Structure-Decidable structure-examples of decidable structures with proof-some examples of undecidable structures

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Abstract—In this paper, decidability of the structures of real, rational, integer and natural numbers will be studied in different languages.

Decidability or undecidability of mathematical structures is one of the fundamental and sometimes very difficult problems of mathematical logic, where several examples of problems in this field are still open and unresolved even after decades. One of the goals of Mathematical Logic is the axiomatization of mathematical theories. Tarski has proved the decidability of the theory of real and complex numbers in the language of addition and multiplication, and it is proved that theories of natural, integer, and rational numbers, in the language of addition and multiplication, are undecidable (Theorems of Gödel and Robinson).

We will proof the following problemes:

The Main Problem 1: \( \langle \mathbb{Q}; \subseteq \rangle \) is decidable?

Problem 2: an explicit axiomatization for \( \langle \mathbb{Z}; \times \rangle \)?

and we will study boolean algebras. Boolean algebras are famous mathematical structures. Tarski showed the decidability of the elementary theory of Boolean algebras. In this paper, we study the different kinds of Boolean algebras and their properties. And we present for the first-order theory of atomic Boolean algebras a quantifier elimination algorithm. The subset relation is a partial order and indeed a lattice order, and I will prove that the theory of atomic Boolean lattice orders is decidable, and furthermore admits elimination of quantifiers. So the theory of the subset relation is decidable. And we will study decidability of atomsless boolean algebra.

II. Structure-decidable structure-examples of decidable structures with proof-some examples of undecidable structures

Decidability: A class of questions is decidable if and only if there is a procedure such that, when given as input any question in the class, the procedure halts and says yes if the answer is positive and no if the answer is negative.

Example: For any natural number n, determining whether n is prime.

The mathematical structure consists of a specific set (usually a set of numbers, like natural, integer, rational, real or complex numbers) in a first-order language that contains some functions, predicates or constants. The theory of a structure is the set of all first-order sentences (in the language of that structure) which are true in that structure.

\[ \text{Structure } A = \langle A; \mathcal{L} \rangle \quad \text{Th}(A) = \{ \theta \in \mathcal{L} \mid A \vdash \theta \} \]

For example, the sentence "any number is equal to the sum of another number with itself" is false in (the domain of) integer numbers, but it is true in (the domain of) rational numbers (e.g., there are no integer \( n \) such that \( n + n = 3 \), but the sum of \( 3/2 \) with itself is \( 3 \)).

\[ \langle Z; + \rangle \not\vdash \forall x \exists y (x = y + y) \quad \langle Q; + \rangle \vdash \forall x \exists y (x = y + y) \]

Decidable: A theory T is decidable if there exists an effective procedure to determine whether \( T \vdash \phi \) where \( \phi \) is any sentence of the language.

Completeness: completeness of a theory T means that for any sentence \( \phi \) in the language of the theory, we have either \( T \vdash \phi \) or \( T \vdash \neg \phi \). If this property does not hold (so if there is some \( \phi \) that the theory says nothing about), we have an incomplete theory.

I. Introduction

Quantifier Elimination and Decidability:

Decision Procedures: The purpose is to produce an algorithm for determining whether or not a formula is valid. So, decision procedure is an algorithm that, given a decision problem, terminates with a yes or no answer.

Quantifier elimination: We say that a theory T has quantifier elimination if for every formula \( \phi \) there is a quantifier-free formula \( \psi \) such that \( \varphi \leftrightarrow \psi \).
only if every formula of the form \( \exists x (\land \alpha_i) \) is (recursively) equivalent with a quantifier -free formula, where each \( \alpha_i \) is either an atomic formula or the negation of an atomic formula.

Proof:
Every formula \( \psi \) can be written (equivalency) in the prenex normal form, say
\[
Q_1x_1Q_2x_2 \cdots Q_nx_n \theta(x_1, x_2, \ldots, x_n)
\]
where \( Q_i \)'s are quantifiers and \( \theta \) is quantifier-free. if \( Q_n = \exists \), then let \( \theta = \theta \), and if \( Q_n = \forall \), then \( \theta = \neg \theta \), (note that in the latter case \( \forall x_n \equiv \neg \exists x_n \theta \). Now, the quantifier- free formula \( \theta \) can be written in the disjunctive normal form, say \( V \bigcup_{j} \alpha_{i,j} \) where each \( \alpha_{i,j} \) is a literal (i.e., an atomic or a negated atomic formula). Noting that \( \exists x (\land \alpha_i) \equiv V \exists \beta_i \) we have
\[
\psi \equiv Q_1x_1Q_2x_2 \cdots Q_{n-1}x_{n-1} \bigcirc \bigvee x_n (\land \alpha_{i,j})
\]
where \( \bigcirc \) is nothing (empty) when \( Q_n = \exists \) and \( \bigcirc = \neg \) when \( Q_n = \forall \). Now, if \( \exists x_n (\land \alpha_{i,j}) \) is equivalent with a quantifier -free formula, \( \psi \) is equivalent with a formula with one less quantifier; counting this way one can show that \( \psi \) equivalent with a formula which has no quantifier.[1]

A. Decidability of Structure of Natural Numbers in Different Languages

Theorem1. the theory \( ThN_s \) where \( N_s = (N, 0, s) \) admits elimination of quantifier.

Proof:
It suffices to consider a formula,
\[
\exists x (\alpha_0 \land \cdots \land \alpha_q)
\]
where each \( \alpha_i \) is atomic or is the negation of an atomic formula. In the language of \( N_s \) the only terms are of the from \( S^ku \) where \( u \) is 0 or a variable. We may suppose that the variable \( x \) occurs in each \( \alpha_i \). For if \( x \) does not occur in \( \alpha \), then
\[
\exists x (\alpha \land \beta) \iff \alpha \land \exists x \beta
\]
Thus each \( \alpha_i \) has the form \( S^m x = S^k u \) or the negation of this equation, where \( u \) is 0 or a variable. We may further suppose \( u \) is different from \( x \) since \( S^m x = S^m x \) could be replaced by \( 0 = 0 \). if \( m = n \) and by \( 0 \neq 0 \) if \( m \neq n \).

Case 1: Each \( \alpha_i \) is the negation of an equation. Then the formula may be replaced by \( 0 = 0 \).

Case 2: There is at least one \( \alpha_i \) not negated; say \( \alpha_0 \) is,
\[
S^m x = t
\]
where the term \( t \) does not contain \( x \). Since the solution for \( x \) must be non-negative, we replace \( \alpha_0 \) by,
\[
t \neq 0 \land \cdots \land t \neq S^{m-1} 0
\]
Then in each other \( \alpha_j \) we replace, say, \( S^k x = u \) first by \( S^{k+m} x = S^m u \) which in turn becomes \( S^k t = S^m u \). We now have a formula in which \( x \) no longer occurs, so the quantifier may be omitted.

Theorem2. The Theory \( N_L = (N, 0, S, <) \) admits elimination quantifier, and so has a decidable theory and is finitely axiomatizable.

S3. \( \forall y (y \neq 0 \rightarrow \exists xy = Sx) \)

L1. \( \forall x \forall y (x < Sy \leftrightarrow x < y) \)

L2. \( x \neq 0 \)

L3. \( \forall x \forall y (x < y \lor y < x \lor x = y) \)

L4. \( \forall x \forall y (x < y \rightarrow y \neq x) \)

L5. \( \forall x \forall y \forall z (x < y \land y < z \rightarrow x < z) \)

Proof:
We consider a formula,
\[
\exists x (\beta_0 \land \cdots \land \beta_n)
\]
where each \( \beta_i \) is atomic or the negation of an atomic formula. The terms are of the form \( S^k u \) where \( u \) is 0 or a variable. There are two possibilities for atomic formula,
\[
S^k u = S^l t, S^k u < S^l t
\]
1. We can eliminate the negation symbol. Replace \( t_1 t_2 \) by \( t_1 t_2 \lor t_1 t_2 \land t_1 \neq t_2 \) by \( t_1 t_2 \land t_2 t_2 \lor t_1 \neq t_2 \) by \( t_1 t_2 \land t_2 t_2 \lor t_1 \neq t_2 \). By regrouping the atomic formulas and noting that
\[
\exists x (\phi \lor \psi) \iff \exists x \phi \lor \exists x \psi
\]
we may again reach formulas of the form,
\[
\exists x (\alpha_0 \land \cdots \land \alpha_q)
\]
where now, each \( \alpha_i \) is atomic or the negation of an atomic formula. This is because if \( x \) does not occur in \( \alpha_i \) then
\[
\exists x (\alpha \land \beta) \iff \alpha \land \exists x \beta
\]
Furthermore, we may suppose that \( x \) occurs on only one side of the equality or inequality \( \alpha_i \).

Case 1: Suppose that some \( \alpha_i \) is an equality. Then we can proceed as in case 2 of the quantifier-elimination proof

Case 2: Otherwise each \( \alpha_i \) is an inequality. Then the formula can be rewritten
\[
\exists x (\land_i t_i < S^m_i x \land \land_j S^m_j x < u_j)
\]
we have lower bounds on \( x \)
If the second conjunction is empty (i.e., if there are no upper bounds on \( x \)) then we can replace the formula by \( 0 = 0 \) If the second conjunction is empty (i.e., if there are no upper bounds on \( x \)) then we an replace the formula by \( \land_j S^{m_j} 0 < u_j \) which asserts that zero satisfies the upper bounds. Otherwise, we rewrite the formula successively as,
\[
\exists x (\land_{i,j} (t_i < S^{m_i} x \land S^{m_j} x < u_j)) (1)
\]
\[
\exists x (\land_{i,j} (S^{m_i} t_i < S^{m_i+j} x < S^{m_i} u_j)) (2)
\]
\[
(\land_{i,j} S^{m_j+1} t_i < S^{m_j} u_j) \land \land_j S^{m_j} 0 < u_j
\]
In each case, we have arrived at a quantifier-free version of the given formula.  

