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Displaying Projectile Motion with Nonlinear Air Resistance Using Caputo’s Definition

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Abstract
Displaying projectile in a resisting medium for two dimension have two forms; An Ordinary differential equation and a fractional differential equation describe its behavior. The two equations include a nonlinear term, which represents the effect of the air resistance on the motion of the projectile. The development of the Ordinary differential equations for all nonlinear cases is obtained using a developed technique that combines the linear operator Laplace transform and the Adomian decomposition method (ADM) to solve the nonlinear part. To discuss the behavior of the projectile motion we plot the obtained results for several values and study the effect of the order of the nonlinear term on the achievable maximum height. On the other hand, the effects of the proportional factor and the projected mass on the motion have been shown by substituting different values and in the equations of motion. In addition to the Laplace Decomposition Method (LDM), we use the Caputo definition of the fractional derivative to investigate the fractional form of the projectile motion equation. The results show that all the results obtained in the ordinary case may be obtained from the fractional case when α=1.

Keywords Projectile motion, Adomian decomposition method, Laplace decomposition method, Fractional Calculus, Caputo Definition.

Mathematics Subject Classification 70K25; 34K37; 44Axx

Introduction
Various methods as; Laplace and Fourier transforms, have been introduced as utilization methods to solve linear and nonlinear differential equations. One of the recently developed techniques is the Laplace Decomposition Method. The technique is constructed of the Laplace transform and the Adomian Decomposition Method, which is established by Adomian in 1980’s [1-3]. It has demonstrated to be a powerful technique, and has successfully applied in several Science and engineering problems. Khuri [4 ] applied this technique to predict an approximate solution of a set of nonlinear ordinary differential equations. Jaradat et al. [5] apply this technique to solve Klein Gordon equation. Handibag and Karande [6] applied this technique for demonstrating linear and nonlinear heat equation. On the other hand, they have developed the LDM to solve nonlinear Fractional differential equations. Shawagfeh [14] has employed Adomian decomposition in the case of the nonlinear fractional differential equation: $D^a y(x) = f(x, y), y^{(k)}(0) = c_k, 0 \leq k \leq [\alpha]$. The LDM was engaged to hold approximate analytical solutions of linear and
nonlinear diffusion-wave equations [15] after substituting the first or second order time derivative by a fractional derivative of order \( \alpha \) with \( 0 < \alpha < 1 \) or \( 1 < \alpha < 2 \) respectively [16]. Ganji, Z. Z. et, al.[17] Implemented the homotopy perturbation method, for demonstrating nonlinear partial differential equation of fractional order arising in fluid mechanics. Kumar et, al.[18] Solved a time fractional Navier-Stokes equation in a type by considering it in Caputo sense.

In this paper, we will apply the modification of the fractional Laplace decomposition method to solve the problem of nonlinear projectile motion. With this method, it is possible to obtain exact solution for nonlinear projectile motion with a fractional Caputo definition.

Following [19], we displaying the behavior of projectile motion in two dimension for two forms in presenting of a resisting medium, considering that the resistive force is proportional to the relative velocity [20].

**Mathematical Tools**

**Laplace Decomposition Method**

This method consider firstly that Laplace transform to a nonlinear differential equation \( h \). Secondly, \( H \) determined as nonlinear differential operator. Thirdly \( L + R \) are the linear part of \( H \), where \( L \) is an operator that have inverse \( L^{-1} \), and \( R \) is the remaining part. Denote the nonlinear term by \( N [21] \), then the equation in standard form is:

\[
Lu + Ru + Nu = h
\]  

(1)

Taking the \( L^{-1} \) to both sides

\[
L^{-1}Lu = L^{-1}h - L^{-1}Ru - L^{-1}Nu
\]

(2)

The basic idea of this technique is to consider the nonlinear term \( Nu \) in the equation (1) as a particular series of polynomials

\[
Nu = \sum_{n=0}^{\infty} A_n ; A_n \equiv \text{Adomian polynomials.}
\]

