Localized wave solutions to a variable-coefficient coupled Hirota equation in inhomogeneous optical fiber

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Localized wave solutions to a variable-coefficient coupled Hirota equation in inhomogeneous optical fiber

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Abstract

Higher-order localized waves for a variable-coefficient coupled Hirota equation describes the vector optical pulses in inhomogeneous optical fiber and are investigated via generalized Darboux transformation in this work. Based on its Lax pair and seed solutions, the localized wave solutions are calculated, evolution plots are constructed, and the dynamics of the obtained localized waves are analyzed through numerical simulation. It is observed that the first- and second-order localized waves interact with dark-bright solitons or breathers, and the functions $\alpha(t)$, $\beta(t)$, and $\delta(t)$ determine the propagation shape of the localized waves. The presented results contribute to enriching the dynamics of localized waves in inhomogeneous optical fiber.

Keywords: variable-coefficient coupled Hirota equation; generalized Darboux transformation; soliton; breather

1 Introduction

As the primary tool for transmitting various types of information, optical fiber communication has a wide range of applications and is developing rapidly [1-5].
recent years, an increasing number of researchers have devoted themselves to studying the dynamics of nonlinear evolution equations in the field of optics, including the nonlinear Schrödinger (NLS) equation [6-8], the Kundu–Eckhaus (KE) equation [9,10], the Radhakrishnan–Kundu–Lakshmanan (RKL) equation [11], the complex cubic–quintic Ginzburg–Landau (CCQGL) equation [12] and the Gerdjikov–Ivanov (GI) equation [13]. Scholars have switched their attention from constant-coefficient equations [14,15] to variable-coefficient equations [16-18], which more effectively account for the inhomogeneity of the medium and its nonuniform boundaries. The variable coefficient equations are applied to describe localized waves, which consist of solitons [19,20], breathers [21], and rogue waves [22,23]. Several methods are used to investigate localized waves, including Darboux transformation (DT) [24-26], Bäcklund transformation [27], and bilinear methods [28,29]. The study of localized waves in nonlinear optical fiber using the variable coefficient equations have provided the theoretical basis for modern communication [30,31].

Motivated by the above considerations, a variable-coefficient coupled Hirota (VCCH) equation is studied in this work [32]:

\[ iq_{1t} + \alpha(t)q_{1xx} + 2\beta(t)(|q_1|^2 + |q_2|^2)q_1 + i\delta(t) \left( \frac{\beta(t)}{\alpha(t)}(6|q_1|^2 + 3|q_2|^2)q_{1x} \right) + 3\frac{\beta(t)}{\alpha(t)}q_1q_2^*q_2 + q_{1xxx} + \frac{1}{2} i \left\{ \frac{[\beta(t)]_t}{\beta(t)} - \frac{[\alpha(t)]_t}{\alpha(t)} \right\} q_1 = 0, \]  
\[ iq_{2t} + \alpha(t)q_{2xx} + 2\beta(t)(|q_1|^2 + |q_2|^2)q_2 + i\delta(t) \left( \frac{\beta(t)}{\alpha(t)}(6|q_2|^2 + 3|q_1|^2)q_{2x} \right) + 3\frac{\beta(t)}{\alpha(t)}q_2q_1^*q_1 + q_{2xxx} + \frac{1}{2} i \left\{ \frac{[\beta(t)]_t}{\beta(t)} - \frac{[\alpha(t)]_t}{\alpha(t)} \right\} q_2 = 0, \]  

where \( q_1 \) and \( q_2 \) are the complex envelope in the electric field, \( x \) is the evolution time, \( t \) is propagation distance, and \( * \) denotes a complex conjugate. \( \alpha(t), \beta(t), \delta(t), \frac{\beta(t)}{\alpha(t)}, \frac{[\beta(t)]_t}{\beta(t)} - \frac{[\alpha(t)]_t}{\alpha(t)} \) are the coefficients of the group velocity dispersion (GVD), nonlinear terms referring to self-phase modulation (SPM) and cross-phase modulation (XPM), third-order dispersion (TOD), nonlinear terms related to self-steepening and delayed nonlinear response, and the gain or absorption modulus, respectively.

