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A dynamical system based on projection operator for solving absolute value equations associated with second-order cone

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Abstract: A new equivalent reformulation of the absolute value equations associated with second-order cone (SOCAVEs) is emphasised, from which a dynamical system based on projection operator for solving SOCAVEs is constructed. Under mild conditions, it is proved that the equilibrium points of the dynamical system exist and could be (globally) asymptotically stable. The effectiveness of the proposed method is illustrated by some numerical simulations.

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Keywords. Absolute value equations; Second-order cone; Dynamical system; Asymptotical stability; Equilibrium point.

1 Introduction

The second-order cone (SOC) in $\mathbb{R}^n$ is defined by

$$K^n = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|x_2\| \leq x_1\},$$

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where \( \| \cdot \| \) denotes the Euclidean norm. If \( n = 1 \), let \( \mathcal{K}^n \) represent the set of nonnegative reals. Moreover, a general SOC \( \mathcal{K} \subset \mathbb{R}^n \) could be the Cartesian product of some SOCs \([7, 8, 13]\), i.e.,

\[
\mathcal{K} = \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_r}
\]

with \( n_1, \cdots, n_r, r \geq 1 \) and \( n_1 + \cdots + n_r = n \). Without loss of generality, we focus on the case that \( r = 1 \) because all the analysis can be carried over to the setting of \( r > 1 \) according to the property of Cartesian product. For any \( x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_1-1} \) and \( y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n_2-1} \), their Jordan product is defined as

\[
x \circ y = (\langle x, y \rangle, y_1 x_2 + x_1 y_2) \in \mathbb{R} \times \mathbb{R}^{n_1-1},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product in \( \mathbb{R}^n \). With this definition, the absolute value vector \( |x| \) in SOC \( \mathcal{K}^n \) is computed by

\[
|x| = \sqrt{x \circ x}.
\]

In this paper, we consider the problem of solving the absolute value equations associated with SOC (SOCAVEs) of the form

\[
Ax - |x| - b = 0
\]

with \( A \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \). Unless otherwise stated, throughout this paper, \( |x| \) is defined as in (1.1). SOCAVEs (1.2) is a special case of the generalized absolute value equations associated with SOC (SOCGAVEs)

\[
Cx + D|x| - c = 0
\]

with \( C, D \in \mathbb{R}^{m \times n} \) and \( c \in \mathbb{R}^m \). To our knowledge, SOCGAVEs (1.3) was formally introduced by Hu, Huang and Zhang \([18]\) and further studied in \([20, 34, 36, 38]\) and the references therein. In addition, SOCAVEs (1.2) is a natural extension of the standard absolute value equations (AVEs)

\[
Ax - |x| = b,
\]

meanwhile, SOCGAVEs (1.3) is an extension of the generalized absolute value equations (GAVEs)

\[
Cx + D|x| = c.
\]

In AVEs (1.4) and GAVEs (1.5), the vector \( |x| \) denotes the componentwise absolute value of the vector \( x \in \mathbb{R}^n \). It is known that GAVEs (1.5) was first introduced by Rohn in \([40]\) and further investigated in \([16, 27, 39]\) and the references therein. Obviously, AVEs (1.4) is a special case of GAVEs (1.5).

Over the past two decades, AVEs (1.4) and GAVEs (1.5) have been widely studied because of their relevance to many mathematical programming problems, such as the linear complementarity problem (LCP), the bimatrix game and others; see e.g. \([27, 30, 39]\). Hence, abundant theoretical results and numerical algorithms for both AVEs (1.4) and GAVEs (1.5) have been established. On the theoretical aspect, for instance, Mangasarian \([27]\) shown that solving GAVEs (1.5) is NP-hard; if GAVEs (1.5) is solvable, checking whether it has a unique solution or multiple solutions is NP-complete \([39]\). Moreover, various sufficient or necessary conditions on solvability and non-solvability of AVEs (1.4) and GAVEs (1.5) were discussed in \([17, 30, 33, 39, 42]\). The latest trend is to investigate the error bound and the condition number of AVEs (1.4) [45]. On the numerical aspect, there are many algorithms for solving AVEs (1.4).
and GAVEs (1.5). For example, the Newton-type methods [3,9,29], the SOR-like method [23], the concave minimization methods [28,46], the exact and inexact Douglas-Rachford splitting methods [5] and others; see e.g. [1,6,31,32,43,44] and the references therein.