(3)

\[
A_n(u_0,u_1,u_2,\ldots,u_n) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}
\]

(4)

The Adomian polynomials from \( A_0 \) to \( A_5 \) for the term \( Nu = f(u) \) are determined by

\[
A_0 = f(u_0) \quad A_1 = u_3 f'(u_0)
\]

\[
A_2 = u_2 f'(u_0) + \frac{1}{2!} u_1^2 f''(u_0). \quad A_3 = y_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!} u_1^3 f^{(3)}(u_0)
\]

\[
A_4 = u_4 f'(u_0) + \left( u_1 u_3 + \frac{1}{2!} u_2^2 \right) f''(u_0) + \frac{1}{2!} u_1^2 u_2 f^{(3)}(u_0) + \frac{1}{4!} u_1^4 f^{(4)}(u_0) \quad \text{and so on}...
\]

**Basic Concepts of Fractional Calculus**

Before applying the Laplace Decomposition Method on a fractional differential equation, one must define accurately the fractional derivative. This definition must allow the determining of physical initial conditions, that expressed \( x(0),x'(0), \ldots \) The Caputo definition of fractional derivatives, satisfies these requirements. The Caputo derivative of a function \( f(t) \) (that \( f(t) = 0 \) for \( t > 0 \)) with \( \gamma > 0 \) was defined by (Caputo M. (1969)) [22]. where this work consider the Caputo fractional derivative for a function of time, \( f(t) \), defined as (Sontakke and Shaikh (2015)) [23].
\[ D^\alpha f(t) = \frac{d}{dt}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \]  

(5)

Where \( n = 1,2, \ldots \in N \) and \( n - 1 < \alpha \leq n \). \( \frac{d}{dt}^{\alpha} \) and \( \Gamma(.) \) is the Euler Gamma function. In case \( n = 1 \), then we have the first derivative in the integrand and the order of the fractional derivative is \( 0 < \alpha \leq 1 \).

The Laplace transform for the Caputo's fractional derivative then:

\[ \mathcal{L}\{D^\alpha f(t);s\} = s^\alpha f(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); (n-1 < \alpha \leq n) \]  

(6)

\( ; R(s) > 0 \) and \( R(\alpha) > 0 \). (Saxena R.K., and Nishimoto, K. (2002)) [24]

The basic idea of the Laplace decomposition method for fractional differential equations is similar to what we illustrated in previous section except the first term in eq.(1) was exchanged with the Caputo fractional derivative definition, then eq.(1) will be

\[ D^\alpha y + Ru + Nu = h \]  

(7)

**Results and Discussion of the Fractional Projectile Motion Problem with High Order Resistance**

This section, shows explicitly engaging fractional differential equations to describe the behavior of a projectile in a resisting medium. The Laplace decomposition calculus of Caputo derivative is introduced to get approximate solutions for these equations. Following [19], we substitute the ordinary derivative operator \( \frac{d}{dt} \) in equations (8) by the fractional operator

\[ \frac{d}{dt} \rightarrow k^{1-\alpha} \frac{d^\alpha}{dt^\alpha} \]  

(8)

Now to get a consistent set of units in equation (8), the parameter \( k \) must have a dimension of \( kg/s \). Then, we catch the following equations of order \( 0 < \alpha \leq 1 \)

\[ m \frac{d^\alpha v}{dt^\alpha} = -k^\alpha v^\gamma \quad \text{Horizontal motion equation} \]  

(9)

\[ m \frac{d^\alpha v}{dt^\alpha} = -mgk^\alpha v - k^\alpha m v^\gamma \quad \text{Vertical motion equation} \]  

(10)

where \( 0 < \alpha \leq 1 \) and \( \gamma = 0,1,2,3,\ldots \)

The followed methodology in this section is not different to what we followed in previous section except that it will be taken into account the properties of the Laplace Transform for the Fractional calculus.