Previous research on Eq. (1) has been carried out. Two types of \( N \)th-order rogue wave solutions were considered in [33], and optical vector breather solutions were obtained via DT and a symbolic iteration technique in [34]. Additionally, Shi et al. obtained the polynomial wave solutions and the rational wave solutions via a
unified method [35], and Yang et al. constructed a Lax pair and Darboux matrix, obtained one- and two-fold soliton and breather solutions, and presented the one- and two-fold breather-to-soliton conversion conditions [36, 37]. However, few studies have investigated the dynamics of higher-order localized waves in Eq. (1). In the following, the localized wave solutions are obtained via generalized DT, and the dynamical characteristics of higher-order wave solutions are discussed.

The remainder of this paper is organized as follows. In Section 2, the generalized DT is derived, and the higher-order localized wave solutions are obtained. Based on numerical simulation, the evolution plots of the higher-order localized waves are given in Section 3, and their dynamical characteristics are discussed. Finally, Section 4 provides several conclusions.

2 Generalized Darboux Transformation

The following Lax pair of Eq. (1) is considered in this section [37]:

\[
\Phi_x = U\Phi, \quad (2a)
\]

\[
\Phi_t = V\Phi, \quad (2b)
\]

where

\[
U = \lambda \sigma + U_1, \quad V = \lambda^3 V_1 + \lambda^2 V_2 + \lambda V_3 + V_4,
\]

\[
\sigma = \begin{pmatrix}
-2i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & i \\
\end{pmatrix}, \quad U_1 = i \sqrt{\frac{\beta(t)}{\alpha(t)}} \begin{pmatrix}
0 & q_1 & q_2 \\
q_1^* & 0 & 0 \\
q_2^* & 0 & 0 \\
\end{pmatrix}, \quad V_1 = 9\delta(t)\sigma, \quad V_2 = 3\alpha(t)\sigma + 9\delta(t)U_1,
\]

\[
V_3 = 3i \begin{pmatrix}
\frac{\beta(t)\delta(t)}{\alpha(t)} (|q_1|^2 + |q_2|^2) & \sqrt{\frac{\beta(t)}{\alpha(t)}} |q_1\alpha(t) + i\delta(t)q_{1x}| & \sqrt{\frac{\beta(t)}{\alpha(t)}} |q_2\alpha(t) + i\delta(t)q_{2x}| \\
\sqrt{\frac{\beta(t)}{\alpha(t)}} |q_1\alpha(t) - i\delta(t)q_{1x}^*| & \frac{\beta(t)\delta(t)}{\alpha(t)} |q_1|^2 & \frac{\beta(t)\delta(t)}{\alpha(t)} q_1^* q_2 \\
\sqrt{\frac{\beta(t)}{\alpha(t)}} |q_2\alpha(t) - i\delta(t)q_{2x}^*| & \frac{\beta(t)\delta(t)}{\alpha(t)} q_2^* q_1 & \frac{\beta(t)\delta(t)}{\alpha(t)} |q_2|^2 \\
\end{pmatrix},
\]

