangle_{Y, Y}, h(t) \rangle dt.
\]

Let us show that for all $t \in [0, T]$, the sequence $y_n$ converges (point by point) to the mild solution $y$ of (1) corresponding to $u$.

By the variation of constants formula, we have for all $t \in [0, T]$

\[
y_n(t) - y(t) = \int_0^t S^Y(t - s) u(s) B_n y_n(s) ds - \int_0^t S^Y(t - s) u(s) By(s) ds
\]

\[
= \int_0^t S^Y(t - s) u(s) n R(n, A^Y) B(y_n(s) - y(s)) ds + \int_0^t S^Y(t - s) u(s) n R(n, A^Y) By(s) ds
\]

\[
- \int_0^t S^Y(t - s) u(s) By(s) ds
\]

\[
= n R(n, A^Y) \int_0^t S^Y(t - s) u(s) B(y_n(s) - y(s)) ds + n R(n, A^Y) \int_0^t S^Y(t - s) u(s) By(s) ds
\]

\[
- \int_0^t S^Y(t - s) u(s) By(s) ds.
\]

Then, we can deduce that

\[
\| y_n(t) - y(t) \| \leq \| n R(n, A^Y) \int_0^t S^Y(t - s) u(s) B(y_n(s) - y(s)) ds \| + \| n R(n, A^Y) \Phi_t(u, y) - \Phi_t(u, y) \|
\]

\[
\leq \| \Phi_T \| \| u \|_{L^p(0,T)} \| y_n - y \|_{L^q(0,T;X)} + \| n R(n, A^Y) \Phi_t(u, y) - \Phi_t(u, y) \|. 
\]

(12)

By the convexity of the function $x \mapsto x^q$, we conclude that for $s \in [0, t]$

\[
\| y_n(s) - y(s) \|^q \leq 2^{q-1} \| n R(n, A) \Phi_s(u, y) - \Phi_s(u, y) \|^q + 2^{q-1} \| \Phi_T \|^q \| u \|^q_{L^p(0,T)} \int_0^t \| y_n(r) - y(r) \|^q_{L^q(0,T;X)} dr.
\]

(13)

By integration of the last inequality over $[0, t]$, we obtain

\[
\int_0^t \| y_n(s) - y(s) \|^q ds \leq 2^{q-1} \int_0^t \| n R(n, A^Y) \Phi_s(u, y) - \Phi_s(u, y) \|^q ds
\]

\[
+ 2^{q-1} \| \Phi_T \|^q \| u \|^q_{L^p(0,T)} t \int_0^t \| y_n(s) - y(s) \|^q_{L^q(0,T;X)} ds.
\]

Then, for $0 \leq t < \alpha := \frac{1}{2^{q-1} \| \Phi_T \|^q \| u \|^q_{L^p(0,T)} t}$ we have

\[
\int_0^t \| y_n(s) - y(s) \|^q ds \leq \frac{2^{q-1}}{(1 - \frac{1}{2})} \int_0^t \| n R(n, A) \Phi_s(u, y) - \Phi_s(u, y) \|^q ds.
\]

8
We conclude that

Thus, we conclude from (14) that for all $n$

Combining (8) and (15), we obtain

Integrating the last inequality over $[0, t]$, we conclude for that

So by the dominated convergence theorem, we conclude for that

Then from [13], we deduce that for all $t \in [0, a)$

Let us show that $y_n(t) \to y(t)$ for all $t \in [\tau, \tau + \alpha)$, where $\tau \in [0, T)$. Similarly to the case where $t \in [0, a)$, we have from [12]

Integrating the last inequality over $[0, t]$, we obtain

Then, for $t \in [\tau, \tau + \alpha)$ we have

and consequently

Thus, we conclude from (14) that for all $t \in [\tau, \tau + \alpha)$

We conclude that

Now, letting $n \to +\infty$ in (11) we deduce that

Combining (8) and (15), we obtain
It follows that the solution \( u^* \) of the unconstrained problem \( (P) \) satisfies the formula

\[
u^*(t) = \text{sign} \left( - \frac{1}{pc}(y^*(t), B^* \phi(t)) \right) \left\| \frac{1}{pc}(y^*(t), B^* \phi(t)) \right\|^{\frac{1}{p-1}} ,
\]

hence, the theorem is proved.

