Cantor Diagonal Method and the Continuum Hypothesis

İBRAHİM KUNT (ibrahimknttt@gmail.com)
ISTANBUL UNIVERSITY

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CANTOR DIAGONAL METHOD AND THE CONTINUUM HYPOTHESIS

ABSTRACT

Using the relationship between the ordinal number \( \omega \) and the Cantor Diagonal Method, I study the cardinality of the digits of real numbers in a defined range. Using the results I have reached, I show the value of \( |\omega + 1| \) in the ZF\( -\)C system, and I present some conclusions regarding the Continuum Hypothesis through these results.

Author: İBRAHİM KUNT

Mail: ibrahimknttt@gmail.com
Introduction

Cauchy Sequences is used when defining and constructing the $\mathbb{R}$.

**Definition 1:** "$\mathbb{R}$ is the set of equivalence classes of Cauchy sequences, and the equivalence class containing $(S_n)$ is denoted by $[S_n]$ or $[S_1, S_2, \ldots, S_n, \ldots]$. For $q \in \mathbb{Q}$, $[q, q, \ldots, q, \ldots]$ will also be denoted by $\hat{q} \in \mathbb{R}$" (Stewart, Tall, 2015, p.204).

The Axiom of Choice is not used when constructing $\mathbb{R}$ using Definition 1.

1. The Relation of the Digits of a Real Number that Exists in the Interval $(0,1)$ with $\omega$

**Definition 2:** $\forall n \left( (n \in \mathbb{N}) \land (x = 0.x_0x_1x_2x_3x_4 \ldots x^n \ldots) \land (0 < x < 1) \right)$

I am using the binary notation method. According to this method, all digits appearing in the number $x$ are 0 or 1. In this number sequence, the subscripts in the symbols of the numbers show that the real number is in the list and show the order of the real number in the list. The superscript $n$ is an index and not power of numbers in the definition.

A real number $y$ that is not in the list in Figure 1 but is in the range $(0,1)$ can be proved using the Cantor diagonal method.

**Definition 3:** $\forall n \left( (n \in \mathbb{N}) \land (y = 0.y_0y_1y_2y_3y_4y_5 \ldots y^n \ldots) \land (0 < y < 1) \right)$

And the equality below is true.

$$y^n = \begin{cases} 1 & \text{if } x_n^n = 0 \\ 0 & \text{if } x_n^n = 1. \end{cases}$$

The number $y$ defined in this way is a real number that exists in the range $(0,1)$ and does not exist in the list in Figure 1.
Figure 1.

\[
\begin{align*}
X_0 &= 0. X_0^0 \ X_1^1 \ X_2^2 \ X_3^3 \ X_4^4 \ \ldots \\ X_n^n \ \ldots \\
X_1 &= 0. X_0^1 \ X_1^2 \ X_2^3 \ X_3^4 \ X_4^5 \ \ldots \\ X_1^1 \ \ldots \\
X_2 &= 0. X_2^0 \ X_2^1 \ X_3^2 \ X_4^3 \ X_5^4 \ \ldots \\ X_2^1 \ \ldots \\
X_3 &= 0. X_3^0 \ X_3^1 \ X_3^2 \ X_3^3 \ X_3^4 \ \ldots \\ X_3^n \ \ldots \\
X_4 &= 0. X_4^0 \ X_4^1 \ X_4^2 \ X_4^3 \ X_4^4 \ \ldots \\ X_4^n \ \ldots \\
X_5 &= 0. X_5^0 \ X_5^1 \ X_5^2 \ X_5^3 \ X_5^4 \ \ldots \\ X_5^n \ \ldots \\
\ldots \\
X_n &= 0. X_n^0 \ X_n^1 \ X_n^2 \ X_n^3 \ X_n^4 \ \ldots \\ X_n^n \ \ldots \\
\ldots \\
\ldots \\
\end{align*}
\]

**Figure 1:** Using the Cantor diagonal method, the existence of a real number \( y \) that is not in this list can be proved.

Let us now examine the consequences of the above proof. I name this proof **Proof 1C**. And I accept the proposition which states that the cardinality of \( \mathbb{N} \) is less than the cardinality of \( \mathbb{R} \) as true.