\[
V_4 = \frac{1}{\alpha(t)} \begin{pmatrix}
\beta(t) [d_5 - \delta(t)(d_{31} + d_{32})] & i \sqrt{\frac{\beta(t)}{\alpha(t)}} [d_{11} - \alpha(t)d_{21}] & i \sqrt{\frac{\beta(t)}{\alpha(t)}} [d_{12} - \alpha(t)d_{22}] \\
i \sqrt{\frac{\beta(t)}{\alpha(t)}} [d_{11} - \alpha(t)d_{21}^*] & \beta(t) [\delta(t)d_{31} - i\alpha(t)|q_1|^2] & \beta(t) [\delta(t)d_{41} - i\alpha(t)q_1^* q_2] \\
i \sqrt{\frac{\beta(t)}{\alpha(t)}} [d_{12} - \alpha(t)d_{22}^*] & \beta(t) [\delta(t)d_{42} - i\alpha(t)q_2^* q_1] & \beta(t) [\delta(t)d_{32} - i\alpha(t)|q_2|^2] \\
\end{pmatrix},
\]
where \( \lambda \) and \( U \) the compatibility condition where
\[
\phi^T_{D\text{T}} \text{ is defined as:}
\]
\[
\begin{bmatrix}
\varphi_1 & \chi_1^* & \phi_1^* \\
\chi_1 & -\varphi_1^* & 0 \\
\phi_1 & 0 & -\varphi_1^*
\end{bmatrix}, \quad \Lambda = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1^* & 0 \\
0 & 0 & \lambda_1^*
\end{bmatrix},
\]
and \( \Phi=(\varphi_1, \chi_1, \phi_1)^T \) is the eigenfunction of Eq. (2) corresponding to the spectral parameter \( \lambda = \lambda_1 \) and the seed solutions \( q_1 = q_1[0] \) and \( q_2 = q_2[0] \). Thus, the classical DT is defined as:

\[
\lambda = \lambda_k, \quad \Phi_k = (\varphi_k, \chi_k, \phi_k)^T, \quad (k = 1, 2, \ldots, N),
\]

\[
\begin{align}
q_1[N] &= q_1[0] - 3 \sqrt{\frac{\alpha(t)}{\beta(t)}} \sum_{k=1}^{N} (\lambda_1 - \lambda_1^*) \frac{\varphi_k[k-1] \chi_k^*[k-1]}{|\varphi_k[k-1]|^2 + |\chi_k[k-1]|^2 + |\phi_k[k-1]|^2}, \\
q_2[N] &= q_2[0] - 3 \sqrt{\frac{\alpha(t)}{\beta(t)}} \sum_{k=1}^{N} (\lambda_1 - \lambda_1^*) \frac{\varphi_k[k-1] \phi_k^*[k-1]}{|\varphi_k[k-1]|^2 + |\chi_k[k-1]|^2 + |\phi_k[k-1]|^2},
\end{align}
\]

where
\[
T[k] = \lambda_{k+1} I - H[k-1] \Lambda[k] H[k-1]^{-1},
\]

\[
\Phi_k[k-1] = (T[k-1] T[k-2] \cdots T[1]) \big|_{\lambda=\lambda_k} \Phi_k,
\]

\[
H[k-1] = \begin{bmatrix}
\varphi_{k[k-1]} & \chi_{k[k-1]}^* & \phi_{k[k-1]}^* \\
\chi_{k[k-1]} & -\varphi_{k[k-1]}^* & 0 \\
\phi_{k[k-1]} & 0 & -\varphi_{k[k-1]}^*
\end{bmatrix}, \quad \Lambda[k] = \begin{bmatrix}
\lambda_k & 0 & 0 \\
0 & \lambda_k^* & 0 \\
0 & 0 & \lambda_k^*
\end{bmatrix}.
\]
The generalized DT of Eq. (1) is constructed based on the above classical DT. Assuming $\Phi_1 = \Phi_1(\lambda, \eta)$ is a solution of Eq. (2) and $\eta$ is a small parameter, the following Taylor expansion of $\eta=0$ is obtained:

$$
\Phi_1 = \Phi_1[0] + \Phi_1[1] \eta + \Phi_1[2] \eta^2 + \cdots + \Phi_1[N] \eta^N + o(\eta^N),
$$

(7)

where

$$
\Phi_1[k] = \frac{1}{k!} \frac{\partial^k}{\partial \lambda^k} \Phi_1(\lambda) \big|_{\lambda=\lambda_1} = \left( \varphi_1[k], \chi_1[k], \phi_1[k] \right)^T, (k = 0, 1, 2, \cdots, N).
$$