We are interested in SOCAVEs (1.2) and SOCGAVEs (1.3) not only because they are extensions of the standard ones, but also because they are equivalent with some LCPs associated with SO (SOCLCPs), which have various applications in engineering, control and finance [18,34,35]. Recently, some numerical methods and theoretical results have been developed for SOCAVEs (1.2) and SOCGAVEs (1.3). For the numerical side, Hu, Huang and Zhang [18] proposed a generalized Newton method for solving SOCGAVEs (1.3) (Here and in the sequel, we assume \( m = n \)). Then, Huang and Ma [20] presented some weaker convergent conditions of the generalized Newton method. Miao et al. [34] proposed a smoothing Newton method for SOCGAVEs (1.3) and a unified way to construct smoothing functions is explored in [38]. Huang and Li [21] proposed a modified SOR-like method for SOCAVEs (1.2). Miao et al. [37] suggested a Levenberg-Marquardt method with Armijo line search for SOCAVEs (1.2). For the theoretical side, Miao et al. [36] studied the existence and nonexistence of solution to SOCAVEs (1.2) and SOCGAVEs (1.3). In addition, the unique solvability for SOCAVEs (1.2) and SOCGAVEs (1.3) was also investigated in [36]. Miao and Chen [35] investigated conditions under which the unique solution of SOCAVEs (1.2) is guaranteed, which are different from those in [36]. Hu, Huang and Zhang proved that SOCGAVEs (1.3) is equivalent to the following problem: find \( x, y \in \mathbb{R}^n \) such that

\[
M x + P y = p,
\]

\[
x, y \in \mathcal{K}^n, \quad \langle x, y \rangle = 0, \tag{1.6}
\]

where \( M, P \in \mathbb{R}^{n \times n} \) and \( p \in \mathbb{R}^n \). However, the problem (1.6) is not a standard SOCLCP, which is in the form of

\[
z \in \mathcal{K}^\ell, \quad w = N z + q \in \mathcal{K}^\ell, \quad \langle z, w \rangle = 0, \tag{1.7}
\]

where \( N \in \mathbb{R}^{\ell \times \ell} \) and \( q \in \mathbb{R}^\ell \). Miao et al. [34] showed that SOCGAVEs (1.3) is equivalent to SOCLCP (1.7) with

\[
N = \begin{bmatrix}
-I & 2I & 0 \\
A & B - A & 0 \\
-A & A - B & 0
\end{bmatrix},
\quad z = \begin{bmatrix}
2x_+ \\
|x| \\
0
\end{bmatrix}
\quad\text{and}\quad
q = \begin{bmatrix}
0 \\
-b \\
b
\end{bmatrix},
\]

where \( x_+ \) is the projection of \( x \) onto the SOC \( \mathcal{K}^n \). Note that the dimension of the above matrix \( N \) is three times the dimension of the matrix \( A \) (or \( B \)) (i.e., \( \ell = 3n \)). More recently, Miao and Chen [35], under the condition that 1 is not an eigenvalue of \( A \) or \( N \), provided the equivalence between SOCAVEs (1.2) and SOCLCP (1.7) without changing the dimension (i.e., \( \ell = n \)).

The goals of this paper are twofold: to highlight another equivalent reformulation of SOCAVEs (1.2) and to present a dynamical system to solve SOCAVEs (1.2). Contrast to the numerical methods mentioned above, our method is from a continuous perspective. Our work here is inspired by recent studies on AVEs (1.4) [6].

The rest of this paper is organized as follows. In section 2, a few relevant basic results on the SOC and the autonomous system are introduced. An equivalent reformulation of SOCAVEs (1.2) and a dynamical system to solve it are developed in section 3. Numerical simulations are given in section 4. Conclusions are made in section 5.

Notation. The set of all \( n \times n \) real matrices is denoted by \( \mathbb{R}^{n \times n} \) and \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \). We use \( I \) to denote the identity matrix with suitable dimension. The transposition of a matrix or a
vector is denoted by $x^\top$. The inner product of two vectors in $\mathbb{R}^n$ is defined as $\langle x, y \rangle \doteq \sum_{i=1}^{n} x_i y_i$ and $\|x\| \doteq \sqrt{\langle x, x \rangle}$. The spectral norm of $A$ is denoted by $\|A\|$ and is defined by the formula $\|A\| \doteq \max \{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$. We use $\text{tridiag}(a, b, c)$ to denote a tridiagonal matrix, which has $a, b, c$ as the subdiagonal, main diagonal and superdiagonal entries, respectively.

2 Preliminaries

In this section, we collect some results which lay the foundation of our later analysis.

We first recall some basic concepts and background materials regarding SOC's, which can be found in [2, 7, 8, 12, 13].