Theorem 4 is a key result to solve the problem of optimal control either with a constrained endpoint or in infinite time horizon.

ii.2 Optimal control with constrained endpoint

The main goal of this subsection is to study the optimal control problem \( (P) \) with the endpoint constraint: \( y(T) = y_d \). More precisely, given some \( T > 0 \), we consider the following problem

\[
(P_c) : \quad \min_{u \in \mathcal{U}_d} J(u) = \int_0^T \|y(t)\|^q dt + \frac{r}{p} \int_0^T |u(t)|^p dt ,
\]

where \( r > 0 \) and \( u, y \) are such that

\[
y(t) = Ay(t) + u(t)By(t), \quad t \in (0,T), \quad y(0) = y_0 \in X, \quad y(T) = y_d
\]

Then, we can take \( u \) in the set of admissible controls \( \mathcal{U}_d = \{ u \in L^p(0,T), \quad y(T) = y_d \} \), where \( y_d \in X \) is the desired state.

In order to solve the problem \( (P_c) \), we consider a decreasing sequence \( \epsilon_n \) converging to 0, and the corresponding solution \( u_n \) of the problem \( (P) \) for \( b = \epsilon_n, c = \frac{r \epsilon_n}{p} \) and \( a = 1 \) (which we will denote in the sequel by \( (P_{c_n}) \)), and let \( y_n \) be the mild solution of the system \( (P) \) corresponding to \( u_n \). In the next theorem, we apply the result of the previous subsection to solve the problem \( (P_c) \).

**Theorem 7** Assume that \( \mathcal{U}_d \neq \emptyset \). Then there exists a solution \( u^* \) of the problem \( (P_c) \). Furthermore, any weak limit value of the solution \( (u_n) \) of \( (P_{c_n}) \) in \( L^p(0,T) \) is a solution of \( (P_c) \).

**Proof:**

Let \( v \in \mathcal{U}_d \). Keeping in mind that \( u_n \) is the solution of the problem \( (P_{c_n}) \) corresponding to \( \epsilon_n \), we can see that

\[
J_{\epsilon_n}(u_n) \leq J_{\epsilon_n}(v) = \epsilon_n J(v) ,
\]

from which it comes

\[
\epsilon_n J(u_n) \leq J_{\epsilon_n}(u_n) \leq \epsilon_n J(v) .
\]

Using the definition of the cost \( J \), the last inequality gives

\[
\frac{r}{p} \int_0^T |u_n(t)|^p dt \leq J(u_n) \leq J(v) .
\]

It follows that the sequence \( (u_n) \) is bounded and admits a sub-sequence which weakly converges to \( u^* \in L^p(0,T) \). Let \( y_n \) be the mild solution of \( (P) \) corresponding to \( u_n \). Then from the definition of \( J(u_n) \) the sequence \( (y_n) \) is bounded, so \( (y_n) \) admits a sub-sequence, also denoted by \( (y_n) \) which weakly converges to \( y^* \in L^q(0,T;X) \).

Now, let \( u^* \) be a weak limit value of \( (u_n) \) in \( L^p(0,T) \). Since \( u_n \to u^* \) in \( L^p(0,T) \), we can show as in the proof of Lemma 6 that \( y^* \) is the mild solution of the system \( (P) \) corresponding to \( u^* \), and
for all \( t \in [0,T] \) we have \( y_n(t) \to y^*(t) \).
Moreover, using the lower semi-continuity of the norm, we conclude that
\[
J(u^*) \leq \liminf_{n \to +\infty} J(u_n).
\] (18)

Combining (17) and (18) we deduce that
\[
J(u^*) \leq J(v).
\]

According to the inequality (16), the sequence \( J(u_n) \) is bounded and we have
\[
\liminf_{n \to +\infty} J(u_n) = \liminf_{n \to +\infty} J(u_n). \]

Hence
\[
\|y^*(T) - y_d\| \leq \liminf_{n \to +\infty} \|y_n(T) - y_d\| \leq \|y_v(T) - y_d\| = 0.
\]
Thus \( u^* \in U_{\text{ad}} \).