It can be easily confirmed that $\Phi_1[0] = \Phi_1[0]$ is a special solution with $\lambda=\lambda_1$, $q_1 = q_1[0]$, and $q_2 = q_2[0]$ of Eq. (2). Therefore, the generalized DT is defined as follows:

$$
\Phi_1[N-1] = \Phi_1[0] + \left[ \sum_{l=1}^{N-1} T_1[l] \right] \Phi_1[1] + \left[ \sum_{l=1}^{N-1} \sum_{h>l} T_1[h] T_1[l] \right] \Phi_1[2] + \cdots + \left[ T_1[N-1] \cdots T_1[2] T_1[1] \right] \Phi_1[N-1],
$$

(8)

$$
q_1[N] = q_1[N-1] - 3 \sqrt{\frac{\alpha(t)}{\beta(t)}} (\lambda_1 - \lambda_1^*) \frac{\varphi_1[N-1] \chi_1[N-1]}{\varphi_1[N-1]^2 + |\chi_1[N-1]|^2 + |\phi_1[N-1]|^2},
$$

(9a)

$$
q_2[N] = q_2[N-1] - 3 \sqrt{\frac{\alpha(t)}{\beta(t)}} (\lambda_1 - \lambda_1^*) \frac{\varphi_1[N-1] \phi_1^*[N-1]}{\varphi_1[N-1]^2 + |\chi_1[N-1]|^2 + |\phi_1[N-1]|^2},
$$

(9b)

where

$$
T_1[k] = \lambda_1 I - H_1[k-1] \Lambda_1 H_1[k-1]^{-1},
$$

$$
\Phi_1[N-1] = (\varphi_1[N-1], \chi_1[N-1], \phi_1[N-1])^T,
$$

$$
H_1[k-1] = \begin{pmatrix} \varphi_1[k-1] & \chi_1[k-1] & \phi_1[k-1] \\ \chi_1[k-1] & -\varphi_1[k-1] & 0 \\ \phi_1[k-1] & 0 & -\varphi_1[k-1] \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}.
$$

3 Localized Wave Solutions

In this section, the first- and second-order localized wave solutions of Eq. (1) are calculated, and the dynamics of localized waves are analyzed on the basis of evolution plots.
Assuming the plane waves $q_1[0] = a_1 \sqrt{\frac{\alpha(t)}{\beta(t)}} e^{i\omega(t)}$ and $q_2[0] = a_2 \sqrt{\frac{\alpha(t)}{\beta(t)}} e^{i\omega(t)}$ are seed solutions of the localized waves, where

$$\omega(t) = \int 2(a_1^2 + a_2^2)\alpha(t)\,dt$$

and $a_1$ and $a_2$ are arbitrary real constants. The corresponding basic vector solution at $\lambda = \left(-\frac{2i}{3} \sqrt{a_1^2 + a_2^2}\right) (1 + \eta^2)$ is calculated as:

$$\Phi_1(\eta) = \begin{pmatrix}
(v_1 e^{\kappa_1 + \kappa_2} - v_2 e^{\kappa_1 - \kappa_2}) e^{i\omega(t)/2} \\
\varsigma_1(v_1 e^{\kappa_1 - \kappa_2} - v_2 e^{\kappa_1 + \kappa_2}) e^{-i\omega(t)/2} + \varpi a_2 e^{\kappa_3} \\
\varsigma_2(v_1 e^{\kappa_1 - \kappa_2} - v_2 e^{\kappa_1 + \kappa_2}) e^{-i\omega(t)/2} - \varpi a_1 e^{\kappa_3}
\end{pmatrix}, \quad (10)$$

where

$$v_1 = \frac{\sqrt{3\lambda - \sqrt{9\lambda^2 + 4(a_1^2 + a_2^2)^2}}}{\sqrt{9\lambda^2 + 4(a_1^2 + a_2^2)}}, \quad v_2 = \frac{\sqrt{3\lambda + \sqrt{9\lambda^2 + 4(a_1^2 + a_2^2)^2}}}{\sqrt{9\lambda^2 + 4(a_1^2 + a_2^2)}},$$