For $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the spectral decomposition of $x$ with respect to SOC is given by

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)},$$

(2.1)

where

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|,$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2} \left(1, (-1)^i \frac{x_2}{\|x_2\|}\right), & \text{if } x_2 \neq 0, \\ \frac{1}{2} \left(1, (-1)^i w\right), & \text{if } x_2 = 0 \end{cases}$$

for $i = 1, 2$ and $w$ is any vector in $\mathbb{R}^{n-1}$ with $\|w\| = 1$. If $x_2 \neq 0$, the spectral decomposition is unique. We call $\lambda_1(x)$ and $\lambda_2(x)$ the eigenvalues of $x$ and $\{u_x^{(1)}, u_x^{(2)}\}$ is called a Jordan frame of $x$. It is known that $\lambda_1(x)$ and $\lambda_2(x)$ are nonnegative if and only if $x \in \mathbb{K}^n$. For any real-valued function $f : \mathbb{R} \to \mathbb{R}$, we define a function on $\mathbb{R}^n$ associated with $\mathbb{K}^n$ by

$$f(x) \doteq f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}$$

if $x \in \mathbb{R}^n$ has the spectral decomposition (2.1). Then we have

$$|x| = \begin{cases} \frac{1}{2} \left(|x_1| + \|x_2\| + |x_1 + \|x_2\||, (|x_1| + \|x_2\|) \frac{x_2}{\|x_2\|}\right), & \text{if } x_2 \neq 0, \\ (0, 0), & \text{if } x_2 = 0. \end{cases}$$

(2.2)

The projection mapping from $\mathbb{R}^n$ onto $\Omega \subset \mathbb{R}^n$, denoted by $P_\Omega$, is defined as

$$P_\Omega(x) \doteq \arg\min\{\|x - y\| : y \in \Omega\}.$$

Given $u \in \mathbb{R}^n$ and a nonempty closed convex subset $\Omega$ of $\mathbb{R}^n$, $w$ is the projection of $u$ onto $\Omega$, i.e., $w = P_\Omega(u)$ if and only if (see e.g. [4, Theorem 1.2.4])

$$\langle u - w, v - w \rangle \leq 0, \quad \forall v \in \Omega.$$  

(2.3)

As mentioned earlier, let $x_+$ be the projection of $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ onto $\mathbb{K}^n$, then we have

$$x_+ = \begin{cases} x, & \text{if } x \in \mathbb{K}^n, \\ 0, & \text{if } x \in -\mathbb{K}^n, \\ u, & \text{otherwise,} \end{cases} \quad \text{where} \quad u = \begin{bmatrix} \frac{x_1 + \|x_2\|}{2} & x_2 \|x_2\| \end{bmatrix}.$$  

(2.4)

The dual cone of $\mathbb{K}^n$ is defined as

$$(\mathbb{K}^n)^* \doteq \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall x \in \mathbb{K}^n\}.$$
It is known that SOC $\mathcal{K}^n$ is a pointed close convex cone and it is self-dual (i.e., $(\mathcal{K}^n)^* = \mathcal{K}^n$).

Now we turn to the autonomous system. Consider the autonomous system
\[
\frac{dx}{dt} = g(x), \tag{2.5}
\]
where $g$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Throughout this paper, we use $x(t; x(t_0))$ to denote the solution of (2.5) determined by the initial value condition $x(t_0) = x_0$. The following results are well-known and can be found in [25, Chapter 2 and Chapter 3].

**Definition 2.1.** The function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz continuous with Lipschitz constant $L > 0$ if
\[
\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n.
\]

**Lemma 2.1.** Assume that $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous in $\mathbb{R}^n$, then for arbitrary $t_0 \geq 0$ and $x(t_0) = x_0 \in \mathbb{R}^n$, the dynamical system (2.5) has a unique solution $x(t; x(t_0)), t \in [t_0, +\infty)$.

**Definition 2.2.** (Equilibrium point). A vector $x^* \in \mathbb{R}^n$ is called an equilibrium point of the dynamical system (2.5) if $g(x^*) = 0$.