Let us start with the special case when \( \gamma = 1 \) in eq.(10) which gives the linear form of the vertical motion equation, then the Newton's equation for the vertical motion is:

\[ \frac{d^\alpha v}{dt^\alpha} = -gk^{\alpha-1} - \frac{k^\alpha}{m} v \quad \text{with initial condition } v(0) = u_0 \]  

(11)

Matching eq.(7) with eq.(11) will get

\[ D^\alpha = \frac{d^\alpha}{dt^\alpha}, \quad R = 0, \quad h = -gk^{\alpha-1} \quad \text{and} \quad Nu = -\frac{k^\alpha}{m} v \]

Apply the Laplace transform on both sides of eq.(11), use the initial condition and the property of the Laplace transform for the Caputo derivative for \( 0 < \alpha \leq 1 \)
\[ \mathcal{L}\left(D_\alpha^a v\right) = s^a v(s) - s^{a-1} v(0) \]

Then, we have

\[ v(s) = \frac{u_0}{s} - \frac{g k^{a-1}}{s^{a+1}} - \frac{k^{a-1}}{m s^a} \mathcal{L}\{v\} \]  \hspace{1cm} (12)

The LDM supposes that the solution of the equation takes the summation form and the last term in eq.(12) represents the summation of the Adomian polynomials. Then,

\[ \sum_{n=0}^{\infty} v_n(s) = \frac{u_0}{s} - \frac{g k^{a-1}}{s^{a+1}} - \frac{k^{a-1}}{m s^a} \mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(u)\right\} \]  \hspace{1cm} (13)

Use the related Adomian polynomials for linear case and match both sides of the last equation and then take the inverse Laplace transform for the result terms to get

\[ v_0(t) = u_0 - g k^{a-1} \frac{t^a}{a!} \]

\[ v_1(t) = -\frac{k^a}{m} \left( u_0 \frac{t^a}{a!} - g k^{a-1} \frac{t^{2a}}{(2a)!} \right) \]  \hspace{1cm} (14)

\[ v_2(t) = \frac{k^{2a}}{m^2} \left( u_0 \frac{t^{2a}}{(2a)!} - g k^{a-1} \frac{t^{3a}}{(3a)!} \right) \]

\[ v_3(t) = -\frac{k^{3a}}{m^3} \left( u_0 \frac{t^{3a}}{(3a)!} - g k^{a-1} \frac{t^{4a}}{(4a)!} \right) \]

Similarly,

\[ v_n(t) = \frac{(-1)^n k^{na}}{m^n} \left( u_0 \frac{t^{na}}{(na)!} - g k^{a-1} \frac{t^{(n+1)a}}{((n+1)a)!} \right) \]  \hspace{1cm} (15)

Take the summation for all terms to obtain the approximate solution

\[ v(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{na}}{m^n} \left( a_1 u_0 t^{na} - a_2 g k^{a-1} t^{(n+1)a} \right) \]  \hspace{1cm} (16)

where \( a_1 \) and \( a_2 \) are constants for the horizontal motion the solution is

\[ v(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{na}}{m^n} (a_1 u_0 t^{na}) \]  \hspace{1cm} (17)

To compare the solutions in classical case with that obtained in fractional case we substitute four values for \( \alpha \) in eq.(17) and plot the results (see Figure 1).
The approximate solution for the linear fractional projectile motion equation with $m = 0.1 \, kg/s$, $v_0 = 20 m/s$, $m = 10g$ and different values for $\alpha$. The time needed to reach the maximum height increases with the increasing in the value of $\alpha$.

When $\gamma = 2,3,4,\ldots$ in eq.(10), we have nonlinear terms in the fractional differential equations. To generalize the solution for any value of $\gamma$ we will take the first three values.