$$\kappa_1 = -\frac{i\lambda}{2} \left[x + 3\lambda(\alpha(t) + 3\lambda\delta(t)) t\right], \quad \kappa_2 = \frac{i}{2} \sqrt{9\lambda^2 + 4(a_1^2 + a_2^2)}(x - \tau t + \Omega(\eta)), \quad \kappa_3 = i\lambda \left[x + 3\lambda(\alpha(t) + 3\lambda\delta(t)) t\right], \quad \tau = -\frac{3}{\lambda^2}[2(a_1^2 + a_2^2) - 9\lambda^2] \delta(t),$$

$$\varsigma_1 = \frac{ia_1}{\sqrt{a_1^2 + a_2^2}}, \quad \varsigma_2 = \frac{ia_2}{\sqrt{a_1^2 + a_2^2}}, \quad \Omega(\eta) = \sum_{j=1}^{N} (m_j + i\eta^2)j \eta^2,$$

and $\varpi$, $m_j$, and $n_j$ are arbitrary real constants.

Let $\gamma = a_1^2 + a_2^2$ and expand function $\Phi_1(\eta)$ as a Taylor series at $\eta = 0$,

$$\Phi_1(\eta) = \Phi_1^{[0]} + \Phi_1^{[1]}\eta^2 + \Phi_1^{[2]}\eta^4 + \Phi_1^{[3]}\eta^6 + \cdots, \quad (11)$$

where

$$\Phi_1(\eta) = \left(\varphi_{1}^{[k]}(0), \chi_{1}^{[k]}(0), \phi_{1}^{[k]}(0)\right)^T = \frac{1}{(2k)!} \frac{\partial^{2k}\Phi_1}{\partial \eta^{2k}} \bigg|_{\eta=0}, \quad (k = 0, 1, 2, \cdots),$$

$$\varphi_{1}^{[0]} = \gamma^{-\frac{3}{4}}(-2 - 2i)((\alpha(t) + 3\sqrt{\delta(t)}) \sqrt{7} - 2x) \sqrt{7} + \frac{1}{4}) e^{i\gamma f(\alpha(t)dt + \frac{\sqrt{2}}{4}((\alpha(t) + 2\sqrt{5\delta(t)})\sqrt{7} - x)},$$

$$\chi_{1}^{[0]} = \gamma^{-\frac{3}{4}}((-2 + 2i)a_1((-\frac{x}{2} + (\alpha(t) + 3\sqrt{\delta(t)}) \sqrt{7} - \frac{1}{4}) e^{-i\gamma f(\alpha(t)dt + \frac{\sqrt{2}}{4}((\alpha(t) + 2\sqrt{5\delta(t)})\sqrt{7} - x)} + 2a_2 \varpi e^{-\frac{2\sqrt{2}}{3}((\alpha(t) + 2\sqrt{5\delta(t)})\sqrt{7})},$$

$$\phi_{1}^{[0]} = \gamma^{-\frac{3}{4}}((-2 + 2i)a_2((-\frac{x}{2} + (\alpha(t) + 3\sqrt{\delta(t)}) \sqrt{7} - \frac{1}{4}) e^{-i\gamma f(\alpha(t)dt + \frac{\sqrt{2}}{4}((\alpha(t) + 2\sqrt{5\delta(t)})\sqrt{7}) - 2a_1 \varpi e^{-\frac{2\sqrt{2}}{3}((\alpha(t) + 2\sqrt{5\delta(t)})\sqrt{7}).$$
As the expression $\Phi_1^{[1]} = \left(\varphi_1^{[1]}, \chi_1^{[1]}, \phi_1^{[1]}\right)^T$ is complicated, its specific form is omitted. The dynamical characteristics of the higher-order localized waves will be discussed subsequently.