Then I write the subscript of the number \( y \) that we wrote as a result of **Proof 1C** as \( \omega \) and add it to the bottom of the numbers in the list in Figure 1 and make a new list. And to do this, I use Cantor's definition of \( \omega \). According to Cantor's definition of \( \omega \), the number \( \omega \) is greater than any natural number \( n \).

As a result, the subindex of the real number \( y \) will be greater than the subscripts of all the real numbers \( x_0, x_1, x_2, x_3, \ldots, x_n \) in Figure 1.

\((X, <)\) is a well-ordered set \( X_\omega = \{x_n : n \in \mathbb{N}\} \approx \omega \). \( X_{\omega+1} = X_\omega \cup \{x_\omega\} \approx \omega + 1 \).

Using these propositions, I define the number \( y \) as follows:

**Definition 4:** \( \forall n \left( (n \in \mathbb{N}) \land (y = x_\omega = 0. x_\omega^0 x_\omega^1 x_\omega^2 x_\omega^3 x_\omega^4 \ldots x_\omega^n \ldots) \right) \land (0 < y < 1) \)

Based on this definition, this list would be as shown in Figure 2.
Figure 2: The new list made by defining \( y = x^\omega \).

\[
\begin{align*}
X_0 &= 0. X_0^0 \ X_1^1 \ X_2^2 \ X_3^3 \ X_4^4 \ldots \\
X_1 &= 0. X_1^0 \ X_1^1 \ X_2^2 \ X_3^3 \ X_4^4 \ldots \\
X_2 &= 0. X_2^0 \ X_2^1 \ X_3^2 \ X_4^3 \ X_5^4 \ldots \\
X_3 &= 0. X_3^0 \ X_3^1 \ X_4^2 \ X_5^3 \ X_6^4 \ldots \\
X_4 &= 0. X_4^0 \ X_4^1 \ X_5^2 \ X_6^3 \ X_7^4 \ldots \\
X_5 &= 0. X_5^0 \ X_5^1 \ X_6^2 \ X_7^3 \ X_8^4 \ldots \\
& \vdots
\end{align*}
\]

\[
X_n = 0. X_n^0 \ X_n^1 \ X_n^2 \ X_n^3 \ X_n^4 \ldots \\
& \vdots
\]

\[
y = X_\omega = 0. X_\omega^0 \ X_\omega^1 \ X_\omega^2 \ X_\omega^3 \ X_\omega^4 \ldots \\
& \vdots
\]

I argue that a number \( z \) that exists in the range \((0,1)\) but is not in the list in Figure 2 cannot be proven to exist using Cantor's diagonal method.

Figure 3: According to the definition of real numbers, the number \( x^\omega \) cannot exist in the expansion of the real number \( x_\omega \).
**Theorem 1:** Using Cantor's diagonal method, the existence of a real number \( z \) that exists in the interval \((0,1)\) but is not in the list in Figure 2 cannot be proved.

**Proof 1:** To prove that such a \( z \) number does not exist, I will use the ordinal number \( \omega \) defined by Cantor. Suppose we construct the list as in Figure 3 using Cantor's definition of the number \( \omega \).

In order for us to prove that such a real number \( z \) exists by using Cantor's diagonal method, the following propositions must be true:

\[
(z_0 \neq x_0^0), (z_1 \neq x_1^1), (z_2 \neq x_2^2), \ldots, (z_n \neq x_n^n), \ldots, (z_\omega^\omega \neq x_\omega^\omega)
\]

However, according to the definition of real numbers in Definition 1: The number \( x_\omega^\omega \) shown in Figure 3 cannot exist in the expansion of the real number \( x_\omega \) that exists in the interval \((0,1)\). Because \( \omega \), the superscript of \( x_\omega^\omega \), is greater than all natural numbers in \( \mathbb{N} \) and is \( \omega \notin \mathbb{N} \). For the same reason, the number \( z_\omega^\omega \) cannot exist in the expansion of the real number \( z \).

According to Definition 1, \( \mathbb{R} \) is the set of equivalence classes of Cauchy sequences. Cauchy sequences are sequences of rational numbers. The cardinality of the terms of each \( (S_n)_{n \in \mathbb{N}} \) rational number sequence and the cardinality of the terms of each Cauchy sequence is equal to the cardinality of \( \mathbb{N} \). So their cardinality is \( \aleph_0 \).