1) The additive theory of natural numbers: Presburger proof decidability of the theory \( \langle N; +, \cdot \rangle \) with quantifier elimination. One common way of quantifier elimination is to extend the language, and we add fixed symbols 0 and 1 and an infinite set of binary relations \( <_n \) for \( n \geq 1 \). Which is defined as follows:

\[
\forall x, y \in N, x <_n y \iff (x < y \land x \equiv y \ (\text{mod} n))
\]

Theorem 3. The following axioms at the Language \( L = \{+, 0, 1, \leq, \{\equiv_m\}_{m \geq 2}\} \), for the structure \( N \) allow quantifier elimination.

(A1) \( x + (y + z) = (x + y) + z \)

(A2) \( x + y = y + x \)

(A3) \( x + 0 = x \)

(A4) \( x + z = y + z \rightarrow x = y \)

(A5) \( x + y = 0 \rightarrow x = y = 0 \)

(O1) \( x \leq y \iff \exists z(x + z = y) \)

(O2) \( x \leq y \land y \leq x \)

(O3) \( 0 \neq 1 \land \forall y(0 \leq y \leq 1 \rightarrow y = 0 \lor y = 1) \)

(D1) \( \forall x \exists y, z(x = n \cdot y + t \land t < \bar{n}) \)

Proof:

Step 1: Identify the terms

In structure \( \langle N; +, \cdot, 0, 1, \{\equiv_n\}_{n \geq 2}\rangle \), every term involving \( x \) is equal to,

\[
n \cdot x + t \quad (n \in N)
\]

where \( x \) does not appear in \( t \)

Step 2: Identify Atomic Formulas and Delete \( \lnot \) if possible

All atomic formulas are,

\[
u \leq v \\
u \equiv_k v
\]

First, we eliminate the inequality behind the atoms. Because,

\[
x = y \iff x \leq y \land y \leq x \]
\[
x \neq y \iff x + 1 \leq y \lor y + 1 \leq x \]
\[
x \leq y \iff y + 1 \leq x \]
\[
x \neq y \iff \bigvee_{0 < i < n} x + i \equiv_n y
\]

So, the following formula admits quantifier elimination.

\[
\exists x(\bigwedge_i u_i \cdot x + t_i \leq m_i \cdot x + s_i \land \bigwedge_j k_j \cdot x + u_j \equiv_q l_j \cdot x + v_j)
\]

Step 3: Simplify atomic formulas

So the following formula must be eliminated quantifier.

\[
\exists x(\bigwedge_i r_i \leq m_i \cdot x + s_i \land \bigwedge_j n_j \cdot x + t_j \leq u_j \land \bigwedge_k k_l \cdot x + v_l \equiv_q w_l)
\]

Step 4: Uniform the coefficients \( x \)

Let \( M \) is Multiply the coefficients by \( x \)

\[
M = \prod_i m_i \prod_j n_j \prod_k l_k
\]

\[
r_i \frac{M}{m_i} \leq Mx + \frac{M}{n_i} s_i
\]

\[
Mx + \frac{M}{l_k} t_j \leq \frac{M}{u_j} v_j
\]

\[
Mx + \frac{M}{l_k} v_l \equiv_q \frac{M}{u_j} \frac{M}{n_i} w_l
\]

So the following formula admits quantifier elimination

\[
\exists x(\bigwedge_i r'_i \leq Mx + s'_i \land \bigwedge_j d'_j Mx + t'_j \leq u'_j \land \bigwedge_k k'_l Mx + v'_l \equiv_q w'_l)
\]

Step 5: Remove the coefficient \( x \)

\[
y = Mx. \text{ So, we have}
\]

\[
\exists y(\bigwedge_i r'_i \leq y + s'_i \land \bigwedge_j d'_j y + t'_j \leq u'_j \land \bigwedge_k k'_l y \equiv_q w'_l)
\]

We use the following equations

\[
t = s \iff ct = cs
\]

\[
t < s \iff ct < cs
\]

\[
\equiv_m s \iff ct \equiv_{cm} cs
\]

so

\[
\exists x(\bigwedge_i r_i \leq x + s_i \land \bigwedge_j x + t_j \leq u_j \land \bigwedge_k x + v_l \equiv_q w_l)
\]

Step 6: Identify Phrases included \( x \)

\[
r_i \leq x + s_i \iff r_i + t_j + v_l \leq x + s_i + t_j + v_l
\]

\[
x + t_j \leq u_j \iff x + s_i + t_j + v_l \leq u_j + s_i + v_l
\]

\[
x + v_l \equiv_q w_l \iff x + s_i + t_j + v_l \equiv_q s_i + t_j + w_l
\]

\[
P = s_i + t_j + v_l
\]

so

\[
\exists x(\bigwedge_i r'_i \leq x + P \land \bigwedge_j x + P \leq u'_j \land \bigwedge_k x + P \equiv_q w'_l)
\]

we put \( y = x + P \)

\[
\exists y(\bigwedge_i r'_i \leq y \land \bigwedge_j y \leq u'_j \land \bigwedge_k y \equiv_q w'_l)
\]

Therefore, it is enough to delete the quantifier in the following formula:

\[
\exists x(\bigwedge_{i=1}^m r_i \leq x \land \bigwedge_{j=1}^n x \leq u_j \land \bigwedge_{k=1}^k x \equiv_q w_l)
\]

Step 7: Reduce Boolean Combination

A: Reduce the order

\[
\exists x(r_0 \leq x \land r_1 \leq x \land \theta(x)) \equiv \left[ r_0 \leq r_1 \land \exists x(r_1 \leq x \land \theta(x)) \right] \lor \left[ r_1 \leq r_0 \land \exists x(r_0 \leq x \land \theta(x)) \right]
\]

B: \( \exists x(x \leq u_0 \land x \leq u_1 \land \theta(x)) \equiv \left[ u_0 \leq u_1 \land \exists x(x \leq \theta(x)) \right] \lor \left[ u_1 \leq u_0 \land \exists x(x \leq \theta(x)) \right]
\]

C: \( \exists x(x \equiv_q w_0 \land x \equiv_q w_1 \land \theta(x)) \equiv \exists x(x \equiv_{q \equiv q_1} w_0 \land \theta(x)) \)

Step 8: Identify the states

\[
\exists x(r \leq x \land x \leq u \land x \equiv_q w) \equiv \bigvee_{i=0}^{q-1} (r + \bar{i} \leq u \land \bar{r} + \bar{i} \equiv_q w)
\]

\[
\exists x(r \leq x \land x \leq u) \equiv r \leq u
\]

\[
\exists x(r \leq x \land x \equiv_q w) \equiv \text{true}
\]
\[ \exists x (x \leq u \land x \equiv_q w) \equiv \bigvee_{i=0}^{q-1} (i \leq u \land i \equiv_q w) \]
\[ \exists x (r \leq x) \equiv true \]
\[ \exists x (x \leq u) \equiv true \]
\[ \exists x (x \equiv_q w) \equiv true \]
\[ \exists x (\quad \quad \quad ) \equiv true \]

B. Introduction to Decidability of the Multiplication Theory of Natural Numbers

Skolem arithmetic: The theory of the structure \((\mathbb{N}, \times)\) is decidable.

Mostowski deals with the notion of direct product in the theory of decision problems. This was well-known to Mostowski, who was able to prove decidability of Skolem Arithmetic through seeing it as a certain weak direct product of Presburger Arithmetic. Such that; the set of positive integers with multiplication is isomorphic to the weak direct product of countably many copies of the non-negative integers. To see this, think of the n-th coordinate of an element of the direct sum as representing the exponent of the n-th prime in the prime decomposition. Now, the decidability of the multiplicative system is obtained from the decidability of complete addition and the decidability of the theory of the boolean algebra of all finite and co-finite subsets of to, together with the idea of finite sets. Skolem had suggested a quantifier elimination for complete multiplication.

Let \(L\) be a language with only constant 0. Let \((Q_i)_{i \in I}\) be non-empty family of \(L\)-structures such that for any \(i \in I\) and any functional symbol \(F\) of \(L\), we have: \(F(0, \ldots , 0) = 0\).

The direct sum of this family is the \(L\)-structure \(A\), denoted by \(\bigoplus_{i \in I} Q_i\), that defined by:

\[ B = \{ f \in \prod_{i \in I} A_i/f(i) = 0 \text{ except for almost finite number of } i \} \]

For \(R\) is \(n\)-ary predicate of \(L\): \(R^A(f_1, \ldots , f_n)\) iff for all \(i \in I\) we have \(R^{Q_i}(f_1(i), \ldots , f_n(i))\)

For \(F\) is \(n\)-ary function symbol of \(L\): \(F^A(f_1, \ldots , f_n) = F^{Q_i}(f_1(i), \ldots , f_n(i))\)

and \(\bigoplus_{i \in I} Q_i\) is an \(L\)-structure, and closed for functions is provided by the conditions of the family of \(L\) -structures. If \(I\) is finite, the direct sum is the same as the direct product. We have \((\mathbb{N}^+, \cdot) = \bigoplus_{n \in \mathbb{N}} Q_n\) where \(Q_n = (\mathbb{N}, +)\) for any \(n\). And we have \((\mathbb{N}^{\geq 0}, |) = \bigoplus_{n \in \mathbb{N}} Q_n\) where \(Q_n = (\mathbb{N}, \leq)\) for any \(n\).

\(p\)-adic numbers were first described by Kurt Hensel in 1897 though, with hindsight, some of Ernst Kummer’s earlier work can be interpreted as implicitly using \(p\)-adic numbers. The \(p\)-adic numbers were motivated primarily by an attempt to bring the ideas and techniques of power series methods into number theory.

- **\(p\)-adic number:** \(p\)-adic number is sums of the form: \(\sum_{i=k}^{\infty} a_ip^i\) where \(k\) is some (not necessarily positive) integer, and each coefficient \(a_i\) \(p\)-adic digit.

\[ 0 \leq a_i \leq p-1 \]

- Fundamental theorem of arithmetic
  
  Every integer greater than 1 can be represented as the product of prime numbers and, moreover, this representation is unique.

- Euclid’s theorem
  
  Euclid’s theorem is a fundamental statement in number theory that asserts that there are infinitely many prime numbers. It was first proved by Euclid.