**Definition 2.3.** The equilibrium point $x^*$ of (2.5) is stable if, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that
\[
\|x(t) - x^*\| < \delta \quad \Rightarrow \quad \|x(t; x(t_0)) - x^*\| < \epsilon, \forall t \geq t_0.
\]
Furthermore, the equilibrium point $x^*$ of (2.5) is asymptotically stable if it is stable and $\delta$ can be chosen such that
\[
\|x(t_0) - x^*\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t; x(t_0)) = x^*.
\]

**Theorem 2.1.** Let $x^*$ be an equilibrium point of (2.5) and $\Omega \subseteq \mathbb{R}^n$ be a domain containing $x^*$. If there is a continuously differentiable function $V : \Omega \to \mathbb{R}$ such that
\[
V(x^*) = 0 \quad \text{and} \quad V(x) > 0, \quad \forall x \in \Omega \setminus \{x^*\},
\]
\[
\frac{dV(x)}{dt} = \nabla V(x)^\top g(x) \leq 0, \quad \forall x \in \Omega,
\]
then $x^*$ is stable. Moreover, if
\[
\frac{dV(x)}{dt} < 0, \quad \forall x \in \Omega \setminus \{x^*\},
\]
then $x^*$ is asymptotically stable.

**Theorem 2.2.** Let $x^*$ be an equilibrium point for (2.5). If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that
\[
V(x^*) = 0 \quad \text{and} \quad V(x) > 0, \quad \forall x \neq x^*,
\]
\[
\frac{dV(x)}{dt} < 0, \quad \forall x \neq x^*,
\]
\[
\|x - x^*\| \to \infty \quad \Rightarrow \quad V(x) \to \infty,
\]
then $x^*$ is globally asymptotically stable.
3 The equivalent reformulation and the dynamical system

In this section, we first highlight that SOCAVEs (1.2) is equivalent to the generalized SOCLCP (SOCGLCP) as follows:

\[ Q(x) = Ax + x - b \in K^n, \quad F(x) = Ax - x - b \in K^n, \quad \langle Q(x), F(x) \rangle = 0. \] (3.1)

Then, a novel dynamical model is presented to solve SOCAVEs (1.2).

In order to claim the equivalence between SOCAVEs (1.2) and SOCGLCP (3.1), we introduce the following two lemmas.

Lemma 3.1. (\cite{11, 27}) Let \( a, b \in \mathbb{R} \). Then \( a \geq 0, b \geq 0 \) and \( ab = 0 \) if and only if \( a + b = |a - b| \).

Lemma 3.2. Let \( s, t \in \mathbb{R}^n \). Then

\[ s \in K^n, \quad t \in K^n \quad \text{and} \quad \langle s, t \rangle = 0 \] (3.2)

if and only if

\[ s + t = |s - t|. \] (3.3)

Proof. We first prove that (3.2) \( \Rightarrow \) (3.3). Since \( s = (s_1, s_2) \in K^n \) and \( t = (t_1, t_2) \in K^n \), we have

\[ s_1 \geq ||s_2|| \quad \text{and} \quad t_1 \geq ||t_2||, \]

which implies that

\[ |\langle s_2, t_2 \rangle| \leq ||s_2|| ||t_2|| \leq s_1 t_1. \]

Thus,

\[ \langle s, t \rangle = s_1 t_1 + s_2^T t_2 \geq s_1 t_1 - s_1 t_1 = 0, \]

and the equality is valid if and only if \( s_2 = kt_2 \) \( (k \geq 0) \), \( s_1 = ||s_2|| \) and \( t_1 = ||t_2|| \). Hence, the vectors \( s \) and \( t \) in (3.2) share the same Jordan frame \([2, 35]\). Let \( s = \lambda_1 e_1 + \lambda_2 e_2 \) and \( t = \mu_1 e_1 + \mu_2 e_2 \), where \( \{e_1, e_2\} \) is the Jordan frame. Then we have \( \lambda_i, \mu_i \geq 0 \) for \( i = 1, 2 \) and \( \lambda_1 \mu_1 = \lambda_2 \mu_2 = 0 \). It then follows from Lemma 3.1 that \( \lambda_1 + \mu_1 = |\lambda_1 - \mu_1| \) and \( \lambda_2 + \mu_2 = |\lambda_2 - \mu_2| \). On the other hand, we have \( s + t = (\lambda_1 + \mu_1) e_1 + (\lambda_2 + \mu_2) e_2 \) and \( |s - t| = |\lambda_1 - \mu_1| e_1 + |\lambda_2 - \mu_2| e_2 \). Hence, we have (3.3).