For these reasons: \( x_\omega^\omega \) cannot exist in the expansion of the real number \( x_\omega \). And the proposition \( z_\omega^\omega \neq x_\omega^\omega \) cannot be true.

In conclusion: We cannot prove that the real number \( z \) exists by using Cantor's diagonal method. \(\square\)

If we put the number \( y \) in Definition 4 at the top of the list, we could prove the existence of the real number \( z \) that is not in the list in Figure 2 using the diagonal method. However, using the definition of the number \( \omega \), we have shown that there is one to one correspondence between the number \( y \) in Definition 4 and \( \omega \), which is greater than all natural numbers (Figure 3).

**2. The Limit of Diagonal Method**

Let's try to use a different method where we can compare the figures in the expansion of the \( z \) number using the diagonal method with the figures in the expansion of the other real numbers in the list in Figure 3.

Since there is no \( x_\omega^\omega \) in the expansion of the real number \( x_\omega \), this new method should ensure that the real number \( z \) is different from the real numbers \( x_0, x_1, x_2, x_3, \ldots, x_n \) and the real number \( x_\omega \). Using the Cantor diagonal method, we can write the real number \( z \) as different from the real numbers \( x_0, x_1, x_2, x_3, \ldots, x_n \) (Proof 1c). For the real number \( z \) to be different from the real number \( x_\omega \), suppose that a \( z^n (n \in \mathbb{N}) \) figure in the expansion of the real number \( z \) is different from a \( x_\omega^n (n \in \mathbb{N}) \) figure in the expansion of the real number \( x_\omega \).

We assumed that all real numbers in the range \((0,1)\) can also be considered as a sequence of 0 and 1 representing the number in the binary expansion. The real number \( z \) that we are trying to write will also consist of 0 and 1 sequences. Therefore, in order for the real number \( z \) to be different from the real numbers
\(x_0, x_1, x_2, x_3, x_4, \ldots, x_n, \ldots\) and the real number \(x_\omega\), we have to define the real number \(z\) as follows:

**Definition 5:** \(\forall n \ [(n \in \mathbb{N}) \land (z = 0.z^0z^1z^2z^3 \ldots z^n \ldots) \land (0 < z < 1)] \rightarrow (z \neq x_0 \neq x_1 \neq x_2 \neq \ldots \neq x_n \neq \ldots \neq x_\omega)\)

And the equalities below are true.

**Theorem 2:** A real number \(z\) as defined in Definition 5 doesn't exist.

**Proof 2:** I will prove the theorem using the Proof by Contradiction. I assume the following hypothesis is true.

**Hypothesis:** A real number \(z\) as defined in definition 5 exists. And the proposition \([((n \in \mathbb{N}) \land (z \neq x_0 \neq x_1 \neq x_2 \neq \ldots \neq x_n \neq \ldots \neq x_\omega))]\) is true.

Due to the definition of \(z\) and the hypothesis above, the following propositions are true:

The proposition \([z \in (0,1)] \land [(n \in \mathbb{N}) \land [(r \in \mathbb{N})]]\) is true.

\(\varphi(n) \overset{\text{def}}{=} [[(x_n^0 = 0) \leftrightarrow (z^n = 1)] \land [(x_n^1 = 1) \leftrightarrow (z^n = 0)]] \rightarrow [(z^n \neq x_n^0) \land (z \neq x_n)]\)

- \(n = 0\)
  
  If \(x_0^0 = 0\) then \(z^0 = 1\), if \(x_0^1 = 1\) then \(z^0 = 0\). Hence \(z^0 \neq x_0^0\) and \(z \neq x_0\).
  
  \(\varphi(0) = [[(x_0^0 = 0) \leftrightarrow (z^0 = 1)] \land [(x_0^1 = 1) \leftrightarrow (z^0 = 0)]] \rightarrow [(z^0 \neq x_0^0) \land (z \neq x_0)]\)

Proposition \(\varphi(0)\) is true for \(n = 0\).

- \(n = r \ (r \in \mathbb{N})\)
  
  If \(x_r^r = 0\) then \(z^r = 1\), if \(x_r^r = 1\) then \(z^r = 0\). Hence \(z^r \neq x_r^r\) and \(z \neq x_r\).
  
  \(\varphi(r) = [[(x_r^r = 0) \leftrightarrow (z^r = 1)] \land [(x_r^r = 1) \leftrightarrow (z^r = 0)]] \rightarrow [(z^r \neq x_r^r) \land (z \neq x_r)]\)

Proposition \(\varphi(r)\) is true for \(n = r\).