- \(p\)-adic valuation:
  
  We define \(p\)-adic valuation of \(x\) with \(V^1\).

  \[ x = p_1^{m_1}p_2^{m_2}p_3^{m_3} \cdots \]
  \[ V(p_2, x) = p_2^{m_2} \]

  - \(p\) is a prime number, and denoted by \(P(p)\) if we have: \(p \neq 1 \land \forall x (x | p \rightarrow (x = 1 \lor x = p))\)
  - \(p\)-primary number: \(x\) is a \(p\)-primary number, and denoted by \(P_R(p, x)\) if we have \(P(p) \land \forall q(P(q) \land q \neq p) \rightarrow q | x\)

  - Truncation:

    \[ \forall x \exists y \exists z \forall p(P(p) \rightarrow (p | x \rightarrow V(p, y) = V(p, z)) \land (p | x \rightarrow V(p, y) = p \cdot V(p, x))) \]

  This \(z\) is unique, and denoted by \(T(x, y) = \prod_{p | x} p^a\).

  So we have:

  \[ x = 2^u \cdot 3^v \cdot 5^w \cdots \]
  \[ y = 2^a \cdot 3^b \cdot 5^c \cdots \]

  \[ T(2^1 \cdot 3^0 \cdot 5^2, 2^2 \cdot 3^1) = 2^2 \cdot 5^1 \]

  - Increment:

    \[ \forall x \exists y \exists z (P(p) \rightarrow (p | x \rightarrow V(p, y) = 1) \land (p | x \rightarrow V(p, y) = p \cdot V(p, x))) \]

  - Separation:

    \[ \forall x \exists y \exists z \forall p(P(p) \rightarrow (p | x \cdot y \land V(p, x) \equiv_n V(p, y)) \rightarrow V(p, z) = p \land (p | x \cdot y \lor V(p, x) \neq_n V(p, y) \rightarrow V(p, z) = 1) \land n \in \mathbb{N} \]

  This \(z\) is unique, and denoted by \(SP_n(x, y) = \prod_{(p^a \equiv p^b)} p | x \cdot y\) so we have:

  \[ x = 2^{3} \times 3^{4} \land y = 2^{5} \times 3^{5} \times 5^{2} \]
  \[ SP_2(x, y) = 2 \times 5 \]
  \[ p = 2, \ 5 \]
  \[ q = 3, \ 2 \]
  \[ n = 5, \ 2 \]
  \[ a = 2, \ 3 \]

  - Divisibility:

    \[ \forall x \exists y \exists z (x = y^n \cdot z \land \forall y \forall \tilde{z} (x = y^n \cdot \tilde{z} \rightarrow z \land \tilde{z})) \]

  \[ 360 = 2^3 \times 3^2 \times 5^1 \]

\(1\)If \(n = \prod_{p | x} p^{v(n, p)}\) but we may not define, \(v(n, p)\) in the theory. Hence \(n = \prod_{p | x} V(n, p)\) (meaning \(V(n, p) = p^{v(n, p)}\)). For little \(v\) we have \(v(p, x, y) = v(p, x) + v(p, y)\) But for big \(V\) we have \(V(p, x, y) = V(p, x).V(p, y)\).
1, 4, 9, 16, 25, ⋯, \(x^2\)

1, 8, 27, 64, 125, ⋯, \(x^3\)

\[1^n, 2^n, 3^n, \cdots\]

\[360 = 2^3 \times 3^2 \times 5^1\]

\[= (2 \times 3)^2 \times 2 \times 5 = 36 \times 10\]

\[360 = 8 \times 45\]

\[x = y^n \cdot z \rightarrow 360 = 6^2 \times 10\]

We must also start that quantifiers can be eliminated. For any model \(\mathcal{A}\) of the given axioms, and any prime \(p\) of the model, one can define

\[A_p = \{x \in \mathcal{A} : x \text{ is } p \text{ - primary}\}\]

and consider the structure,

\[\mathcal{A}_p = (A_p, \cdot, 1)\]

The axioms given for \(Th(N_{>0}; 1)\) guarantee each \(\mathcal{A}_p\) to be a model of \(T_h(N;+, 0)\). In fact, for the model \((N;+, 1)\), each \(\mathcal{A}_p\) is isomorphic to \((N;+, 0)\). \(\mathcal{A}_p\) is a model of the theory of addition, and \(\mathcal{A}_p\) is definable in \(\mathcal{A}\) in terms of the parameter \(p\):

\[v_0 \in A_p; PR(p, v_0)\]

For any formula \(\phi v_1 \cdots v_n\) of the language of \(\mathcal{A}\), we can find a formula \(\phi^p\) such that, for all \(x_0, \cdots, x_{n-1} \in \mathcal{A}\),

\[\mathcal{A} \models \phi^p(x_0, \cdots, x_{n-1}) \iff \mathcal{A} \models \phi(y_0, \cdots, y_{n-1})\]

where \(y_i = V(p, x_i)\). To define \(\phi^p\), first relativise \(\phi\) to \(A_p\) and then replace each free variable \(v_i\) in \(\phi\) by \(V(p, v_i)\).

The construction of \(\phi^p\) is uniform in the constant \(p\), i.e., for each \(\phi\), there is a single formula \(\phi^0 v_1 \cdots v_n\) from which each \(\phi^p\) is obtained by substituting the constant \(p\) for the variable \(v_i\), so we use the additive notation (means 0, 1, +, ≼, S, ⋯) or use the multiplicative notation (means 1, p, +, ⋆, I, ⋯) for the elements of \(A_p\) The results are the same.

Consider the additive formula

\[y_0 < y_1, y_i = V(p, x_i)\]

\[A_p \models y_0 < y_1 \iff \mathcal{A} \models V(p, x_0) \land V(p, x_1)\]

\[\iff A \models V(p, SP_1(x_0, x_1)) = \bar{p}\]

Mostowski observed was that \(Th(N_{>0}; 1)\) is the weak direct power of \(T_h(N;+, 0)\), The relevant theorem of Mostowski:

Let \(T\) be a decidable theory with a unique distinguished constant. The theory of weak direct powers of models of \(T\) is decidable.[10, Theorem 5.2]

So, by The Feferman-Vaught Theorem every formula of the language \((\cdot, 1)\) is equivalent to a propositional combination of formula of the form,

\[\exists p_1 \cdots \exists p_k (\Lambda_{1 \leq i \leq k} p_i \neq p_i \land \Lambda_{1 \leq i \leq k} P(p_i) \land \theta(p))\]

\(\theta\) is formula of the language \((\cdot, 1)\)

\(T_h(N_{>0}; 1)\) admits a quantifier elimination when language is augmented by the function symbols \(I, T, SP_n (n \geq 0)\), and the unary symbols \(E_n (n \geq 1)\).

Theorem 4, the Theory \((N, \cdot)\) admits quantifier elimination, and so has decidable theory and is axiomatizable

Proof:

in the article [3] ² has been proven.

Axiomatizing and decidability of the theory of \((N;\cdot, x)\):

\[(A_1) \forall x \forall y \forall z (x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)\]

\[(A_2) \exists x \forall y x = y \cdot x = y\]

\[(A_3) \forall x \forall y x \cdot y = y \cdot x\]

\[(A_4) \forall x \forall y \forall z (x \cdot z = y \cdot z) = x = y\]

\[(A_5) \forall x \forall y x \cdot y = 1 \rightarrow x = y = 1\]

\[(A_6_n) \forall x \forall y x^n = y^n \rightarrow x = y \quad (n \in N^*)\]

\[(A_{7, n}) \forall x \exists y \forall z (x = n \cdot y + z \land z \leq n \land z \neq n) \land \forall n \in N^*\]

\[(A_8) \forall x \exists y \forall z (x = y^n \cdot z \land \forall y (y = y^n \cdot z \rightarrow z \land z))\]

\[(A_9) \forall p \exists P (P(p) \land px)\]

\[(A_{10}) \forall \exists P (PR(x) \land PR(p)) \rightarrow x \lor y \lor x\]

\[(A_{11}) \forall p \exists p (P(p) \rightarrow \exists y (y = V(p, x)))\]

\[(A_{12}) x = y = \forall p (P(p) \rightarrow \exists y (V(p, x) = V(p, y)))\]

\[(A_{13}) \forall x \forall y p(P(p) \rightarrow V(p, x) = V(p, y))\]

\[(A_{14}) \forall x \forall y p(P(p) \rightarrow V(p, x) = V(p, y) \land (p \land px) \rightarrow V(p, x) = 1))\]

\[(A_{15}) \forall x \forall y p(P(p) \rightarrow (p \land px \rightarrow V(p, y) = 1) \land (p \land px \rightarrow V(p, x) = 1)))\]

\[(A_{16}) \forall x \forall y P(p) (P(p) \rightarrow (p \land px \land V(p, y) = 1) \land px))\]

\[(A_{17}) \forall x \forall y P(p) (P(p) \rightarrow (p \land px \land V(p, y) = 1) \land px))\]

The \(T_h(N_{>0}; 1)\) is complete and decidable.

because:

Let \(L' = (\cdot, V)\) which is the language obtained from \(L\) by adding a binary function sign \(V\), for any formula \(\varphi\) in \(L\), there is the associated formula \(\varphi^p\) in \(L\)' whose set of free variables is the ones in \(\varphi\) plus the variables \(p\), that is \(\varphi^p\) is obtained by replacing each free variable \(x\) in \(\varphi\) with the term \(V(p, x)\), we let \(M = T_h(N_{>0}; 1)\) and \(\mathcal{A}\) be a model of \(M\), and \(\varphi\) be an \(n\)– formula in \(L\) and \(p\) be a prime number and \(\bar{f} \in A^n\)

\[\mathcal{A} \models \varphi^p (\bar{f}) \iff \mathcal{A} \models \varphi (V(p, \bar{f}))\]

and \(M'\) denoted to a theory in the language \(L'' = (\cdot, V, p)\). Let \(\theta\) be a formula of \(L\) and \(k \in N > 0\) ,then we denoted \(R_k(\theta)\) to the formula in \(L\) as follows:

\[\exists p_1 \cdots \exists p_k (\Lambda_{1 \leq i \leq k} p_i \neq p_i \land \Lambda_{1 \leq i \leq k} P(p_i) \land \theta(p))\]

Any formula \(\phi\) of \(L\) is \(M''\)– equivalent to a combination to a boolean formula of the form \(R_k(\theta)\); because it is \(M''\)– equivalent to a formula of \(L\) and \(M''\) is an extension of \(M\) so it is sufficient to prove complete and decidable for \(M''\). When \(\varphi\) is a statement we can effectively reduce it to a boolean combination of formulas of the type \(R_k(\theta)\)

² Cęgielski
where is a statement, $R_6(\theta)$ is true if and only if, it is true in the theory of addiction, since addition theory is complete and decidable, so The theory of multiplication of natural numbers is complete and decidable. And The multiplication theory of natural numbers is not finitely axiomatizable, because the theory of addition is not finitely axiomatizable.