**Remark 8** It is worth noting that there are several works that are interested in the question \( U_{\text{ad}} \neq \emptyset \) (see for instance [3, 4, 14, 24]).

### iii. Optimal control problem in infinite time-horizon

We consider the following problem of optimal control problem \((P)\) in infinite time-horizon, which we denote by \((P_\infty)\)
\[
(P_\infty) : \quad \min_{u \in U_{\text{ad}}} J(u) = \min_{u \in U_{\text{ad}}} \int_0^{+\infty} \|y(t)\|^q dt + \frac{r}{p} \int_0^{+\infty} |u(t)|^p dt,
\]
where \( r > 0 \) and \( u, y \) are such that
\[
\dot{y}(t) = Ay(t) + u(t)By(t), \quad t > 0, \quad y(0) = y_0 \in X.
\]

Here the set of admissible controls is given by \( U_{\text{ad}} = \{ u \in L^p(0, +\infty), \quad J(u) < +\infty \} \).

In order to solve the problem \((P_\infty)\), we consider an increasing sequence \((T_n)\) such that \( T_n \to +\infty \), and let \( u_n \) be the solution of the problem \((P)\) in \([0,T_n]\) with \( a = 0, \quad b = 1 \) and \( c = \frac{r}{p} \) (which we denote by \((P_n)\)). Let \( y_n \) be the mild solution of the system \((1)\) corresponding to \( u_n \), and let \( J_n \) be the indicated cost function in \([0,T_n]\).

Let us define the following sequence of controls that extend \( u_n \) in \( \mathbb{R}^+ \):
\[
v_n(t) = \begin{cases} 
  u_n(t), & \text{if } t \leq T_n \\
  0, & \text{if } t > T_n
\end{cases}
\] (19)

The next result provides a solution of the problem \((P_\infty)\).

**Theorem 9** Assume that \( U_{\text{ad}} \) is not empty. Then the problem \((P_\infty)\) possesses a solution \( u^* \). Furthermore, any weak limit value of \((v_n)\) (defined by (19)) in \( L^p(0, +\infty) \) is a solution of the problem \((P_\infty)\).

**Proof:**
Let us set \( R(u) = \frac{r}{p} \int_0^{+\infty} |u(t)|^p dt, \forall u \in L^p(0, +\infty) \), and let \( v \in U_{\text{ad}} \) be fixed. Since \( u_n \in L^p(0, T_n) \),
it follows that \( v_n \in L^p(0, +\infty) \).
Since \( u_n \) is a solution of the problem \((P)\) in \([0, T_n]\), it comes that
\[
R(v_n) = \frac{r}{p} \int_0^{T_n} |u_n|^p dt \leq J_n(u_n) \leq J_n(v) \leq f(v).
\]

It follows that \((v_n)\) is bounded. We deduce that the sequence \((v_n)\) admits a subsequence, still denoted by \((v_n)\), that weakly converges to \( u^* \in L^p(0, +\infty) \).
Since \( v_n \rightharpoonup u^* \) in \( L^p(0, +\infty) \), we can show as in the proof of Lemma [5] that \( u^* \) is the mild solution of the system \([1]\) corresponding to \( u^* \) and for all \( t \geq 0 \) we have \( y_{v_n}(t) \rightharpoonup y^*(t) \).
The continuity of the mapping \( R \) implies the lower semi-continuity w.r.t to the weak topology (see Corollary III.8 in [13]). We deduce that
\[
v_n \rightharpoonup u^* \quad \text{weakly} \Rightarrow R(u^*) \leq \liminf_{n \to +\infty} (R(v_n)). \tag{20}
\]

The norm \(|.|\) being lower semi-continuous, it follows that
\[
\int_0^{+\infty} \|y^*(t)\|^q dt \leq \liminf_{n \to +\infty} \int_0^{+\infty} \|y_{v_n}(t)1_{T_n}\|^q dt. \tag{21}
\]