Obviously, when $q_1 = q_1[0]$, $q_2 = q_2[0]$, and $\lambda = -\frac{2i}{3}\sqrt{\gamma}$, $\Phi_1^{[0]} = (\varphi_1^{[0]}, \chi_1^{[0]}, \phi_1^{[0]})^T$ is the solution of the Lax pair. According to Eqs. (8) and (9), the first-order localized wave solutions of Eq. (1) are obtained as:

$$q_1[1] = q_1[0] - 3\sqrt{\frac{\alpha(t)}{\beta(t)}}(\lambda_1 - \lambda_1^*) \frac{\varphi_1[0]\chi_1^*[0]}{|\varphi_1[0]|^2 + |\chi_1[0]|^2 + |\phi_1[0]|^2}, \quad (12a)$$

$$q_2[1] = q_2[0] - 3\sqrt{\frac{\alpha(t)}{\beta(t)}}(\lambda_1 - \lambda_1^*) \frac{\varphi_1[0]\phi_1^*[0]}{|\varphi_1[0]|^2 + |\chi_1[0]|^2 + |\phi_1[0]|^2}, \quad (12b)$$

The evolution plots of first-order localized waves are obtained by altering the values of the free parameters. The dynamics of the first-order localized waves are then discussed.

Figure 1 depicts the interactions between first-order rogue waves and dark-bright solitons. When $\alpha(t)$, $\beta(t)$, and $\delta(t)$ are constants, a rogue wave in the component $q_1[1]$ interacts with dark solitons and the velocity of the dark solitons remains constant during propagation, as shown in Figure 1(a). When $\alpha(t)$ and $\beta(t)$ are constants and $\delta(t) = \cos(20t)$, the first-order rogue wave will interact with a periodic dark soliton, as shown in Figure 1(b). When $\alpha(t)$ and $\beta(t)$ are variable coefficients and $\delta(t) = \cos(20t)$, unlike the previous figure, the rogue wave changes into an S-shape, as shown in Figure 1(c). Figure 1(d)-(f) shows that the rogue waves in the component $q_2[1]$ are not easily observed in a background of zero amplitude.
Figure 1. The first-order localized waves with $a_1 = 1, a_2 = 0, \omega = \frac{1}{10}$ and (a)(d) $\alpha(t) = 1, \beta(t) = 2, \delta(t) = \frac{1}{50}$; (b)(e) $\alpha(t) = 1, \beta(t) = 2, \delta(t) = \frac{\cos(t)}{20}$; (c)(f) $\alpha(t) = \frac{t}{20}, \beta(t) = 5t, \delta(t) = \frac{\cos(t)}{20}$.

Figure 2 shows the collision between a first-order rogue wave and a breather. Here, $\alpha(t)$ and $\beta(t)$ are constants, and $\delta(t)$ is a linear function. Figure 2(a) and (d) show a rogue wave and a parabolic breather. When $\alpha(t)$ and $\beta(t)$ are the same as the former and $\delta(t)$ is a trigonometric function, the evolution figures of a first-order rogue wave interacting with a periodic breather is obtained, as shown in Figure 2(b) and (e). When $\alpha(t)$ and $\beta(t)$ are trigonometric functions and $\delta(t) = \frac{t}{100}$, periodic rogue waves are obtained, as shown in Figure 2(c) and (f). Moreover, Figure 2 illustrates that the amplitude of $q_1[1]$ is greater than the amplitude of $q_2[1]$, which are influenced by $a_1$ and $a_2$. 
Figure 2. The first-order localized waves with \( a_1 = \frac{4}{5}, a_2 = 1, \varpi = \frac{1}{100} \) and (a)(d) \( \alpha(t) = \frac{3}{2}, \beta(t) = \frac{3}{2}, \delta(t) = \frac{t}{100}; \) (b)(e) \( \alpha(t) = \frac{3}{2}, \beta(t) = \frac{3}{2}, \delta(t) = \frac{\sin(t)}{30}; \) (c)(f) \( \alpha(t) = \frac{\sin(t)}{3}, \beta(t) = \frac{\sin(t)}{3}, \delta(t) = \frac{t}{100}. \)