- \(n = r + 1 \ [(r + 1) \in \mathbb{N}]\)
  
  If \(x_{r+1}^{r+1} = 0\) then \(z^{r+1} = 1\), if \(x_{r+1}^{r+1} = 1\) then \(z^{r+1} = 0\). Hence \(z^{r+1} \neq x_{r+1}^{r+1}\) and \(z \neq x_{r+1}\).
\[ \varphi(r + 1) = [(x_{r+1}^r = 0) \leftrightarrow (z^{r+1} = 1)] \land [(x_{r+1}^r = 1) \leftrightarrow (z^{r+1} = 0)] \rightarrow [(z^{r+1} \neq x_{r+1}^{r+1}) \land (z \neq x_{r+1})] \]

Proposition \( \varphi(r + 1) \) is true for \( n = r + 1 \).

Proposition \((\varphi(0) \land \forall n( \varphi(n) \rightarrow \varphi(S_n))) \rightarrow \forall n \varphi(n) \) is true for \( n \in \mathbb{N} \).

As a result, it is proven that for all indices \( n \), there are \( z^n \neq x_n^n \) and \( z \neq x_n \). (1)

The proposition \([(z \in (0,1)] \land [(n \in \mathbb{N})] \land [(r \in \mathbb{N})]] \) is true.

\[ \varphi(n) \overset{\text{def}}{=} \left[ (z^n = 0) \leftrightarrow (z^n = 1) \right] \land \left[ (x^n_\omega = 1) \leftrightarrow (z^n = 0) \right] \rightarrow \left[ (z^n \neq x^n_\omega) \land (z \neq x_\omega) \right] \]

- \( n = 0 \)
  - If \( x_0^n = 0 \) then \( z^0 = 1 \), if \( x_0^n = 1 \) then \( z^0 = 0 \). Hence \( z^0 \neq x_0^n \) and \( z \neq x_\omega \).
  
  \( \varphi(0) = \left[ (z_\omega^0 = 0) \leftrightarrow (z_\omega^0 = 1) \right] \land \left[ (x_\omega^0 = 1) \leftrightarrow (z_\omega^0 = 0) \right] \rightarrow \left[ (z_\omega \neq x_\omega^0) \land (z \neq x_\omega) \right] \)

Proposition \( \varphi(0) \) is true for \( n = 0 \).

- \( n = r (r \in \mathbb{N}) \)
  - If \( x_r^n = 0 \) then \( z^r = 1 \), if \( x_r^n = 1 \) then \( z^r = 0 \). Hence \( z^r \neq x_r^n \) and \( z \neq x_\omega \).
  
  \( \varphi(r) = \left[ (z_\omega^r = 0) \leftrightarrow (z_\omega^r = 1) \right] \land \left[ (x_\omega^r = 1) \leftrightarrow (z_\omega^r = 0) \right] \rightarrow \left[ (z_\omega \neq x_\omega^r) \land (z \neq x_\omega) \right] \)

Proposition \( \varphi(r) \) is true for \( n = r \).

- \( n = r + 1 \ (r + 1 \in \mathbb{N}) \)
  - If \( x_{r+1}^{r+1} = 0 \) then \( z^{r+1} = 1 \), if \( x_{r+1}^{r+1} = 1 \) then \( z^{r+1} = 0 \). Hence \( z^{r+1} \neq x_{r+1}^{r+1} \) and \( z \neq x_\omega \).
  
  \( \varphi(r + 1) = \left[ (z_\omega^{r+1} = 0) \leftrightarrow (z_\omega^{r+1} = 1) \right] \land \left[ (x_\omega^{r+1} = 1) \leftrightarrow (z_\omega^{r+1} = 0) \right] \rightarrow \left[ (z_\omega \neq x_\omega^{r+1}) \land (z \neq x_\omega) \right] \)

Proposition \( \varphi(r + 1) \) is true for \( n = r + 1 \).

Proposition \((\varphi(0) \land \forall n( \varphi(n) \rightarrow \varphi(S_n))) \rightarrow \forall n \varphi(n) \) is true for \( n \in \mathbb{N} \).