Observing that
\[
J_{T_n}(u_n) = \int_0^{+\infty} \|y_{v_n} 1_{T_n}(t)\|^q dt + \frac{r}{p} \int_0^{+\infty} |v_n(t)|^p dt,
\]
we derive, via (20) and (21), that
\[
J(u^*) \leq \liminf_{n \to +\infty} (J_{T_n}(u_n)). \tag{22}
\]

Let us show that the sequence \((J_{T_n}(u_n))\) converges to \( J(u^*) \). For this end, we will show that the sequence \((J_{T_n}(u_n))\) is increasing and upper bounded by \( J(u^*) \).
We have
\[
J_{T_n}(u_n) \leq J_{T_n}(u_{n+1}) \leq J_{T_n+1}(u_{n+1})
\]
and
\[
J_{T_n}(u_n) \leq J_{T_n}(u^*) \leq J(u^*),
\]
and hence
\[
\lim_{n \to +\infty} J_{T_n}(u_n) \leq J(u^*). \tag{23}
\]

Combining (22) and (23), we conclude that
\[
\lim_{n \to +\infty} J_{T_n}(u_n) = J(u^*).
\]

Having in mind that \( u_n \) is the solution of the problem \((P_\epsilon)\) on \([0, T_n]\), we conclude that
\[
J_n(u_n) - J(v) = \int_0^{T_n} \left( \frac{r}{p} |u_n(t)|^p + \|y_n(t)\|q \right) dt - \int_0^{+\infty} \left( \frac{r}{p} |v(t)|^p + \|y_v(t)\|q \right) dt
\]
\[
= J_n(u_n) - J_n(v) - \int_{T_n}^{+\infty} \left( \frac{r}{p} |v(t)|^p + \|y_v(t)\|q \right) dt
\]
\[
\leq 0
\]
Thus, letting \( n \to +\infty \), we get
\[
J(u^*) - J(v) = \lim_{n \to +\infty} J_n(u_n) - J(v) \leq 0.
\]
This shows that \( u^* \) is a solution of the problem \((P_\infty)\).
Remark 10

- We refer the readers to [6, 8, 23] for some sufficient conditions that ensure that \( U_{ad} \neq \emptyset \).
- In [12], the authors assume that \( A \) is exponentially stable, which implies that \( u = 0 \in U_{ad} \).

III. Applications

In this section, we will study the optimal control problem for two examples of PDEs. Let us first show that there is always an extension \( Y \) of \( X \) for which the existence of the extension \( A^Y \) and \( S^Y(t) \) is guaranteed. Let \( X_{-1} \) be the extrapolation space of \( X \) defined as the completion of \( X \) w.r.t norm ||\( R(\lambda;A)x || \), for all \( x \in X \). We can easily show that the operator \( A \) is bounded from \( (D(A), ||.||_X) \) into the Banach space \( (X_{-1}, ||.||_{-1}) \), hence it can be extended to an operator \( A_{-1} \in \mathcal{L}(X, X_{-1}) \) which generates a \( C_0 \) semi-group on \( X_{-1} \) given by \( S_{-1}(t) = (\lambda - A_{-1})^{-1} \).

Indeed, we can easily verify that \( S_{-1}(t) \) is a \( C_0 \)-semi-group on \( X_{-1} \). Moreover, to show that its generator is \( A_{-1} \), we show that for all \( x \in X \)

\[
\lim_{t \to 0^+} \frac{S(t)x - x}{t} = A_{-1}x.
\]

Indeed, the case \( x \in D(A) \) follows from the fact that

\[
\lim_{t \to 0^+} \frac{S(t)x - x}{t} = Ax.
\]

And for \( x \in X \), we use the fact that \( R(\lambda, A)x \in D(A) \), to deduce that

\[
R(\lambda, A_{-1}) \lim_{t \to 0^+} \frac{S(t)x - x}{t} = R(\lambda, A_{-1})A_{-1}x.
\]

We also refer to [17] for more details about this space.

i. Transport equation

Let us consider the following system

\[
\begin{align*}
y_t(x,t) &= y_x(x,t) + u(t)By(x,t), \quad (x,t) \in (0,1) \times (0,T] \\
y(1,t) &= 0, \quad t \in [0,T] \\
y(0,0) &= y_0 \in L^2(0,1),
\end{align*}
\]