Skolem claimed the decidability of the theory $(\mathbb{N}; x, =)$ by using the quantifier elimination. The first decidability proof appeared in the work of Mostowski. Cegielski axiomatized multiplication theory and proved quantifier elimination.

### III. Peano Arithmetic

#### Peano's Axiomatic System:

- 1. $\forall x \neg (S(x) = 0)$
- 2. $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$
- 3. $\forall x (x + 0 = x)$
- 4. $\forall x \forall y (x + S(y) = S(x + y))$
- 5. $\forall x (x \cdot 0 = 0)$
- 6. $\forall x \forall y (x \cdot S(y) = x \cdot y + x)$
- 7. $\forall x \exists y (x < S(y) \leftrightarrow x < y \land x \neq y)$
- 8. $\forall x \forall y (x < S(y) \leftrightarrow x < y \land x = y)$
- 9. $\forall x \forall y (x < y \land x \neq y)$
- 10. $\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x \varphi(x)$

Peano Arithmetic $\mathbf{PA}$ is undecidable.

The decidability of the structures of natural numbers in different languages is shown in the following tables so that the theories that admit QE by $\sqrt{\land}$ and, the theories do not admit QE by $\times$ is shown.

<table>
<thead>
<tr>
<th>Structures</th>
<th>Decidability of the structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mathbb{N}; &lt;)$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$(\mathbb{N}; +)$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$(\mathbb{N}; \times)$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$(\mathbb{N}; &lt;, +)$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$(\mathbb{N}; &lt;, \times)$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$(\mathbb{N}; +, \times)$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$(\mathbb{N}; +, &lt;, \times)$</td>
<td>$\checkmark$</td>
</tr>
</tbody>
</table>

### Table I

<table>
<thead>
<tr>
<th>Structures</th>
<th>Decidability of the structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mathbb{N}; &lt;)$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$(\mathbb{N}; +)$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$(\mathbb{N}; \times)$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$(\mathbb{N}; &lt;, +)$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$(\mathbb{N}; &lt;, \times)$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$(\mathbb{N}; +, \times)$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$(\mathbb{N}; +, &lt;, \times)$</td>
<td>$\checkmark$</td>
</tr>
</tbody>
</table>

### Table II

1) Decidability of The theory of $(\mathbb{N}; \sqsubseteq)$: Theorem 5. The following completely axiomatizes the structure $(\mathbb{N}; \sqsubseteq)$ and, moreover, its theory admits quantifier elimination, and so is decidable.

- $[1]\forall x (x \sqsubseteq x)$

---

- $[2]\forall x, y (x \sqsubseteq y \land y \sqsubseteq x \rightarrow x = y)$
- $[3]\forall x, y, z (x \sqsubseteq y \land z \sqsubseteq x \rightarrow z \sqsubseteq y)$
- $[4]\forall x, y \exists z (z \sqsubseteq x \land \forall t (t \sqsubseteq x, y \rightarrow t \sqsubseteq z)), z = x \cap y$
- $[5]\forall x, y \exists z (x \sqsubseteq z \land \forall t (t \sqsubseteq x, y \rightarrow t \sqsubseteq z)), z = x \cup y$
- $[6]\forall x (1 \subseteq x)$

Definition 1. An element $x$ of a lattice is join-irreducible if it satisfies:

$\forall a, b (x = a \lor b \rightarrow (x = a) \lor (x = b))$ This is denoted by $SI(x)$ (or $SI^*(x)$ if $x$ is not zero).

- $[7]\forall x, y, z (SI^*(x) \land SI^*(y) \land SI^*(z) \land [(x \subseteq z) \land (z \subseteq y)] \rightarrow x \subseteq y \lor y \subseteq x)$
- $[8]\forall x, y, z (SI^*(x) \land SI^*(y) \land SI^*(z) \land (x = z) \land (z \subseteq y)) \rightarrow x \subseteq y \lor y \subseteq x)$
- $[9]\forall x, a ([SI^*(a) \land a \subseteq x] \rightarrow \exists b (SI(b) \land a \subseteq b \land x \land \forall c (SI(c) \land c \subseteq x, a) \rightarrow c \subseteq b))$

$b$ is called a valuation of $x$.

- $[10]VAL(x, a) \land VAL(y, b) \land [(a = b = 1) \lor (a = b = 1 \land \forall x [SI^*(x) \land b \rightarrow x \subseteq y \land x \subseteq y \land x \subseteq y]) \leftrightarrow VAL(x \cap y, a) \lor VAL(x \cap y, a)$

- $[11]\forall x (x \neq 0 \rightarrow \exists a (P(x) \land a \subseteq x))$

- $[12]\forall x (x \neq 0 \rightarrow \exists a (P(x) \land a \subseteq x))$

- $[13]\forall x \exists y \forall a (P(a) \rightarrow (V(a, x) \neq 0 \rightarrow V(a, s) \neq a) \land (V(a, x) = 0 \rightarrow V(a, s) = 0))$

This $s$ which is unique, is denoted by $\mathbf{SUPP}(x)$.

- $[14]\forall x \forall y \exists a (P(a) \rightarrow (a \subseteq x \rightarrow V(a, z) = V(a, y)) \land a \subseteq x \rightarrow V(a, z) = 0))$

This $z$ which is unique, is denoted by $\mathbf{T}(x, y)$.

- $[15]15 - 1\forall y a, x (SI(a, x) \rightarrow \exists y (SI(a, y) \land x \subseteq y \land x \subseteq y \land x \subseteq y) \land \forall z ((SI(a, z) \land x \subseteq y \land x \subseteq y) \land \forall z ((SI(a, z) \land x \subseteq y \land x \subseteq y)))$

This $y$ which is unique, is denoted by $\mathbf{S}_a(x)$.

- $[16]15 - 2\forall x, a, x (SI(x, a) \land x \neq 0 \rightarrow (V(a, x) = 0) \land (a \subseteq x \rightarrow V(a, y) = 0))$

This $y$ which is unique, is denoted by $\mathbf{I}(x)$.

- $[17]\forall x \forall y \exists a (P(a) \rightarrow (V(a, z) = 0) \lor (a \sqsubseteq x \rightarrow V(a, y) = 0) \land (a \sqsubseteq x \rightarrow V(a, x) = 0))$

proof: [4].

The Quantifier Elimination of the structure of natural numbers in different languages is shown in the following tables so that the theories that admit QE by $\sqrt{\land}$ and, the
theories do not admit QE by \( \times \) is shown.

### Table (III) A Quantifier Elimination Procedure for the Natural numbers at different Language:

<table>
<thead>
<tr>
<th>Theory of</th>
<th>admit QE</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\mathbb{N}, &lt;))</td>
<td>(\times)</td>
</tr>
<tr>
<td>((\mathbb{N}, 0, &lt;))</td>
<td>(\times)</td>
</tr>
<tr>
<td>((\mathbb{N}, 0, S, &lt;))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>((\mathbb{N}, 0, 1, +, \leq))</td>
<td>(\times)</td>
</tr>
<tr>
<td>((\mathbb{N}, 0, 1, +, \leq, (\exists n)_{n \geq 2}))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>((\mathbb{N}, \leq))</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

IV. Decidability of Structure of Integer Numbers in Different Languages

Theorem 1. The structure \((\mathbb{Z}, 0, s)\) admits elimination of quantifier, and it has decidable theory.

**Proof:**

It suffices to consider a formula,

\[ \exists x (\alpha_0 \land \cdots \land \alpha_q) \]

where each \(\alpha_i\) is atomic or is the negation of an atomic formula. In the language of \(\mathcal{N}_0\) the only terms are of the from \(S^ku\) where \( u \) is 0 or a variable. we may suppose that the variable \( x \) occurs in each \(\alpha_i\).

For if \( x \) does not occur in \(\alpha\) then,

\[ \exists x (\alpha \land \beta) \leftrightarrow \alpha \land \exists x \beta \]

Thus each \(\alpha_i\) has the form \(S^mu = S^n u\) or the negation of this equation, where \( u \) is 0 or a variable. We may further suppose \( u \) is different from \( x \) since \( S^m x = S^n x \) could be replaced by \( 0 = 0 \) if \( m = n \) and by \( 0 \neq 0 \) if \( m \neq n \).

Case 1: Each \(\alpha_i\) is the negation of an equation. Then the formula may be replaced by \( 0 = 0 \).

Case 2: There is at least one \(\alpha_i\) not negated; say \(\alpha_0\) is,

\[ S^m x = t \]

where the term \( t \) does not contain \( x \). Since the solution for \( x \) must be non-negative, we replace \(\alpha_0\) by,

\[ t \neq 0 \land \cdots \land t \neq S^{m-1}0 \]

Then in each other \(\alpha_j\) we replace, say, \( S^k x = u \) first by \( S^{k+m} x = S^m u \) which in turn becomes \( S^{k+l} = S^m u \). We now have a formula in which \( x \) no longer occurs, so the quantifier may be omitted. \(\blacksquare\)

Theorem 2. The Theory \((\mathbb{Z}, 0, S, <)\) admits elimination quantifier, and so has a decidable theory and is finitely axiomatizable.