Next, we prove that (3.3) \( \Rightarrow \) (3.2). By (3.3), we know that \( s + t \) and \( s - t \) have the same Jordan frame, from which we obtain that \( s \) and \( t \) have the same Jordan frame. Indeed, it follows from the fact that

\[ 2s = (s + t) + (s - t) \quad \text{and} \quad 2t = (s + t) - (s - t). \]

Let \( s = \lambda_1 e_1 + \lambda_2 e_2 \) and \( t = \mu_1 e_1 + \mu_2 e_2 \). Then it follows from (3.3) that \( \lambda_1 + \mu_1 = |\lambda_1 - \mu_1| \) and \( \lambda_2 + \mu_2 = |\lambda_2 - \mu_2| \), which combine with Lemma 3.1 implies that \( \lambda_i, \mu_i \geq 0 \) for \( i = 1, 2 \) and \( \lambda_1 \mu_1 = \lambda_2 \mu_2 = 0 \). Then we can obtain (3.2). \( \square \)

Remark 3.1. Lemma 3.2 can be found in \cite{13, Proposition 4.1} and \cite{10, Proposition 2.3}. Here we give a new proof based on the Jordan frame, which is different from that of \cite{13, Proposition 4.1}.
According to Lemma 3.2, if we set \( s+t = Ax-b \) and \( s-t = x \), we can obtain the equivalence between SOCAVEs (1.2) and SOCGLCP (3.1). We should point out that the equivalence between SOCAVEs (1.2) and SOCGLCP (3.1) is implicit in the proof of [35, Theorem 4.1] and the proof of Lemma 3.2 is also inspired by that of [35, Theorem 4.1]. Moreover, SOCGLCP (3.1) is equivalent to the generalized linear variational inequality problem associated with SOC (SOCGLVI) [11]:

find an \( x^* \in \mathbb{R}^n \), such that \( Q(x^*) \in K^n \), \( \langle v - Q(x^*), F(x^*) \rangle \geq 0, \forall v \in K^n \),

which is also equivalent to

\[
Q(x) = P_{K^n} [Q(x) - F(x)].
\]

We should point out that the equivalence between SOCAVEs (1.2) and SOCGLCP (3.1) is implicit in the proof of [35, Theorem 4.1] and the proof of Lemma 3.2 is also inspired by that of [35, Theorem 4.1]. Moreover, SOCGLCP (3.1) is equivalent to the generalized linear variational inequality problem associated with SOC (SOCGLVI) [11]:

\[
\text{find an } x^* \in \mathbb{R}^n, \text{ such that } Q(x^*) \in K^n, \langle v - Q(x^*), F(x^*) \rangle \geq 0, \forall v \in K^n,
\]

(3.4)

Indeed, we have the following theorem.

**Theorem 3.1.** The vector \( x^* \) solves SOCAVEs (1.2) if and only if \( r(x^*) = 0 \), where

\[
r(x) = Q(x) - P_{K^n} [Q(x) - F(x)].
\]

(3.6)

Furthermore, it can be proved that

\[
r(x) = Ax - |x| - b.
\]

(3.7)

**Proof.** It is enough to prove (3.7). Note that \( Q(x) - F(x) = 2x \). We will split the proof into three cases.

(a) If \( x \in K^n \), then \( 2x \in K^n \). It follows from \( |x| = x \) and \( P_{K^n}(2x) = 2x \) that

\[
Q(x) - P_{K^n} [Q(x) - F(x)] = Ax + x - b - 2x = Ax - x - b = Ax - |x| - b.
\]

(b) If \( x \in -K^n \), \( 2x \in -K^n \). Then it follows from \( |x| = -x \) and \( P_{K^n}(2x) = 0 \) that

\[
Q(x) - P_{K^n} [Q(x) - F(x)] = Ax + x - b - 0 = Ax - (-x) - b = Ax - |x| - b.
\]

(c) If \( x \notin K^n \) and \( x \notin -K^n \), it follows from (2.4) that

\[
P_{K^n}(2x) = \left[ \frac{x_1 + \| x_2 \|}{\| x_2 \|} x_2 + x_2 \right].
\]

In addition, it follows from (2.2) that

\[
|x| = \left[ \frac{\| x_2 \|}{\| x_2 \|} x_2 \right].
\]

Then

\[
Q(x) - P_{K^n} [Q(x) - F(x)] = Ax + x - b \left[ \frac{x_1 + \| x_2 \|}{\| x_2 \|} x_2 + x_2 \right] = Ax - \left[ \frac{\| x_2 \|}{\| x_2 \|} x_2 \right] - b = Ax - |x| - b.
\]

The proof is completed. \( \square \)
Now we are in the position to develop a dynamical system to solve SOCAVEs (1.2). Inspired by [14, 19, 26, 41], we propose the following projection-type dynamical system:

\[
\frac{dx}{dt} = \gamma A^\top \{P_{K^n} [Q(x) - F(x)] - Q(x)\},
\]  

(3.8)

where \( \gamma > 0 \) is a constant. According to (3.6) and (3.7), the dynamical system (3.8) can be reduced to

\[
\frac{dx}{dt} = \gamma A^\top (b + |x| - Ax) \doteq h(x).
\]

(3.9)

Based on Theorem 3.1, we have the following theorem.