As a result, it is proven that for all indices \( n \), there are \( z^n \neq x^n_\omega \) and \( z \neq x_\omega \). (2)

From Definition 4 and **Proof 1** is: \( x^n_\omega \neq x^n_n \). (3)

From (1), (2), and (3), the result \( z^n \neq x^n_\omega \neq x^n_n \) is obtained. But according to Definition 5, for example, this result is wrong for \( n = 1 \) value:

- If \( z^1 = 1 \) then \( x_1^1 = 0 \).
- If \( z^1 = 1 \) then \( x_1^1 = 0 \). These result in: \( x_1^1 = x_1^1 \). (4)
The results of (3) and (4) are contradictory to each other. The hypothesis caused a contradiction. So our hypothesis is wrong and therefore Theorem 2 is true. □

3. The values of $|\omega|$ and $|\omega + 1|

**Theorem 3:** The proposition $|\omega| = |\omega + 1| = 2^{\aleph_0} = \aleph_0$ is true in ZFC.

**Proof 3:** According to ZFC, the following propositions are true:

- Subsets of $\mathbb{N}$ can be written as strings of numbers 0 and 1. If we define the set $F$ as the set of sequences $0\cdot1$, the following proposition is true:
  
  $$(0,1) \approx \wp(\mathbb{N}) \approx \mathbb{R} \approx F.$$

- Propositions $|\mathbb{R}| = 2^{\aleph_0}$, $|\mathbb{N}| = \aleph_0$ are true. The set of all real numbers is equipollent to the set $2^\omega$ (Suppes,1972, p.190). (II)

- $\omega + 1 \neq \omega$, $1 + \omega = \omega$

- $|\omega + 1| = |\omega| = \aleph_0$, $\aleph_0 = \mathcal{K} (\omega)$ (Suppes,1972, p.156). (III)

- $X_\omega = \{x_n : n \in \mathbb{N}\} \approx \omega$ (IV)

According to the result we obtained in **Proof 2**, the real numbers $x_0, x_1, x_2, x_3, x_4, \ldots, x_n, \ldots$, $x_\omega$ in Figure 3 are correspondence one-to-one with $\omega + 1$. And therefore:

- $\mathbb{N} \cup \{\omega\} = \{n : n \in \mathbb{N} \lor n = \omega\} = Y$

- $S(\omega) = \omega \cup \{\omega\} = Y = \omega + 1$

- $X_{\omega + 1} = X_\omega \cup \{x_\omega\} \approx \omega + 1$ (V)

I explained earlier in the article that each of the real numbers $x_0, x_1, x_2, x_3, x_4, \ldots, x_n, \ldots$, $x_\omega$ can be considered as sequences consisting of the numbers 0 and 1. The proposition $\wp(\mathbb{N}) \approx \mathbb{R} \approx (0,1)$ is true in ZFC. In **Proof 2**, it is proven that every number in the range $(0,1)$ is one-to-one correspondence with $\omega + 1$. (VI)

Therefore, the proposition $|\omega| = |\omega + 1| = 2^{\aleph_0} = \aleph_0$ is proved true due to (I), (II), (III), (IV), (V), (VI) and due to **Proof 2**. □

Now, considering the consequences of the above proof, we can make the following proof:

**Theorem 4:** The proposition $\mathfrak{K}_1 = 2^{\aleph_0}$ is true in ZF–C.

**Proof 4:** The proposition $|\omega + 1| = |\omega|$ cannot be proved true without the Axiom of choice. Paul Cohen has proven that if ZF is consistent, then ZF–C is also consistent. In that case, we can accept the proposition $|\omega + 1| \neq |\omega|$ as true in the ZF–C system. We do not need to use the Axiom of Choice to prove that $Y$ is a set because it has been proven in **Proof 2** that the set $Y$ is an uncountable set.

According to **Proof 2** and (IV), propositions $|\omega + 1| > \omega$ and $|\omega + 1| = 2^{\aleph_0}$ are true if $\neg C$ is true. If $\neg C$ is true, then according to **Proof 2**, there is no cardinal $K$ that makes $\aleph_0 < K < 2^{\aleph_0}$ true. Therefore, if we define $|\omega + 1| = \mathfrak{K}_1$:

If $|\omega| = \aleph_0$ is true, then $\mathfrak{K}_1 = 2^{\aleph_0}$ is true. □
REFERENCE