S3. \[ \forall y (y \neq 0 \rightarrow \exists xy = S x) \]

L1. \[ \forall x \forall y (x < S y \leftrightarrow x < y) \]

L2. \[ x \leq 0 \]

L3. \[ \forall x \forall y (x < y \lor y < x \land x = y) \]

L4. \[ \forall x \forall y (x < y \rightarrow y \neq x) \]

L5. \[ \forall x \forall y \forall z (x < y \rightarrow y < z \rightarrow x < z) \]

**Proof:**

We consider a formula,

\[ \exists x (\beta_0 \land \cdots \land \beta_n) \]

where each \(\beta_i\) is atomic or the negation of an atomic formula. The terms are of the form \( S^k u \). Where \( u \) is 0 or a variable. There are two possibilities for atomic formula,

\[ S^k u = S^l t, S^k u < S^l t \]

1. We can eliminate the negation symbol. Replace \( t_1 t_2 \) by \( t_1 = t_2 \lor t_2 < t_1 \) and replace \( t_1 \neq t_2 \) by \( t_1 < t_2 \lor t_2 < t_1 \). By regrouping the atomic formulas and noting that

\[ \exists x (\phi \lor \psi) \leftrightarrow \exists x \phi \lor \exists x \psi \]

we may again reach formulas of the form,

\[ \exists x (\alpha_0 \land \cdots \land \alpha_q) \]
where now, each $\alpha_i$ is atomic
2. We may suppose that the variable $x$ occur in each $\alpha_i$
This is because if $x$ does not occur in $\alpha_i$ then

$$\exists x(\alpha \land \beta) \iff \alpha \land \exists x\beta$$
Furthermore, we may suppose that $x$ occurs on only one side of the equality or inequality $\alpha_i$.

Case 1: Suppose that some $\alpha_i$ is an equality. Then we can proceed as in case 2 of the quantifier-elimination proof

Previous theory

Case 2: Otherwise each $\alpha_i$ is an inequality. Then the formula can be rewritten

$$\exists x(\bigwedge_{i,j} t_i < S^{m_i}x \land \bigwedge_j S^{n_j}x < u_j)$$
we have lower bounds on $x$
If the second conjunction is empty (i.e., if there are no upper bounds on $x$) then we can replace the formula by

$$0 = 0$$
If the second conjunction is empty (i.e., if there are no upper bounds on $x$) then we can replace the formula by

$$\bigwedge_j S^{n_j}0 < u_j$$
which asserts that zero satisfies the upper bounds. Otherwise, we rewrite the formula successively as,

$$\exists x \bigwedge_{i,j} (t_i < S^{m_i}x \land S^{n_j}x < u_j)$$
$$\exists x \bigwedge_{i,j} (S^{m_i}t_i < S^{m_i+n_j}x \land S^{n_j}u_j)$$
$$\bigwedge_{i,j} S^{m_i+n_j}t_i < S^{m_i}u_j \land \bigwedge_j S^{n_j}0 < u_j$$

In each case, we have arrived at a quantifier-free version of the given formula.

V. The additive theory of Integer numbers

Theorem 3. The theory of the structure $\mathbb{Z} = \{+, 0, 1, \leq, \{=\} \}_{\geq 2}$, admits quantifier elimination, and this theory is decidable.

(A1) $x + (y + z) = (x + y) + z$

(A2) $x + y = y + x$

(A3) $x + 0 = x$

(A4) $x + z = y + z \rightarrow x = y$

(A5) $x + y = 0 \rightarrow x = y = 0$

(O1) $x \leq y \iff \exists z (x + z = y)$

(O2) $x \leq y \lor y \leq x$

(O3) $0 \neq 1 \land \forall y (0 \leq y \leq 1 \rightarrow y = 0 \lor y = 1)$

(D1) $\forall x \exists y, z (x = n \cdot y + t \land t < \bar{n})$

Proof: at structure $(\mathbb{Z}; +, 0, 1, \{\leq\})_{\geq 2}$ every term involving $x$ is equal to

$$n \cdot x + t \quad (n \in \mathbb{N})$$

for some $x$-free term $t$ and $n \geq 1$. Therefore, every atomic formula involving $x$ is equall to the following formulas:

$$u = v$$

whence $\phi$ is an atomic formula and $x$ is a variable.

$\phi$ is of the form: $t_0 = s_0$ or $t_0Rs_0$ such that $R \in (\mathbb{Z}; <, \{=\})_{\geq 2}$.

If $\phi$ is $L$ atomic formula with variables $x$ then $\phi$

at $(\mathbb{N}; +, 0, 1, \{\equiv\})_{\geq 2}$ is equivalent to one of the following formulas

$$ax + t = s; ax + t < s; ax + t > s; ax + t \equiv s, n > 1$$

because:

$$n \cdot x + t = m \cdot x + k \Rightarrow a \cdot x + t = s$$

$$n \cdot x + t < m \cdot x + k \Rightarrow a \cdot x + t < s \land s < a \cdot x + t$$

and

$$n \cdot x + t \equiv p \cdot m \cdot x + k \Rightarrow a \cdot x + t \equiv p \cdot s$$

first we remove the following negations signs. Because:

$$t \neq s \iff t < s \lor s < t;$$

$$t \neq s \iff t \equiv s \lor s \equiv t \lor t \equiv s \lor s \equiv t$$

We have $\phi = (\alpha_1 \land \cdots \land \alpha_k) \lor \cdots \lor (\beta_1 \land \cdots \land \beta_k)$ so

$$\exists \phi(x, x_1, \cdots, x_n) \iff \exists \phi((\alpha_1 \land \cdots \land \alpha_k) \lor \cdots \lor (\beta_1 \land \cdots \land \beta_k))$$

Then, we can assume,

$$\exists \phi((\alpha_1 \land \cdots \land \alpha_k) \lor \cdots \lor \exists \phi((\beta_1 \land \cdots \land \beta_k)))$$

and

$$\alpha_i$$

is of the form

$$ax + t \Delta s; \Delta \in \{=, <, >, \equiv\}$$
It can be assumed that any $\alpha_i$ is of the form,

$$ax + t = s; ax + t < s; ax + t > s; ax + t \equiv s, n > 1$$

Thus, by the Main Lemma of Quantifier Elimination it suffices to show that every formula of the form

$$\exists x \bigwedge_{i \in L} x + a_i = b_i \land \bigwedge_{j < p} c_j < L_j x + d_j \land \bigwedge_{k < q} k_e x + s_k < t_k \land \bigwedge_{l < r} M_l x + u_l \equiv m_l, v_l)$$

is equivalent with a quantifier-free formula, Step 1: Unification of coefficients $x$.

We assume $A$ be the least common multiple of the coefficients $x$.

$$\exists x \bigwedge_{i < h} A x + a_i = b_i \land \bigwedge_{j < p} c_j < A x + d_j \land \bigwedge_{k < q} A x + s_k < t_k \land \bigwedge_{l < r} M_l x + u_l \equiv m_l, v_l)$$

Step 2: substituting $A \cdot x$ with $y$:

$$\exists y \bigwedge_{i < h} y + a_i = b_i \land \bigwedge_{j < p} c_j < y + d_j \land \bigwedge_{k < q} y + s_k < \bigwedge_{l < r} y + u_l \equiv m_l, v_l \equiv A 0)$$

By the equivalences

$$t = s \iff ct = cs$$

$$t < s \iff ct < cs$$

$$t \equiv m \land s \equiv m \land ct \equiv cm$$

which are provable, where $c > 0$ is an integer and $s, t$ both $L$ are terms. By adding $x \equiv A 0$, It suffices to eliminate the quantifier of

$$\exists x \bigwedge_{i < h} x + a_i = b_i \land \bigwedge_{j < p} c_j < x + d_j \land \bigwedge_{k < q} x + s_k < t_k \land \bigwedge_{l < r} x + u_l \equiv m_l, v_l)$$
We can assume $h = 0$ Because if $h \neq 0$ then

$$\phi \equiv a_0 < b_0 \land \bigwedge_{i < h} b_0 + a_i = a_0 + b_i \land \bigwedge_{j < p} c_j + a_0 < b_0 + d_j \land \bigwedge_{k < q} b_0 + s_k < a_0 + k_l \land \bigwedge_{l < r} b_0 + u_l \equiv m_l, v_l + a_0$$
quantifier was removed.

Now that $h = 0$ we have:
\[ \exists x (x \neq 0 \land x \neq 0) \]

We can assume that \( p \leq 1 \) because for \( p > 1 \), we have,

\[ \exists x (c_0 < x + d_0 \land d_1 < x + d_1 + \psi(x)) \]

This is decidable. Then, the following theorem completely axiomatizes the structure \((\mathbb{Z}^*, \cdot, 1)\) and, moreover, its theory admits quantifier elimination, and so the theory \( \langle Z, \cdot \rangle \) is.

VI. The Decidability of The multiplicative theory of integers

The multiplicative theory of integers, is called Skolem arithmetic. Skolem claimed decidability for the Theory of \((\mathbb{N}; \times, =)\), using quantifier elimination technique. The first decidability proof appeared in Mostowski’s work on a product of theories of decision problems. Feferman and Vaught generalize this idea. Later Cegielski axiomatized the multiplicative theory and proved quantifier elimination.

Theorem 5. The following theorem completely axiomatizes the structure \((\mathbb{Z}^*, \cdot, 1)\) and, moreover, its theory admits quantifier elimination, and so the theory \( \langle Z, \cdot \rangle \) is.