**Theorem 3.2.** Let \( A \) be nonsingular, then \( x^* \) is a solution of SOCAVEs (1.2) if and only if \( x^* \) is an equilibrium point of the dynamical system (3.9).

Before ending this section, we will study the existence of the solutions and the stability of the equilibrium points of the dynamical system (3.9).

**Lemma 3.3.** The function \( h \) defined as in (3.9) is Lipschitz continuous in \( \mathbb{R}^n \) with Lipschitz constant \( \gamma \|A^\top\| (\|A\| + 1) \).

**Proof.** Since \( \|x_1 - x_2\| \leq \|x_1 - x_2\| \) [20, 24, 36], the proof is trivial according to [6, Lemma 3.1].

Based on Lemma 2.1 and Lemma 3.3, we have the following theorem.

**Theorem 3.3.** For a given initial value \( x(t_0) = x_0 \), there exists a unique solution \( x(t; x(t_0)), t \in [0, \infty) \) for the dynamical system (3.9).

In order to consider the stability of the equilibrium points of the dynamical system (3.9), we need the following theorem.

**Theorem 3.4.** If \( x^* \) is a solution of SOCAVEs (1.2) and \( \|A^{-1}\| \leq 1 \), then

\[
(x - x^*)^\top A^\top r(x) \geq \frac{1}{2} \|r(x)\|^2, \quad \forall x \in \mathbb{R}^n.
\]

(3.10)

**Proof.** The proof is inspired by that of [15, Theorem 2].

Since \( K^n \) is a closed convex set and \( Q(x^*) \in K^n \), it follows from (2.3) that

\[
[v - P_{K^n}(v)]^\top [P_{K^n}(v) - Q(x^*)] \geq 0, \quad \forall v \in \mathbb{R}^n.
\]

Let \( v \doteq Q(x) - F(x) \), we have

\[
[r(x) - F(x)]^\top [P_{K^n} [Q(x) - F(x)] - Q(x^*)] \geq 0.
\]

(3.11)

On the other hand, it follows from \( P_{K^n}(\cdot) \in K^n \), \( F(x^*) \in K^n = (K^n)^* \) and \( Q(x^*)^\top F(x^*) = 0 \) that

\[
F(x^*)^\top [P_{K^n} [Q(x) - F(x)] - Q(x^*)] \geq 0.
\]

(3.12)

It follows from (3.11), (3.12) and

\[
P_{K^n}[Q(x) - F(x)] - Q(x^*) = [Q(x) - Q(x^*)] - r(x)
\]
that
\[
\{[Q(x) - Q(x^*)] + [F(x) - F(x^*)]\}^\top r(x) \\
\geq \|r(x)\|^2 + [Q(x) - Q(x^*)]^\top [F(x) - F(x^*)],
\]
which combines the definitions of $Q$ and $F$ in (3.1) implies
\[
2r(x)^\top A(x - x^*) \geq \|r(x)\|^2 + (x - x^*)^\top (A^\top A - I)(x - x^*), \quad \forall x \in \mathbb{R}^n.
\]
Then the proof is completed with $\|A^{-1}\| \leq 1$. \hfill \blacksquare

Now we can give the following stability theorem.

**Theorem 3.5.** Let $\|A^{-1}\| \leq 1$, then the equilibrium point $x^*$ (if it exists) of the dynamical system (3.9) is asymptotically stable. In particular, if $\|A^{-1}\| < 1$, then the unique equilibrium point $x^*$ of the dynamical system (3.9) is globally asymptotically stable.

**Proof.** Let $x = x(t; x(t_0))$ be the solution of (3.9) with initial value $x(t_0) = x_0$ and $x^*$ is the equilibrium point nearby $x_0$. Let
\[
V(x) = \frac{1}{2}\|x - x^*\|^2, \quad x \in \mathbb{R}^n.
\]
It is easy to check that $V(x^*) = 0$ and $V(x) > 0$ for all $x \neq x^*$. Moreover, it follows from (3.10) that
\[
\frac{d}{dt} V(x) = \frac{dV}{dx} \frac{dx}{dt} \\
= -\gamma (x - x^*)^\top A^\top r(x) \\
\leq -\frac{\gamma}{2} \|r(x)\|^2 < 0, \quad \forall x \neq x^*.
\]
Hence, the first part of the theorem follows from Theorem 2.1.