(24)

Here, we have \( X = L^2(0,1) \) and the operator \( A = \frac{\partial}{\partial x} \) with domain \( D(A) = W_0^{1,1}(0,1) = \{ y \in W^{1,1}(0,1)/y(1) = 0 \} \) generates the semigroup \( S(t) \) defined for all \( y \in X \) by

\[
S(t)y(x) = \begin{cases} 
y(t + x) & \text{if } t + x < 1 \\
0 & \text{if } t + x \geq 1.
\end{cases}
\]

Moreover, \( B \) is defined by \( By = \xi(y)b \) for all \( y \in X \), where \( \xi : X \to \mathbb{R} \) is a non-zero linear functional of \( X \) i.e. there exists \( h \in X \) such that \( \xi(y) = \langle y, h \rangle \) and \( b \in X_{-1}/X \) (for example \( b = A_{-1}1_{[0,1]} \)), we can observe that \( B^*y = \langle b, y \rangle_{X_{-1}/X_{-1}} h \). Such systems serve as useful models for the practical description of various real problems such as air pollution or traffic flow (see [2, 19, 28]). It is clear
that $B$ is a bounded operator from $X$ to $X_{-1}$ and it is $(p,q)$-admissible for $p,q \geq 2$ (see [1]).

Now, let us consider the following quadratic cost function $J$

$$J(u) = \int_0^T \|y(t)\|^2 dt + \frac{T}{2} \int_0^T |u(t)|^2 dt.$$  

We will study optimal control in both finite time horizon.

According to Theorem 4, the solution of the problem (P) is given by

$$u^*(t) = -\frac{1}{p} \langle y^*(t), B^*\phi(t) \rangle$$

where $\phi$ is the mild solution of the adjoint system given by

$$\left\{ \begin{array}{ll}
\phi(t) = -A^*\phi(t) - u^*(t)B^*\phi(t) - 2y^*(t), & t \in [0,T] \\
\phi(T) = 0 &
\end{array} \right.$$

ii. Fractional diffusion equation

In this example, we study the optimal control of the following fractional equation of diffusion type

$$\left\{ \begin{array}{ll}
y_t(x,t) = \Delta y(x,t) + u(t)(-\Delta)\alpha y(x,t), & \text{in } Q = \Omega \times (0, +\infty) \\
y = 0, & \text{on } \Sigma = \partial\Omega \times (0, +\infty) \\
y(., 0) = y_0 & \text{in } \Omega
\end{array} \right. \quad (25)$$

where $\Omega$ is an open bounded domain in $\mathbb{R}^N$, $0 < \alpha < 1$ and $u \in L^p(0, +\infty)$ is a control function. Such systems serve as useful models for the description of transport processes in complex systems, slower than the Brownian diffusion, etc... see [20, 22]. This system takes the form of system [1] by setting $X = L^2(\Omega)$, $A = \Delta$ with $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $B = (-\Delta)^\alpha$.

Let $(\alpha_n)$ be the increasing sequence of the eigenvalues of $-A$ and let $(\Phi_n)$ be the corresponding orthonormal basis in $X$. Then, the operators $B$ and $S(t)$ can be expressed as $S(t)y = \sum_{n=1}^{+\infty} \exp(-\alpha_n t) \langle y, \Phi_n \rangle \Phi_n$ and $B y = \sum_{n=1}^{+\infty} \alpha_n^\alpha \langle y, \Phi_n \rangle \Phi_n$. It is clear that $B$ is an unbounded operator from $X$ to $X$, but is a bounded operator from $X$ to $X_{-1}$.

**Lemma 11** Let $0 < \alpha < 1$. Then for any $y_0 \in X$ and $u \in L^p_{loc}(0, \infty)$, the system (25) admits a unique mild solution.