Proof. [3]

Table (I) : A Quantifier Elimination Procedure for the integers:

<table>
<thead>
<tr>
<th>Theory of</th>
<th>Language</th>
<th>admit QE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle \mathbb{Z}, + \rangle )</td>
<td>( L = (+) )</td>
<td>( \times )</td>
</tr>
<tr>
<td>( \langle \mathbb{Z}, +, \cdot, 1 \rangle )</td>
<td>( L = (0; 1; +; \cdot; \leq; &lt;) )</td>
<td>( \times )</td>
</tr>
<tr>
<td>( \langle \mathbb{Z}, 0, 1, +, \leq, \equiv_{n \geq 2} \rangle )</td>
<td>( L = (0; 1; +, \leq, \equiv_{n \geq 2}) )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( \langle \mathbb{Z}, x \rangle )</td>
<td>( L = (\cdot, x, p) )</td>
<td>( \checkmark )</td>
</tr>
</tbody>
</table>
VII. Decidability of Structure of rational Numbers in Different Languages

Theorem 1. Theory \((Q, <)\) admits elimination of quantifier.

Proof:
1: Identify the terms

In structure \(\langle Q; < \rangle\), every term involving \(x\) is equal to,
\[
  n \cdot x + t \quad (n \in N)
\]
where \(x\) does not appear in \(t\)

2: Identify Atomic Formulas and Delete \(~\) if possible

All atomic formulas are,
\[
  u < v \quad u = v
\]
First, we eliminate the inequality behind the atoms. Because,
\[
  x \neq y \leftrightarrow x < y \land y < x
\]

3: Simplify atomic formulas

So the following formula must be eliminated quantifier.
\[
  \exists x(\bigwedge_i r_i < m_i \cdot x + s_i \land \bigwedge_j n_j \cdot x + t_j < u_j \land \bigwedge_k l_k \cdot x + v_l = w_l)
\]
4: Uniform the coefficients \(x\)

Let \(M\) is Multiply the coefficients by \(x\)

\[
  M = \prod_i m_i \prod_j n_j \prod_k l_k
\]

\[
  r_i = \frac{M}{m_i} \cdot r_i < Mx + \frac{M}{m_i} s_i
\]

\[
  Mx + \frac{M}{m_i} t_j < \frac{M}{m_i} u_j
\]

\[
  Mx + \frac{M}{m_i} v_l = \frac{M}{m_i} w_l
\]

So the following formula admits quantifier elimination
\[
  \exists x(\bigwedge_i r_i < Mx + s_i \land \bigwedge_j n_j Mx + t_j < u_j \land \bigwedge_k l_k Mx + v_l = w_l)
\]

5: Remove the coefficient \(x\)

\(y = Mx\). So, we have
\[
  \exists y(\bigwedge_i r_i < y + s_i \land \bigwedge_j n_j y + t_j < u_j \land \bigwedge_k l_k y + v_l = w_l)
\]
We use the following equations
\[
  t = s \leftrightarrow ct = cs
\]
so
\[
  \exists x(\bigwedge_i r_i < x + s_i \land \bigwedge_j x + t_j < u_j \land \bigwedge_i x + v_l = w_l)
\]
6: Identification Phrases included \(x\)

\[
  r_i < x + s_i \leftrightarrow r_i + t_j + v_l < x + s_i + t_j + v_l
\]
\[
  x + t_j < u_j \leftrightarrow x + s_i + t_j + v_l < u_j + s_i + v_l
\]
\[
  x + v_l = w_l \leftrightarrow x + s_i + t_j + v_l = s_i + t_j + w_l
\]
\[
  P = s_i + t_j + v_l
\]
so
\[
  \exists x(\bigwedge_i r_i < x + P \land \bigwedge_j x + P < u_j \land \bigwedge_i x + P = w_l)
\]

7: Identify the states

\(l \neq 0 \equiv \bigwedge_i r_i < w_0 \land \bigwedge_j w_0 < u_j \land \bigwedge_i w_0 = w_l \equiv True\)

\(l = j = 0 \equiv \bigwedge_i r_i < x \land \bigwedge_j x < u_j \equiv True\)

\(l = i = 0 \equiv \bigwedge_i r_i < x \land \bigwedge_j x < u_j \equiv True\)

\(l, i, j \neq 0 \equiv \exists x(\bigwedge_i r_i < x \land \bigwedge_j x < u_j)\)

Decidability Mathematical Structures: Structures The Theory of Addition \((Q, +)\):

The Theory of Addition \((Q, +)\) admits elimination of quantifier.

Proof:
Step 1: Identify the terms

In structure \(\langle Q, + \rangle\), every term involving \(x\) is equal to,
\[
  n \cdot x + t \quad (n \in N)
\]
where \(x\) does not appear in \(t\)

Step 2: Identify Atomic Formulas

All atomic formulas are,
\[
  u = v \quad u \neq v
\]

Step 3: Simplify atomic formulas

So the following formula must be eliminated quantifier.
\[
  \exists x(\bigwedge_i k_i \cdot x + v_l = w_l \land \bigwedge_i m_j \cdot x + n_j \neq s_j)
\]

\(\equiv \exists x(\bigwedge_i k_i \cdot x = u_l \land \bigwedge_i m_j \cdot x \neq t_j)\)

Step 4: Uniform the coefficients \(x\)

Let \(M\) is Multiply the coefficients by \(x\)

\[
  M = \prod_i k_i \prod_j m_j
\]

So the following formula admits quantifier elimination
\[
  \exists x(\bigwedge_i M \cdot x = u_l \land \bigwedge_i M \cdot x \neq t_j)
\]

Step 5: Remove the coefficient \(x\)

\(y = Mx\). So, we have
\[
  \exists y(\bigwedge_i y = u_l \land \bigwedge_i y \neq t_j)
\]
We use the following equations
\[
  t = s \leftrightarrow ct = cs
\]
\(t \neq s \leftrightarrow ct \neq cs\)
so
\[
  \exists x(\bigwedge_i x = u_l \land \bigwedge_i x \neq t_j)\]
Step 6: Identify the states

\[ i \neq 0 \equiv \bigwedge_i u_0 = u_i \land \bigwedge_j u_0 \neq t_j \]

\[ i = 0, j \neq 0 \equiv True \]

Theorem 4. The Theory \( \langle Q; +, -0, < \rangle \) admits quantifier elimination. And so has decidable theory. Proof: The following formula must be eliminated quantifier.

\[ \exists x (\bigwedge_i n_i \cdot x = t_i \land \bigwedge_j n_j \cdot x < m_j \cdot x + s_j) \]

Similar to previous proofs, admits quantifier elimination. And so has decidable theory.

Theorem 5. The Theory \( \langle Q^+; \times, 1.0^{-1}, \{ R_n \}_{n \geq 2} < \rangle \) admits quantifier elimination. And so has decidable theory.

Proof: [1]

Similar to previous proofs, admits quantifier elimination. And so has decidable theory.

Theorem 6. The theory of the rational numbers \( \langle Q, \subseteq \rangle \) is decidable, and moreover axiomatizable.

Proof: quantifier elimination for The theory of the rational numbers \( \langle Q^+, \subseteq \rangle \): 

\[ p \subseteq q \leftrightarrow \exists m \in \mathbb{N}^+ (p \cdot m = q) \]

Structure \( \langle Q^+, \subseteq \rangle \) is equivalent with structure \( \langle Q^+, \times \rangle \). First, we conclude decidability \( \langle Q^+, \times \rangle \) of paper [1] so, the structure \( \langle Q^+, \subseteq \rangle \) based on the article [1] is decidable.

We will express the axioms of rational numbers as follows:

Positive rational numbers are formed from two parts, the integer part whose denominator is one, and the Intrevel Algebra of rational numbers. The positive part of all the properties of natural numbers. So we have the axioms of \( \langle \mathbb{N}, \subseteq \rangle \) and atomless Boolean Algebra and the axioms of \( \langle Q^+, \times \rangle \).

so we have the following axioms for \( \langle Q^+, \subseteq \rangle \):

1. \[ \forall x (x \subseteq x) \]
2. \[ \forall x, y (x \subseteq y \land x \rightarrow y = y) \]
3. \[ \forall x, y, z (x \subseteq y \subseteq z \rightarrow x \subseteq z) \]
4. \[ \forall x, y, z (x \subseteq y \land \forall t (x \subseteq y \rightarrow t \subseteq z) \rightarrow z = x \land y) \]
5. \[ \forall x, y, z (x \subseteq y \land \forall t (x \subseteq y \rightarrow t \subseteq z) \rightarrow z = x \land y) \]
6. \[ \forall x (\bot \subseteq x) \]
7. \[ \forall x, y (\forall (\exists (S_I (z) \subseteq x \rightarrow z \subseteq y))) \rightarrow x \subseteq y \]
8. \[ \forall x, y, z (S_I (x) \subseteq S_I (y) \subseteq S_I (z) \land [x \subseteq y \subseteq z \subseteq x, y, y]) \rightarrow x \subseteq y \lor y \subseteq x \]
9. \[ \forall x, a (S_I (a) \cdot a \subseteq x) \rightarrow \exists b (S_I (a) \cdot a \subseteq b \subseteq x \land \forall (S_I (c) \cdot c \subseteq x, a \subseteq c \subseteq b)) \]
10. \[ \forall a, b (a = b = 1) \rightarrow (a = a \lor b = b) \]
11. \[ \forall x (x \neq 0 \rightarrow \exists a (P(a) \land a \subseteq x)) \]
12. \[ \forall x (x \neq 0 \rightarrow \exists a (P(a) \land a \subseteq x)) \]
13. \[ \forall x (x \neq 0 \rightarrow \exists a (P(a) \land a \subseteq x)) \]

\( x \) is a strict partial order.