**Case 1: $\gamma = 2$**

Apply the Laplace transform on both sides of eq.(10) for $\gamma=2$. After taking the Laplace transform and substituting the initial condition, then the summation form for the result equation is

$$\sum_{n=0}^{\infty} v_n(s) = \frac{u_0}{s} - \frac{g k^{\alpha-1}}{m s^\alpha} \sum_{n=0}^{\infty} A_n(u)$$  \hspace{1cm} (18)

where $v^2 = \sum_{n=0}^{\infty} A_n(u)$ and the related Adomian polynomials for this case are mentioned in the previous section, then the first terms of the solution are

$$v_0(t) = u_0 - \frac{g k^{\alpha-1} t^\alpha}{\alpha!}$$

$$v_1(t) = -\frac{k^\alpha}{m} \left( \frac{1}{(\alpha)!} u_0^2 t^\alpha \right) - \frac{2}{(2\alpha)!} g u_0 k^{\alpha-1} t^{2\alpha} + \frac{(2\alpha)!}{((\alpha)!)^2 (3\alpha)!} g^2 k^{2\alpha-2} t^{3\alpha}$$

$$v_2(t) = \frac{k^{2\alpha}}{m^2} \left( \frac{2}{(2\alpha)!} u_0^3 t^{2\alpha} \right) - \frac{4((\alpha)!)^2}{((\alpha)!)^2 (3\alpha)!} g u_0^2 k^{\alpha-1} t^{3\alpha}$$

$$+ \frac{2(2\alpha)!}{((\alpha)!)^2 (2\alpha)! (4\alpha)!} g^2 u_0 k^{2\alpha-2} t^{4\alpha} - \frac{2(2\alpha)! (4\alpha)!}{((\alpha)!)^3 (3\alpha)! (5\alpha)!} g^3 k^{3\alpha-3} t^{5\alpha}$$

$$\vdots$$

$$v_n(t) = \frac{(-1)^n k^{n\alpha}}{m^n} \left( u_0^{n+1} g^0 k^{0} t^{n\alpha} - u_0^n g^1 k^{\alpha-1} t^{(n+1)\alpha} + u_0^{n-1} g^2 k^{2\alpha-2} t^{(n+2)\alpha} - \ldots \right)$$

Then, the solution for vertical motion is
\[ v(t) = \sum_{n=0}^{\infty} (-1)^n \frac{k^{n\alpha}}{m^n} \left( b_1 u_0^{n+1} g^{0 \alpha} t^{n\alpha} - b_2 u_0^n g^{1 \alpha} k t^{(n+1)\alpha} + b_3 u_0^{n-1} g^{2 \alpha} 2^{(n+2)\alpha} t^{(n+2)\alpha} - \ldots \right) + b_p u_0^0 g^{n+1 \alpha} (n+1)\alpha^2 \alpha t^{(2n+1)\alpha} \]  

(21)

And the solution for the horizontal motion equation is

\[ v(t) = \sum_{n=0}^{\infty} (-1)^n \frac{k^{n\alpha}}{m^n} (b_1 u_0^{n+1} g^{0 \alpha} t^{n\alpha}) \quad ; \quad b_1 \text{ to } b_\alpha \text{ are constants.} \]  

(22)

Similarly, we substitute four values for \( \alpha \) in eq.(22) and plot the results (see Figure 2).

![Figure 2](image)

**Fig. 2** The approximate solution for the Quadratic fractional projectile motion equation with \( \gamma = 0.1 \text{ kg/s, } v_0 = 20 \text{ m/s, } m = 10 \text{ g} \) and different values for \( \alpha \). From the figure one can observe that the behavior of the projectile approaches to classical behavior when the value of \( \alpha \) approaches to 1.