**Proof**

Let $(u, y) \in L^p(0, +\infty) \times L^q(0, +\infty; X)$, then for every $t > 0$, we have

$$\phi_t(u, y) = \int_0^t u(s) \sum_{n=1}^{+\infty} \alpha_n^\alpha \exp(-\alpha_n(t-s)) \langle y(s), \Phi_n \rangle \Phi_n ds.$$  

Moreover, we have, for $0 < s < t$

$$\| \sum_{n=1}^{+\infty} \alpha_n^\alpha \exp(-\alpha_n(t-s)) \langle y(s), \Phi_n \rangle \Phi_n \|^2 = \frac{1}{(2(t-s))^{2\alpha}} \sum_{n=1}^{+\infty} (2\alpha_n(t-s))^{2\alpha} \exp(-2\alpha_n(t-s)) \langle y(s), \Phi_n \rangle^2.$$  

Let us take $M > 0$ such that for all $s \geq 0$, $s^{2\alpha} \exp(-s) < M$. Then we find

$$\| \sum_{n=1}^{+\infty} \alpha_n^\alpha \exp(-\alpha_n(t-s)) \langle y(s), \Phi_n \rangle \Phi_n \|^2 \leq \frac{M}{(2(t-s))^{2\alpha}} \|y(s)\|^2.$$
We will discuss the two cases:

\[ \varphi \]

where

\[ \varphi \]

According to Theorem 4, the problem (25) admits a unique mild solution.

This implies

\[ \|\varphi(t, y)\| \leq \int_{0}^{T} \frac{M^1}{(2(t - s))^r} |u(s)| \|y(s)\| ds. \]

Let \( p, q > 1 \) be such that \( \frac{1}{p} + \frac{1}{q} < 1 - \alpha \) and let \( \frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q} \), so that \( 0 < ar < 1 \). Hence, using the Hölder inequality, we conclude that

\[ \|\varphi(t, y)\| \leq M^2 \left( \int_{0}^{T} \frac{1}{(2(t - s))^{ar}} ds \right)^{\frac{1}{r}} \|u\|_{L^p(0, T)} \|y\|_{L^q(0, e)} < +\infty, \]

which means that \( \varphi(t, y) \in X \). We deduce that \( B \) is \( (p, q) \)-admissible and hence the system (25) admits a unique mild solution.

Now, let us consider the following quadratic cost function

\[ J(u) = \int_{0}^{T} |y(t)|^2 dt + \int_{0}^{T} |u(t)|^p dt, \]

where \( u \in L^p(0, T) \) and \( y \) is the corresponding mild solution of the system (25).

We will discuss the two cases: \( T < +\infty \) and \( T = +\infty \).

**Finite time horizon** \( (T < +\infty) \).

**Corollary 12** The solution of the problem (P) is given by the following time-varying feedback law:

\[ u^*(t) = -\left( \frac{4}{p^r} \int_{t}^{T} (y^*(s), B^* y^*(s)) ds \right)^{\frac{1}{p}} \]

**Proof**

According to Theorem 4, the problem (P) admits a solution \( u^* \in U_{ad} = L^p(0, T) \), which is given by (4). We will show that \( u^* \) can be expressed as a time-varying feedback law.

Let us consider the two bounded operators \( A_n = nA(nI - A^Y)^{-1} \) and \( B_n = n(nI - A^Y)^{-1}B \) with \( Y = X_{-1} \). Here, we will reproduce the proof of Theorem 4. Let \( u \in L^p(0, T) \) and let \( y_{m,n} \) and \( \varphi_{m,n} \) be the respective solutions to (1) and (9) with \( A_m \) instead of \( A \) and \( B_n \) instead of \( B \). Then we have

\[ \langle \varphi_{m,n}(t), B_n y_{m,n}(t) \rangle + \langle \varphi_{m,n}(t), B_n y_{m,n}(t) \rangle = \langle -A_m^* \varphi_{m,n}(t) - u(t)B_n^* \varphi_{m,n}(t) - 2y_{m,n}(t), B_n y_{m,n}(t) \rangle \]

\[ = \langle B_n^* \varphi_{m,n}(t), A_m y_{m,n}(t) + u(t)B_n y_{m,n}(t) \rangle \]

\[ = \langle B_n^* \varphi_{m,n}(t), A_m y_{m,n}(t) \rangle - \langle A_m \varphi_{m,n}(t), B_n y_{m,n}(t) \rangle \]

\[ - 2 \langle y_{m,n}(t), B_n y_{m,n}(t) \rangle. \]