Based on the following limit formula

\[
\Phi_1[1] = \lim_{\eta \to 0} \frac{T[1]|_{\lambda=\lambda_1(1+\eta^2)}}{\eta^2} \Phi_{1}\nonumber = \lim_{\eta \to 0} \frac{(\lambda_1 \eta^2 + T_{1}[1]|_{\lambda=\lambda_1})\Phi_{1}}{\eta^2} = \lambda_1 \Phi_{1}[0] + T_{1}[1] \Phi_{1}[1],
\]

and Eqs. (8) and (9), the second-order localized wave solutions can be obtained as:

\[
q_1[2] = q_1[1] - 3 \sqrt{\frac{\alpha(t)}{\beta(t)}} (\lambda_1 - \lambda_1^*) \frac{\varphi_1[0] \chi_1[0]}{|\varphi_1[0]|^2 + |\chi_1[0]|^2 + |\phi_1[0]|^2},
\]

\[
q_2[2] = q_2[1] - 3 \sqrt{\frac{\alpha(t)}{\beta(t)}} (\lambda_1 - \lambda_1^*) \frac{\varphi_1[0] \phi_1[0]}{|\varphi_1[0]|^2 + |\chi_1[0]|^2 + |\phi_1[0]|^2},
\]

where

\[
\Phi_{1}[1] = (\varphi_{1}[1], \chi_{1}[1], \phi_{1}[1])^T, \nonumber
\]

\[
T_{1}[1] = \lambda_1 I - H_{1}[0] \Lambda_1 H_{1}[0]^{-1},
\]

\[
H_{1}[0] = \begin{pmatrix} \varphi_{1}[0] & \chi_{1}[0] & \phi_{1}[0] \\ \chi_{1}[0] & -\varphi_{1}^*[0] & 0 \\ \phi_{1}[0] & 0 & -\varphi_{1}^*[0] \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.
\]
Similarly, the dynamical characteristics of the second-order localized wave solutions are analyzed by altering the values of the free parameters in the following cases. The second-order localized waves have the same dynamics as the first-order localized waves.

Figure 3(a) shows the second-order rogue wave interacting with two parabolic dark-solitons when \( \alpha(t) \) and \( \beta(t) \) are constants, \( \delta(t) = \frac{t}{100}, m_1 = 0, \) and \( n_1 = 0. \) When \( \delta(t) = \frac{t^2}{100}, m_1 = 0, \) and \( n_1 = 0, \) the dark solitons change from parabolic to cubic, as shown in Figure 3(b). Unlike the former, in Figure 3(c), the second-order rogue wave is separated into three first-order rogue waves when \( m_1 = 30 \) and \( n_1 = 30. \) Furthermore, as shown in Figure 3(d)-(f), the rogue wave in the component \( q_2[2] \) is difficult to observe in a background of zero amplitude.

\[ \begin{align*}
\text{(a)} & \quad \text{(b)} & \quad \text{(c)} \\
\text{(d)} & \quad \text{(e)} & \quad \text{(f)}
\end{align*} \]

Figure 3. The second-order localized waves with \( a_1 = 1, a_2 = 0, \) \( \alpha(t) = 1, \) \( \beta(t) = \frac{1}{3}, \) \( \omega = \frac{1}{100}, \) and \( (a)(d) \) \( \delta(t) = \frac{t}{100}, m_1 = 0, n_1 = 0; \) \( (b)(e) \) \( \delta(t) = \frac{t^2}{100}, m_1 = 0, n_1 = 0; \) \( (c)(f) \) \( \delta(t) = \frac{t^2}{100}, m_1 = 30, n_1 = 30. \)

Figure 4 shows the dynamics of the second-order rogue wave and breathers when \( \alpha(t) \) and \( \beta(t) \) are constants and \( \delta(t) \) is a trigonometric function. Figure 4(a) and (d) display the second-order rogue wave interacting with two periodic breathers when
\[ \delta(t) = \frac{\cos(t)}{50}, \quad m_1 = 0, \quad n_1 = 0. \]

Based on the above parameters, the period of the two breathers decreases and their propagation velocity becomes faster when \( \delta(t) = \frac{\cos(3t)}{50} \), as illustrated in Figure 4(b) and (e). In addition, Figure 4(c) and (f) show that separation phenomenon occurs in the second-order rogue waves when changing the values of parameters \( m_1 \) and \( n_1 \).