The axiomatization is a modeling in the first order language \( L, \langle Q^+, \subseteq \rangle \), will be a model of this language.

\[ \forall x (x \subseteq x) \]
\[ \forall x, y (x \subseteq y \rightarrow x = y) \]
\[ \forall x, y, z (x \subseteq y \subseteq z \rightarrow x \subseteq z) \]
\[ \forall x, y, z (x \subseteq y \subseteq z \subseteq x, y, y) \rightarrow x \subseteq y \lor y \subseteq x \]
\[ \forall x, a (S_I (a) \cdot a \subseteq x) \rightarrow \exists b (S_I (a) \cdot a \subseteq b \subseteq x \land \forall (S_I (c) \cdot c \subseteq x, a \subseteq c \subseteq b)) \]
\[ \forall a, b (a = b = 1) \rightarrow (a = a \lor b = b) \]
\[ \forall x (x \neq 0 \rightarrow \exists a (P(a) \land a \subseteq x)) \]

Axioms 4 and 5 are equivalent to the following axioms in set theory:

\[ \forall A, B \subseteq C (C \subseteq A, B \land \forall T (T \subseteq A, B \rightarrow T \subseteq C)), C = A \land B \]
\[ \forall A, B \subseteq C (C \subseteq A, B \land \forall T (T \subseteq A, B \rightarrow T \subseteq C)), C = A \land B \]

Definition: \( x \) is p-prime and denoted by \( PR(p, x) \) iff we have \( P(p) \land \forall q ((P(q) \land p \neq q) \rightarrow q^2x = p^n) \)

Definition: An element \( x \) is join-irreducible iff it satisfies \( \forall a, b (a = a \lor b \rightarrow (a = a) \lor (b = b)) \). This is denoted by \( S_I (x) \) or \( S^* (x) \) if \( x \neq 1 \)

Lemma 1: \( x \) is p-prime number why?
If $x$ is not $p$-primary number then we have:

$$x = \prod_i p_i^{\alpha_i} = \prod_j p_j^{\alpha_j}$$

then $x$ is not join-irreducible.

$[7]\forall x, y, \forall z(SI(z)[z \subseteq x \rightarrow z \subseteq y]) \rightarrow x \subseteq y$

Proposition 1: $\forall x, y, z \forall SI(z)[z \subseteq x \leftrightarrow z \subseteq y]$)

$[8]\forall x, y, z(SI^*_x(x) \land SI^*_y(y) \land SI^*_z(z) \land [(x, y \subseteq z \subseteq x, y)] \rightarrow x \subseteq y \land y \subseteq x$)

Proposition 2: $\exists x \subseteq y \lor y \subseteq x$

$[9]\forall x, y, z(SI^*_x(a) \land a \subseteq x) \rightarrow \exists b(SI^*_y(b) \land a \subseteq b \subseteq x \land \forall c(SI^*_c(c) \land c \subseteq x \rightarrow c \subseteq b)$

Proposition 3: $\forall x, y, z(x \subseteq y \leftrightarrow \forall a VAL(x, a) \land a \subseteq y)$

Proof:

$[10]\forall x, y, z(x \subseteq y \leftrightarrow \forall a VAL(x, a) \land a \subseteq y)$

Atom: $a \neq 0\forall x \leq a \rightarrow (x \neq a \land x = a)$, we denotes by $A(a)$.

$[11]\forall x, y, z(x \subseteq y \leftrightarrow (x \land y) \subseteq (x \land y)$

Lemma 2. $\exists a(P(a)a \subseteq x) \Rightarrow SI^*_x(a)$

Proof:

$[12]\forall a, b, c \Rightarrow b \subseteq a \Rightarrow b = a \lor b = 1$

Definition. A nonempty set $L$ with two binary operations read "meet", "join" respectively on $L$ is called a lattice if it satisfies the following identities:

$L_1: (a \land y) \lor (b \land y) = y \lor x$  
$L_2: (a \land y) \lor (y \land z) = (x \lor y) \land z$  
$L_3: (a \land y) \lor (b \land x) = (a \land x) \lor (b \land y)$

And it is said that a lattice is complete if every subset of $L$ has an infimum and a supremum. And it is said that a lattice $L$ is distributive if the following conditions hold for any $x, y, z \in L$:

$$x \land (y \lor z) = (x \land y) \lor (x \land z)$$

$$x \lor (y \land z) = (x \lor y) \land (x \lor z)$$

Definition. We say $A$ Boolean Algebra is a completed distributive lattice. A Boolean Algebra is a structure $(B, \land, \lor, \land, 0, 1)$ that satisfies the following identities:

$B_1: (B, \land, \lor)$ lattice is distributive

$B_2: x \land 0 = 0, x \lor 1 = 1$

$B_3: x \land x = 0, x \lor x = 1$

Boolean algebras was first introduced by Boole 3 in an effort to automate reasoning. Since that they have been extensively studied, and have proved fundamental in numerous application areas.

We consider the structure $(B, \land, \lor, 0, 1)$ where $B$ is a set.

$\land, \lor$ are binary operations in $B$, is a monary operation in $B$ and $0, 1 \in B$. So, we have for $x, y, z \in B$:

$$x \land y = y \land x$$

$\land (y \land z) = (x \land y) \land z$

$$x \land (y \land z) = (x \land y) \land (x \land z)$$

Definition. An atom in Boolean algebra is an element $x$ such that $x \neq 0$ and if $y < x$, then $y = 0$. We will define a first order formula $At(x)$ with meaning $x$ is an atom:

$$x \neq 0 \land y < x \rightarrow x \land y = 0$$

And if $B$ be a Boolean Algebra, it is said that $B$ is atomic iff for every $y \in B, y \neq 0$, there is $x \in B$ such that $x$ is an atom and $x \leq y$.

I am trying to describe the concept of an atom in a Boolean algebra. Let $I = \{a, b\}$ be a set, and $P(I) = \emptyset, \{a\}, \{b\}, \{a, b\}$ be one of the possible algebras of subsets of $I$. We have $A$ being an algebra of set, it is also a Boolean algebra. $\{a\}$ is atom. In any partially ordered set with a minimum element, an atom is element that covers the minimum element.

And let $X = \{a, b, c\}$ be a set, and $A = \emptyset, \{a\}, \{b, c\}, X$ be one of the algebras of subsets of $X$. Now, an element in is a atom if, for every $y \in A$, either $x \land y = x$ or $x \land y = 0$. So $\{a\}$ and $\{b, c\}$ are the atoms of $A$ so, every singleton set is an atom.

Definition 5. A Boolean Algebra is atomless if it has no atoms. Every atomless Boolean algebras with more than one element must be infinite. Indeed, the unit $1$ is different from zero, so there is a non-zero element $p_1$ strictly below $1$ otherwise, $1$ would be an atom. Because $p_1$ is not zero, there must be a non-zero element $p_2$ strictly below $p_1$; otherwise, $p_1$ decreasing sequence of elements $1 > p_1 > p_2 > \cdots$.

The interval algebra of the real numbers is atomless. Also the interval algebra of rational numbers is atomless, or the regular open algebra of the sace of real numbers is atomless.

A. Mereology

Mereology is sometimes understood as a formal, or logical, analysis of the part-to-whole relation. A theory of parts and wholes should tell us what items can be parts. Since something is a part only if it is a part of a whole, a

3George Boole (/buːl/; 2 November 1815 – 8 December 1864) was a largely self-taught English mathematician, philosopher and logician, most of whose short career was spent as the first professor of mathematics at Queen’s College, Cork in Ireland. He worked in the fields of differential equations and algebraic logic, and is best known as the author of The Laws of Thought (1854) which contains Boolean algebra.
mereology will tell us what items can be wholes. Classic extensial mereology was introduced by S.Lesniewski 1916 and developed by him in the years thereafter. It was reformulated as calculus of individuals by H.Leonard and N.Goodman in 1940.

Classic mereology is equivalent (isomorphic) to Boolean Algebra complete (without 0 ) complete (without 0 ) \(^4\).

Mereology is about the relation of part to whole between objects or individuals. Any whole is a part of itself. And the empty object is a part of every whole. Any part of a whole different from the empty part and from the whole itself is a proper part.

1) Axiomatizations short of classical mereology:

Classical mereology is a formal theory of the part-whole relation. There are various different definitions, and various axiomatizations for classical mereology.

Let \( M \) be a non-empty set and \( \subseteq \) a binary relation in \( M \). The first axioms state that the relation \( \subseteq \) is a strict partial order in the set \( M \). That the relation \( \subseteq \) is asymmetric, transitive, and irreflexive in \( M \), i.e., the relation \( \subseteq \) satisfies in \( M \) conditions (as \( \subseteq \)), (t \( \sqcap \)), and (irr \( \subseteq \)):

for arbitrary \( x, y \):

relation \( \subseteq \) is reflexive and antisymmetric, i.e., we have:

\[
\begin{align*}
(A_1) & \forall x \in M, \ (r \subseteq) \\
(A_2) & \forall x, y \in M, \ x \subseteq y \land y \subseteq x \implies x = y, \ (antis \subseteq) \\
(A_3) & \forall x, y, z \in M, \ x \subseteq y \land y \subseteq z \implies x \subseteq z, \ (t \subseteq) \).
\end{align*}
\]

If every proper part of \( x \) is part of \( y \), and every proper part of \( y \) is part of \( x \), then \( x = y \).

The conjunction of \((A_1)\) and \((A_3)\) is logically equivalent to the

\[
x = y \iff (x \subseteq y \land y \subseteq x)
\]

Which follows from above axioms.

There was close connection between mereologies and complete Boolean algebras. The connection was known to Tarski. We have a first-order (or higher) language that includes a special 2-place predicate \( \sqsubseteq \), meant to represent “is part of” or “is a part of”.

The axioms of mereology are these of complete Boolean algebra, provided with the following interpretation:

- \( x \sqsubseteq y : \) a part of \( y \)
- \( x \cup y : \) Mereological sum or union of \( x \) and \( y \)
- \( x \cap y : \) Mereological product or overlap of \( x \) and \( y \)

0: the empty individual
1: the universal individual
1 \(- x : \) complement of \( x \), the universal individual minus \( x \)
2) Mereological structures and complete Boolean algebras: By a Boolean algebra we mean an algebraic structure \( \mathcal{A} = \langle A, +, \cdot, 0, 1 \rangle \), in which \( A \) is a non-empty set, + and \( \cdot \) are binary operations on \( A \), – is a unary operation on \( A \), while 0 and 1 are elements of \( A \); moreover, the following axioms are satisﬁes

\[
\begin{align*}
x + y &= y + x \\
x \cdot y &= y \cdot x \\
x + (y + z) &= (x + y) + z \\
x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\
x \cdot (y + z) &= (x \cdot y) + (x \cdot z) \\
x + (y + z) &= (x + y) + (x + z) \\
1 + 0 &= 1 \\
x + (-x) &= 0
\end{align*}
\]

Boolean algebra is a set partially ordered by the relation \( \leq \) by means of conditions (def \( \leq \)) \( \leq \) \( a \leq b \iff a + b = b \iff a \cdot b = a \).