If $\|A^{-1}\| < 1$, SOCAVEs (1.2) has a unique solution [36] and thus the equilibrium point of (3.9) is unique. Since $V(x) \to \infty$ as $\|x - x^*\| \to \infty$, it follows from Theorem 2.2 that the unique equilibrium point is globally asymptotically stable. \hfill \blacksquare

**Remark 3.2.** If $\|A^{-1}\| = 1$, then SOCAVEs (1.2) may have no solutions, more than one solutions and a unique solution. See the next section for more details.

4 Numerical simulations

In this section, we will present four examples to illustrate the effectiveness of the proposed method. All experiments are implemented in MATLAB R2018b with a machine precision $2.22 \times 10^{-16}$ on a PC Windows 10 operating system with an Intel i7-9700 CPU and 8GB RAM. We use “ode23” to solve the system of ordinary differential equations. More concretely, we use the MATLAB built-in expression
\[
[t, y] = \text{ode23}(\text{odefun}, \text{tspan}, y_0),
\]
which integrates the system of differential equations from $t_0$ to $t_f$ with $\text{tspan} = [t_0, t_f]$. 9
Example 4.1 ([20]). Consider SOCAVEs (1.2) with \( A = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{n \times n} \) and \( b = Ax^* - |x^*|, \) where \( x^* = (-1, 1, -1, 1, \ldots, -1, 1)^\top \in \mathbb{R}^n. \)

In this example, \( \|A^{-1}\| < 1 \) and thus SOCAVEs (1.2) has a unique solution for any \( b \in \mathbb{R}^n. \) Equivalently, the dynamical system (3.9) has a unique equilibrium point and its globally asymptotical stability will be numerically checked. We set \( t_0 = 0 \) and \( t_f = 0.1 \) for this example. In Figure 1, we show the phase diagram of the state \( x(t) \) with different initial points for \( n = 2 \) and \( n = 3, \) which visually display the globally asymptotical stability of the equilibrium point.

In the following examples, \( \gamma = 2 \) is used.

Example 4.2. Consider SOCAVEs (1.2) with

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Obviously, SOCAVEs (1.2) has infinitely many solutions for this example. Thus the dynamical system (3.9) has infinitely many equilibrium points. In fact, \( x = (a, b)^\top (a \geq 0, b = 0) \) are equilibrium points of the dynamical system (3.9). Specifically, we have the following monotone properties of the solution of (3.9).

\[(a)\] If \( x_1 \geq |x_2| \geq 0, \) then

\[
\frac{dx_1}{dt} = 0, \quad \frac{dx_2}{dt} = -2\gamma x_2 \quad \Rightarrow \quad \begin{cases} \frac{dx_2}{dt} \geq 0, & \text{if } x_2 \leq 0, \\ \frac{dx_2}{dt} < 0, & \text{if } x_2 > 0. \end{cases}
\]
(b) If $-|x_2| < x_1 < |x_2|$, then
\[
\frac{dx_1}{dt} = \gamma (|x_2| - x_1) > 0,
\]
\[
\frac{dx_2}{dt} = -\gamma (1 + \frac{x_1}{|x_2|})x_2 \quad \Rightarrow \quad \begin{cases} 
\frac{dx_2}{dt} \geq 0, & \text{if } x_2 \leq 0, \\
\frac{dx_2}{dt} < 0, & \text{if } x_2 > 0.
\end{cases}
\]

(c) If $x_1 \leq -|x_2| \leq 0$, then
\[
\frac{dx_1}{dt} = -2\gamma x_1 \geq 0,
\]
\[
\frac{dx_2}{dt} = 0.
\]

Figure 3 displays the transient behaviors of $x(t) = (x_1(t), x_2(t))^\top$ with 7 different initial points, from which we find that each trajectory generated by the dynamical system (3.9) approaches to a solution of SOCAVEs (1.2).

**Example 4.3.** Consider SOCAVEs (1.2) with
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.
\]

Obviously, SOCAVEs (1.2) has a unique solution $x^* = (0, 1)^\top$ for this example. Thus the dynamical system (3.9) also has a unique equilibrium point. In addition, we have the following monotone properties of the solution of (3.9).