**Case 2 : \( \gamma = 3 \)**

In this case eq.(10) will be

\[ \frac{d^\alpha v}{dt^\alpha} = -g k^{\alpha-1} \frac{k^\alpha}{m} v^3 \quad \text{ with initial condition } v(0) = u_0 \]  

(23)

The result equation as a summation form after applying the Laplace transform on both sides of eq.(23) can be written as

\[ \sum_{n=0}^{\infty} v_n(s) = \frac{u_0}{s} - \frac{g k^{\alpha-1} k^\alpha}{m s^{\alpha+1}} \left\{ \sum_{n=0}^{\infty} A_n(u) \right\} \]  

(24)

Where the nonlinear term in eq.(23) is presented by the last term in eq.(24), and the Adomian polynomials were found in the previous section. Match both sides of eq.(23) and take the inverse Laplace transform for the result terms of the solution, we have

\[ v_0(t) = u_0 - g k^{\alpha-1} \frac{t^\alpha}{(\alpha)!} \]

\[ v_1(t) = - \frac{k^\alpha}{m} \left( \frac{u_0^3 t^\alpha}{(\alpha)!} \right) - \frac{3 g u_0^2 k^{\alpha-1} t^{2\alpha}}{(2\alpha)!} \left( \frac{(\alpha)!}{(3\alpha)!} \right) - \frac{3 g^2 u_0 k^{2\alpha-2} (2\alpha)! t^{3\alpha}}{(4\alpha)!} \left( \frac{(\alpha)!}{(4\alpha)!} \right) \]
$$v_2(t) = \frac{k^{2\alpha}}{m^2} \left( \frac{3}{(2\alpha)!} u_0^5 t^{2\alpha} - \left( \frac{6(2\alpha)! + 9((\alpha))!^2}{((\alpha))!^3 (3\alpha)!} \right) g u_0^4 k^{\alpha-1} t^{3\alpha} + \right. \right.$$ 

$$\left. \left( \frac{3(2\alpha)! (3\alpha)! + 18((\alpha))!^2 (3\alpha)! + 9(\alpha!) (2\alpha)!^2}{((\alpha))!^3 (2\alpha)! (4\alpha)!} \right) g^3 u_0^2 k^{2\alpha-2} t^{4\alpha} - \right.$$ 

$$\left. \left( \frac{9(\alpha!) (3\alpha)! (4\alpha)! + 18((\alpha))!^2 (4\alpha)! + 3((3\alpha)!^2 (2\alpha)!}{((\alpha))!^3 (3\alpha)! (5\alpha)!} \right) g^4 u_0 k^{4\alpha-4} t^{6\alpha} - \right.$$ 

$$\left. \left( \frac{3(3\alpha)! (6\alpha)!}{((\alpha))!^5 (4\alpha)!} \right) g^5 k^{5\alpha-5} t^{7\alpha} \right)$$ 

$$v_n(t) = \frac{(-1)^n k^{n\alpha}}{m^n} \left( u_0^{2n+1} g^0 k^0 t^{n\alpha} - u_0^{2n} g^1 k t^{(n+1)\alpha} + u_0^{2n-1} g^2 k^2 t^{(n+2)\alpha} - \right. \right.$$ 

$$\left. \cdots + u_0^0 g^{2n+1} k(2n+1) t^{(2n+1)\alpha} \right)$$

Then the solution for the vertical motion is

$$v(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{n\alpha}}{m^n} \left( c_1 u_0^{2n+1} g^0 k^0 t^{n\alpha} - c_2 u_0^{2n} g^1 k t^{(n+1)\alpha} + c_3 u_0^{2n-1} g^2 k^2 t^{(n+2)\alpha} - \right. \right.$$ 

$$\left. \cdots + c_5 u_0^0 g^{2n+1} k(2n+1) t^{(2n+1)\alpha} \right) ; c_1 \text{to } c_5 \text{ are constants.}$$

And the approximate solution for the horizontal motion is

$$v(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{n\alpha}}{m^n} \left( c_1 u_0^{2n+1} g^0 k^0 t^{n\alpha} \right)$$

**Fig. 3** The approximate solution for the fractional projectile motion equation when air resistance depends on $v^3$ with $\alpha = 0.7, 0.8, 0.9, 1$. Again one can observe that the result obtained in the normal case may be obtained from the fractional case when $\alpha = 1$. 