![Figure 4](image)

Figure 4. The second-order localized waves with \( a_1 = 1, a_2 = 1, \alpha(t) = 1, \beta(t) = 1, \omega = \frac{1}{1000} \) and (a)(d) \( \delta(t) = \frac{\cos(t)}{50}, m_1 = 0, n_1 = 0; \) (b)(e) \( \delta(t) = \frac{\cos(3t)}{50}, m_1 = 0, n_1 = 0; \) (c)(f) \( \delta(t) = \frac{\cos(3t)}{50}, m_1 = 20, n_1 = 20. \)

Figure 5(a) demonstrates that the second-order rogue wave coexists with the two periodic parabolic dark solitons when \( \alpha(t) \) and \( \beta(t) \) are constants and \( \delta(t) = \frac{t + \cos(5t)}{30} \). Furthermore, it is found that \( \delta(t) \) determines the type of dark soliton. The second-order rogue wave and the two dark solitons are periodic when \( \alpha(t), \beta(t), \) and \( \delta(t) \) are trigonometric functions, and the height of the rogue waves’ peak decreases along the positive and negative directions of the \( x \) axis (as shown in Figure 5(b)). As illustrated in Figure 5(c) and (d), it is hard to observe the second-order rogue in the component \( q_2[2]. \)
Figure 5. The second-order localized waves with $a_1 = 1, a_2 = 0, \omega = \frac{1}{200}, m_1 = 0, n_1 = 0$ and (a)(c) $\alpha(t) = 1, \beta(t) = 1, \delta(t) = \frac{t + \cos(5t)}{30};$ (b)(d) $\alpha(t) = \cos(t), \beta(t) = \frac{\cos(t)}{2}, \delta(t) = \frac{\cos(t)}{80}.$

Figure 6(a) and (c) shown that when $\alpha(t)$ and $\beta(t)$ are variable coefficients and $\delta(t)$ is a constant, the second-order rogue wave interacts with the $K$-shape dark-bright solitons when $m_1 = 0$ and $n_1 = 0$. If the values of $m_1$ and $n_1$ are changed, the second-order rogue waves are separated into three first-order rogue waves, as shown in Figure 6(b).
4 Conclusions

This work studied a VCCH equation by constructing generalized DT on the basis of classical DT and Taylor expansion, which was used to obtain the first- and second-order localized wave solutions. Localized wave evolution plots were then given via numerical simulation. It was found that parameters had an important effect on the dynamics of the localized waves. The parameters $a_1$ and $a_2$ played an important role in the type of localized waves. If $a_1 \neq 0$ and $a_2 = 0$, it was seen that the interaction between rogue waves and dark-bright solitons. If $a_1$ and $a_2$ were not equal to 0, the rogue waves interacted with breathers. The parameters $m_j$ and $n_j$ ($j = 1, 2, \ldots, N - 1$) determined the separation of the rogue waves. When the parameters $m_1$ and $n_1$ were not equal to 0, the second-order rogue waves separated into three first-order rogue waves. Moreover, it was observed that the functions $\alpha(t)$, $\beta(t)$, and $\delta(t)$ influenced the propagation shape of localized waves. When $\alpha(t)$,
\( \beta(t) \), and \( \delta(t) \) were constants, common localized waves occurred. When \( \alpha(t) \) and \( \beta(t) \) were constants and \( \delta(t) \) was a linear, quadratic, or trigonometric function, the rogue waves interacted with the parabolic, cubic, or periodic dark-bright solitons and breathers. When \( \alpha(t), \beta(t), \) and \( \delta(t) \) were trigonometric functions, the localized waves had periodicity all along the propagation direction. These results contribute to the understanding of localized wave propagation in inhomogeneous optical fibers.

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**Conflict of interest**

The authors declare that they have no conflicts of interest to report regarding the present study.

**References**