(a) If $x_1 \geq |x_2| \geq 0$, then
\[
\frac{dx_1}{dt} = -\gamma < 0,
\]
\[
\frac{dx_2}{dt} = -\gamma (1 + 2x_2) \quad \Rightarrow \quad \begin{cases} 
\frac{dx_2}{dt} \geq 0, & \text{if } x_2 \leq \frac{1}{2}, \\
\frac{dx_2}{dt} < 0, & \text{if } x_2 > \frac{1}{2}.
\end{cases}
\]
(b) If $-|x_2| < x_1 < |x_2|$, then
\[
\begin{align*}
\frac{dx_1}{dt} &= \gamma(-1 + |x_2| - x_1) \Rightarrow \left\{ \begin{array}{ll}
\frac{dx_1}{dt} \geq 0, & \text{if } x_1 \geq 1 - |x_2|, \\
\frac{dx_1}{dt} < 0, & \text{if } x_1 < 1 - |x_2|,
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
\frac{dx_2}{dt} &= -\gamma \left[ -1 + \frac{x_1}{|x_2|}x_2 \right] \Rightarrow \left\{ \begin{array}{ll}
\frac{dx_2}{dt} \geq 0, & \text{if } x_2 \leq \frac{|x_2|}{x_1 + |x_2|}, \\
\frac{dx_2}{dt} < 0, & \text{if } x_2 > \frac{|x_2|}{x_1 + |x_2|}.
\end{array} \right.
\end{align*}
\]

(c) If $x_1 \leq -|x_2| \leq 0$, then
\[
\begin{align*}
\frac{dx_1}{dt} &= \gamma(-1 - 2x_1) \Rightarrow \left\{ \begin{array}{ll}
\frac{dx_1}{dt} \geq 0, & \text{if } x_1 \leq -\frac{1}{2}, \\
\frac{dx_1}{dt} < 0, & \text{if } x_1 > -\frac{1}{2},
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
\frac{dx_2}{dt} &= \gamma > 0.
\end{align*}
\]

Figure 4 displays the transient behaviors of $x(t) = (x_1(t), x_2(t))^T$ with 8 different initial points, from which we find that all of the trajectories generated by the dynamical system (3.9) approach to the unique solution of SOCAVES (1.2).

Example 4.4. Consider SOCAVES (1.2) with
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

SOCAVES (1.2) has no solutions for this example. Thus the dynamical system (3.9) has no equilibrium points. Indeed, we have

(a) If $x_1 \geq |x_2| \geq 0$, then
\[
\begin{align*}
\frac{dx_1}{dt} &= \gamma > 0,
\end{align*}
\]
\[
\begin{align*}
\frac{dx_2}{dt} &= -\gamma(1 + 2x_2) \Rightarrow \left\{ \begin{array}{ll}
\frac{dx_2}{dt} \geq 0, & \text{if } x_2 \leq -\frac{1}{2}, \\
\frac{dx_2}{dt} < 0, & \text{if } x_2 > -\frac{1}{2}.
\end{array} \right.
\end{align*}
\]

(b) If $-|x_2| < x_1 < |x_2|$, then
\[
\begin{align*}
\frac{dx_1}{dt} &= \gamma(1 + |x_2| - x_1) > 0,
\end{align*}
\]
\[
\begin{align*}
\frac{dx_2}{dt} &= -\gamma \left[ 1 + \frac{x_1}{|x_2|}x_2 \right] \Rightarrow \left\{ \begin{array}{ll}
\frac{dx_2}{dt} \geq 0, & \text{if } x_2 \leq -\frac{|x_2|}{x_1 + |x_2|}, \\
\frac{dx_2}{dt} < 0, & \text{if } x_2 > -\frac{|x_2|}{x_1 + |x_2|}.
\end{array} \right.
\end{align*}
\]

(c) If $x_1 \leq -|x_2| \leq 0$, then
\[
\begin{align*}
\frac{dx_1}{dt} &= \gamma(1 - 2x_1) > 0,
\end{align*}
\]
\[
\begin{align*}
\frac{dx_2}{dt} &= -\gamma < 0.
\end{align*}
\]

Thus, for any initial value $x_0$, at least $x_1(t)$ in the solution of (3.9) is monotonically increasing. Figure 5 displays the transient behaviors of $x(t) = (x_1(t), x_2(t))^T$ with 8 different initial points, which illustrates our claims.
5 Brief conclusion

In this paper, a novel dynamical system is proposed to solve SOCAVEs (1.2), which is different from the existing conventional optimization methods. The results of this paper can be considered as extensions of those in [6].

6 Declarations

Ethical Approval
Not applicable

Competing interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors’ contributions
Cairong Chen: conceptualization, methodology, software, writing–reviewing & editing.
Dongmei Yu: methodology, software, writing–original draft. Deren Han: software, validation.
Chang feng Ma: writing–reviewing & editing.

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References


Figure 3: Transient behaviors of $x(t)$ for Example 4.2 ($tspan = [0, 5]$).
Figure 4: Transient behaviors of $x(t)$ for Example 4.3 ($t\text{span} = [0, 5]$).
Figure 5: Transient behaviors of $x(t)$ for Example 4.4 ($tspan = [0, 10]$).