$v = 0.1 \text{ kg/s}, v_0 = 20 \text{ m/s}, m = 10 g$ and different values for $\alpha; \alpha = 0.7, 0.8, 0.9, 1$. 

[Velocity vs. Time Graph]
Case 3 : $\gamma = 4$

The last case will be solved is the fractional projectile motion equation when air resistance depends on $v^4$. Then, the equation of motion is

$$\frac{d^a v}{dt^a} = -g k^{a-1} \frac{k^a}{m} v^4$$

with initial condition $v(0) = u_0$ (29)

And the first terms of the solution are

$$v_0(t) = u_0 - g k^{a-1} \frac{k^a}{m} \frac{t^a}{(a)!}$$

$$v_1(t) = -\frac{k^a}{m} \left( u_0 \frac{t^a}{(a)!} - \frac{4 u_0^3 g k^{a-1} t^{2a}}{(2a)!} + \frac{6 u_0^2 g^2 k^{2a-2} (2a)! t^{3a}}{(a!)^2 (3a)!} - \frac{4 u_0 g^3 k^{3a-3} (3a)! t^{4a}}{(a!)^3 (4a)!} \right)$$

$$+ \frac{4}{(2a)!} u_0^2 \frac{t^{2a}}{(a)!^2 (3a)!} - \frac{16 (a)!^2 + 12 (2a)!}{(a!)^2 (3a)!} u_0^3 g k^{a-1} t^{3a}$$

$$+ \frac{24 (a)! (2a)! (2a)! + 48 (a)! (a)! (3a)! + 12 (2a)! (3a)!}{(a)! (a)! (a)! (4a)!} u_0^4 g^2 k^{2a-2} t^{4a}$$

$$- \left( \frac{16 (a)! (2a)! (3a)! (3a)! + 72 (a)! (2a)! (2a)!}{(a)! (a)! (a)! (3a)!} u_0^4 g^3 k^{3a-3} t^{5a} \right)$$

$$+ \frac{48 (a)! (a)! (3a)! (4a)! + 4 (2a)! (3a)! (4a)!}{(a)! (a)! (a)! (2a)! (3a)!} u_0^4 g^3 k^{3a-3} t^{5a}$$

$$+ \left( \frac{(2a)! (3a)! (4a)! (4a)! + 4 (2a)! (3a)! (4a)!}{(a)! (a)! (a)! (2a)! (3a)!} u_0^4 g^3 k^{3a-3} t^{5a} \right)$$

$$- \left( \frac{3 (3a)! (4a)! (4a)! + 48 (3a)! (3a)! (5a)!}{(a)!^5 (3a)! (4a)!} u_0^4 g^3 k^{3a-3} t^{5a} \right)$$

$$+ \frac{24 (2a)! (4a)! (5a)!}{(a)!^5 (3a)! (4a)! (5a)!} u_0^4 g^3 k^{3a-3} t^{5a}$$

$$+ \frac{3 (4a)! (4a)! + 16 (3a)! (5a)!}{(a)!^6 (4a)! (5a)!} u_0^4 g^3 k^{3a-3} t^{5a}$$

$$- \left( \frac{(4a)!}{(a)!^7 (5a)!} \frac{8! g^7 k^{7a-7} t^{9a}}{(9a)!} \right)$$

$$\vdots$$

$$v_n(t) = \frac{(-1)^n k^{na}}{m^n} \left( u_0^{3n+1} g^0 k^n t^{na} - u_0^{3n} g^1 k^n t^{(n+1)a} + u_0^{3n-1} g^2 k^n t^{(n+2)a} - \cdots \right)$$

$$+ u_0^3 g^3 k^{3(a-3)} t^{(n+4)a}$$

(30)