Thus

\[ \langle \varphi_{m,n}(t), B_n y_{m,n}(t) \rangle + \langle \varphi_{m,n}(t), B_n y_{m,n}(t) \rangle = -2 \langle y_{m,n}(t), B_n^* y_{m,n}(t) \rangle. \]

Integrating the last equality over \([t, T]\), we get, by letting \( m, n \to +\infty \)

\[ \langle \varphi(t), B y(t) \rangle = \int_{t}^{T} 2 \langle y(s), B^* y(s) \rangle ds \]
We conclude that the solution \( u^* \) of the problem \((P)\) is given by the following time-varying feedback law:

\[
u^*(t) = - \left( \frac{4}{pr} \int_t^T (y^*(s), B^* y^*(s)) ds \right)^{1/r} \]

**Infinite time horizon (i.e. \( T = +\infty \)).**

We consider the optimal control problem \((P_\infty)\) with \( U_{ad} = \{ u \in L^p(0, +\infty), \quad \text{if} (u) < +\infty \}\). Here we have \( v = 0 \in U_{ad} \), as the uncontrolled system (i.e. \( u = 0 \)) is exponentially stable, so according to Theorem 9, there exists a solution \( u^* \) to the problem \((P_\infty)\). Furthermore, any weak limit value of the sequence \((v_n)\) (given by (19) in \( L^p(0, +\infty) \), is a solution of the problem \((P_\infty)\).

In the proposition below we show that the problem \((P_\infty)\) admits a solution that can be expressed as a time-varying feedback, which guarantee the stability of system (25).

**Proposition 13** The solution of the problem \((P_\infty)\) is given by the following time-varying feedback law:

\[
u^*(t) = - \left( \frac{4}{pr} \int_t^{+\infty} (y^*(s), B y^*(s)) ds \right)^{1/r} \]

Moreover, \( u^* \) strongly stabilises the system (25).

**Proof:**

Let \( y_n \) be the solution of the system (1), corresponding to the solution \( u^*_n \) of the problem \((P)\), then we have for \( t > 0 \)

\[
\langle B^* y_n(t), y_n(t) \rangle \leq \|B^* y_n(t)\| \|y_n(t)\| \leq C \|y_n(t)\| \|y_n(t)\| \leq C \|y_n(t)\|^2.
\]

This together with Theorem IV.9 in (13) guarantees the existence of a subsequence, still denoted by \( u_{n^*} \), such that for a.e. \( t \geq 0 \), we have

\[
\langle y_n(t), B^* y_n(t) \rangle \leq g(t).
\]

for some \( g \in L^1(0, +\infty) \). Then by the dominated convergence theorem (recall that \( \langle B^* y^*, y^* \rangle \geq 0 \)), we conclude that

\[
u^*(t) = - \left( \frac{4}{pr} \int_t^{+\infty} (y^*(s), B^* y^*(s)) ds \right)^{1/r},
\]

where \( y^* \) is the mild solution of the system (25) corresponding to \( u^* \).

Observing that the feedback control \( u^* \) leads to a dissipative closed-loop system, we can see that \( t \mapsto \|y^*(t)\|^2 \) is a decreasing function. This together with the fact that \( \int_0^{+\infty} \|y^*(t)\|^2 dt < +\infty \), implies that \( \|y^*(t)\| \to 0 \), as \( t \to +\infty \). In other words, the optimal control \( u^* \) is a stabilizing one for the system (25).

**IV. Conclusion**

In this paper, we studied the optimal control problem for an abstract unbounded bilinear system, where the control operator is \((p,q)\)-admissible. We formulated optimality conditions with a constrained and unconstrained endpoint in the context of finite time-horizon. Moreover, we have extended these results to infinite time-horizon. The established results are applied to transport and fractional diffusion equations.
Declarations

- **Data availability:** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
- **conflict of interest statement:** The authors have no competing interests to declare that are relevant to the content of this article.

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