The approximate solution for the vertical motion is
\[ v(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{n \alpha}}{m^n} \left( d_1 u_0 (3n+1) g^0 k^0 t^{n \alpha} - d_2 u_0 (3n+2) g^1 k^1 t^{(n+1)\alpha} + d_3 u_0 (3n-1) g^2 k^2 t^{(n+2)\alpha} - \ldots \right. \\
\left. + g^{3n+1} k^{(3n+1)\alpha - (3n+1) t} (4n+1) g^0 k^0 t^{n \alpha} \right) ; \ d_1 \text{ to } d_\alpha \text{ are constants} \]  

(31)

And for the horizontal motion is

\[ v(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{n \alpha}}{m^n} \left( d_1 u_0 (3n+1) g^0 k^0 t^{n \alpha} \right) \]

(32)

![Fig. 4 The approximate solution for the fractional projectile motion equation when air resistance depends on \( v^4 \) with \( k = 0.1 \text{ kg/s} \), \( v_0 = 20 \text{ m/s} \), \( m = 10 \text{ g} \) and different values for \( \alpha \).](image)

The solution for any order of nonlinearity for the fractional projectile motion equation could be investigated from the solutions for the previous four cases (linear and nonlinear). Then the generalized solution for order \( \gamma \) of velocity is

\[ v_\gamma(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{n \alpha}}{m^n} \left( \mu_1 u_0 (y-1)n+1 g^0 k^0 t^{n \alpha} - \mu_2 u_0 (y-1)n g^1 k^1 t^{(n+1)\alpha} + \mu_3 u_0 (y-1)n-1 g^2 k^2 t^{(n+2)\alpha} - \ldots \right. \\
\left. + \mu_\alpha u_0 (y-1)n+1 k^{((y-1)n+1)\alpha - ((y-1)n+1) t} (y\alpha) \right) \]

(33)

Where \( \mu_1, \mu_2, \mu_3 \) are allied to \( \gamma \).

Similarly, for the horizontal motion

\[ v_\gamma(t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{n \alpha}}{m^n} \left( \mu_1 u_0 (y-1)n+1 t^{n \alpha} \right) \]

(34)

For studied cases where the object is projected with initial velocity equal to \( 20 \text{ m/s} \) with \( k = 0.1 \text{ kg/s} \); figures (1, 2, 3, and 4) show the behavior of the projectile through flying for several powers to the nonlinear term of the fractional projectile motion equations, also they describe the behavior of the same projectile for several values of the fractional orders (\( \alpha = 0.7, 0.8, 0.9, 1 \)).

It can be demonstrated from the figures that the trajectories of the projectile at several values of \( \alpha \) and several regular values of velocity \( v_0 \) demonstrated using the fractional approach, are always less than the classical approach (normally one).

**Conclusion**

The Laplace Decomposition Method is a significant technique, that can catch linear and nonlinear Ordinary Fractional differential equations. In this paper, this method has successfully applied to two forms of projectile motion equations;
Ordinary form and Fractional form. The solutions have obtained and plotted for several orders of nonlinearity and got a general formula for any order. It is clear from the results that the behavior of the motion for the various orders showed that an increasing in the power of air resistance causes decreasing at the time needed to reach the maximum height. In addition, the increasing in the projectile mass or decreasing on the value of $k$ cause increasing at the time needed to reach maximum height and then increasing in the achievable maximum height. On the other hand, it was noted from solving the fractional projectile motion equation that all the results displayed in the normal case are identical to those displayed from the fractional case when $\alpha = 1$. This successful of our technique in displaying projectile encourage us spread it to demonstrate more nonlinear science phenomena or problems.

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**Authors' contributions:**
EKJ, ADA and WAH analyzed, derive the equation and draw results, where the major contributor are EJJ and WAH. On the other hand ANA and OKJ revise the calculation and write the abstract and introduction. All authors determine the conclusion section, read and approved the final manuscript.

**References**