Let us remind that for any \( a, b \in A \):

\( a \leq b \iff -b \leq -a, -(a) = a \).

We will prove now that from a complete Boolean algebra by deleting zero we will obtain a mereological structure:

Theorem 1. Let \( \mathcal{A} = \langle A, +, \cdot, 0, 1 \rangle \), be a complete Boolean algebra. Assume \( \sqsubseteq \leq |A| \setminus \{0\} \), where the relation \( \leq \) is deﬁned by (def \( \leq \)). Then \( \langle A \setminus \{0\}, \sqsubseteq \rangle \) is a mereological structure. After adding zero element to some mereological structure we will «turn» it into a complete Boolean algebra.

The decidability of the structure of rational numbers in different languages is shown in the following tables so that the theories of decidable by \( \sqrt{\ } \) and, undecidable theories by \( \times \) is shown.

<table>
<thead>
<tr>
<th>Structures</th>
<th>The decidability of the structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>Q, &lt;</td>
</tr>
<tr>
<td>(</td>
<td>Q, +</td>
</tr>
<tr>
<td>(</td>
<td>Q, \times</td>
</tr>
<tr>
<td>(</td>
<td>Q, &lt;+, +</td>
</tr>
<tr>
<td>(</td>
<td>Q, \sqcap, +</td>
</tr>
</tbody>
</table>

\( \text{Table I} \)
VIII. Decidability of structures of real numbers at different Language

Theorem 1. The structure \((\mathbb{R}; <)\) admits quantifier elimination, so has a decidable theory.

Proof: Quantifier Elimination Procedure for \((\mathbb{R}; <)\)

\[ \exists x \left[ \bigwedge_{i=1}^{m} x = x_i \land \bigwedge_{j=1}^{n} z_j < x \land \bigwedge_{k=1}^{p} x < u_k \right] \]

If \(m > 0\):

\( R \models \varphi \iff \bigwedge_{i=1}^{m} x = x_i \land \bigwedge_{j=1}^{n} z_j < x \land \bigwedge_{k=1}^{p} x < u_k \)

If \(m = 0\) then we distinguish 3 subcases:

1. If \(n = 0\), then \( R \models \varphi \iff \text{true, because } R \text{ has no minimum.} \)
2. If \(p = 0\), then \( R \models \varphi \iff \text{true, because } R \text{ has no maximum.} \)
3. If \(n > 0\) and \(p > 0\), then

\( R \models \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} x < u_k \)

Proof:

\( \rightarrow \) is transitive.

\( \leftarrow \) there exists \( x \in R \) with \( \max_j z_j < x < \min_k u_k \).

Theorem 2. The structure \((\mathbb{R}; <, +)\) admits quantifier elimination, and so is decidable. [1]

Proof:

It suffices to prove that the following formula is equivalent to a formula without quantifier.

\[ \exists y \left[ \bigwedge_{i=1}^{m} p_i \cdot x \land \bigwedge_{j=1}^{n} q_j \cdot x < s_j \land \bigwedge_{k=1}^{p} r_k \cdot x = u_k \right] \]

Consider the coefficients \( p_i, q_j, r_k \) are equal. As a result, we have the following equivalence

\[ \exists y \left[ \bigwedge_{i=1}^{m} x \land \bigwedge_{j=1}^{n} y < s_j \land \bigwedge_{k=1}^{p} y = u_k \right] \]

Now the quantifier of this formula is easily removed.

Theorem 3. The structure \((\mathbb{R}; +)\) admits quantifier elimination, and so is decidable. [1]

Proof:

Theorem 4. The structure \((\mathbb{R}; \times)\) admits quantifier elimination, and so is decidable. [1]

Theorem 5. The structure \((\mathbb{R}, \times, 0^{-1}, 0, 1, -1, P^5)\) admits quantifier elimination, and so is decidable. [1]

Table (III): A Quantifier Elimination Procedure for the Reals Numbers at different Language:

<table>
<thead>
<tr>
<th>Theory of</th>
<th>Language</th>
<th>admit QE</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\mathbb{R}; &lt;))</td>
<td>( L = (\langle) )</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>((\mathbb{R}; 0, +, -))</td>
<td>( L = (0; +; -) )</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>((\mathbb{R}; 0, +, -))</td>
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<td>(\checkmark)</td>
</tr>
<tr>
<td>((\mathbb{R}; \times))</td>
<td>( L = (\times, 0^{-1}, 0, 1, -1, P^5) )</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

TABLE I

IX. Deciding Boolean Algebras:

Boolean algebras were first introduced by Boole in an effort to automate reasoning. Since that they have been extensively studied, and have proved fundamental in numerous application areas.

At the study of Boolean algebras, we show decidability and undecidability questions for the class of Boolean algebras. And we describe an algorithm for deciding the Boolean algebras. A basic result of Tarski is that the elementary theory of Boolean algebras is decidable. Even the theory of Boolean algebras with a distinguished ideal is decidable. On the other hand, the theory of a Boolean algebra with a distinguished subalgebra is undecidable. Both the decidability results and undecidability results extend in various ways to Boolean algebras in extensions of first-order logic.

Definition: Atoms are exactly the minimal nonzero elements, i.e. \(a\) is an atom iff \(0 \leq a\) and \(0 < x \iff x = a\).

An algorithm for deciding the theory Atomic Boolean algebras: We present an algorithm and show how decide. We have some definitions:

- \( L = \{\cap, \cup, \setminus, A \setminus B, =, 0, C_n, E_n, n \in \mathbb{N}^+\} \)

- \( A(a) \iff \forall x [x \subseteq a \rightarrow x = 0 \lor x = a] \land a \neq 0 \)

- \( C_n(x) \equiv \exists a_1 \cdots a_n (\bigwedge_{i<j} a_i \neq a_j \land \bigwedge_{i=1}^{n} A(a_i) \land \bigwedge_{i=1}^{n} a_i \subseteq X) \)

- \( E_n(x) \equiv C_n(x) \land \neg C_{n+1}(x) \)

- The next step of the algorithm is eliminate =:

  - Because: \(a = b \iff a \subseteq b \land b \subseteq a\)

  - eliminate \(\subseteq\):

    - Because: \(a \subseteq b \iff a \setminus b = 0 \iff E_0(a-b)\)

  - And eliminate: \(\neg\):

    - Because: \(\neg C_n(x) \iff \bigvee_i E_i(x) \iff \neg E_n(x) \iff C_{n+1}(x) \lor \bigvee_i E_i(x) \)

Quantifier-Elimination for Boolean formulas is as follows:

\( L = \{\cap, \cup, \setminus, =, \{C_n\}, \{E_n\}, n \in \mathbb{N}^+\} \)

We have the following

\( R = \{|C_n|_{n \geq 0} \mid |E_n|_{n \geq 0}\} \)

\( F = \{A \mid F_1 \land F_2 \lor F_1 \lor F_2 \lor F_2 \land F \lor F \} \)

\( A = \{B_1 = B_2 \mid B_1 \subseteq B_2 \} \land E_n(B) \}

\( B = \{|x| \mid B_1 \cap B_2 \mid E_n \mid B_2 \mid B^c\} \)

\( n = \{0, 1, \ldots\} \)

So it is enough to consider only the following formulas:

\( C_n(x) = |x| \geq n, E_n(x) = |x| = n \). Contradictions of literals
are eliminated according to the above definitions.

\[-|x| = n \iff |x| = 0 \lor \cdots \lor |x| = n - 1 \lor |x| \geq n + 1\]
\[-|x| \geq n \iff |x| = 0 \lor \cdots \lor |x| = n - 1\]

So at this step we've removed some of the relationships as follow:
1. Eliminate equality
   \[a = b \iff a \subseteq b \land b \subseteq a\]
2. Delete inclusion
   \[a \subseteq b \iff |a \cap b| = 0\]
3. Eliminate contradictions
   \[-C_n(x) \iff V_{i<n} E_i(x)\]
   \[-E_n(x) \iff C_{n+1}(x) \lor V_{i<n} E_i(x)\]

Language to Quantifier-Elimination
\[\cap, \cup, \emptyset, \{C_n\}_{n \geq 0}, \{E_n\}_{n \geq 0}\]

Quantifier Elimination:
In the resulting formula, each set variable \(x\) occurs in some term \(|t(x)|\), each set expression \(|t(x)|\) as a union of cubes (regions in the Venn diagram). The cubes have the form \(C_{n+1}(x)\) where \(x^c\) is either \(x^c\) or \(x_i^c\); there are \(m = 2^n\) cubes. The resulting formula is then equivalent to
\[\exists x(\land_i C_{n_i}(t_i(x)) \land (\land_j E_{m_j}(t'_j(x)))\]
for example:
\[\exists x([x \cap c] \geq 3 \land [x \cap c] \geq 7 \land [c - x] = 2)\]
\[\exists x(C_3(x \land c) \land C_7(x \land c) \land E_2(c - x)) \equiv C_9(c)\]
\[\exists x(C_5(x \land c) \land C_7(x \land d) \land E_6(c - x)) \equiv C_{11}(c) \land C_7(d)\]

More explained in the table below

<table>
<thead>
<tr>
<th>The main formula</th>
<th>Deleted form</th>
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<tbody>
<tr>
<td>[\exists y \cdots [x \cap y] \geq k \land [x \cap y'] \geq l \cdots]</td>
<td>[</td>
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<tr>
<td>[\exists y \cdots [x \cap y] = k \land [x \cap y'] \geq l \cdots]</td>
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<td>[</td>
</tr>
<tr>
<td>[\exists y \cdots [x \cap y] = k \land [x \cap y'] = l \cdots]</td>
<td>[</td>
</tr>
</tbody>
</table>

TABLE I

A Boolean Algebra is atomless if it has no atoms. Every atomless Boolean algebra with more than one element must be infinite. Indeed, the unit 1 is different from zero, so there is a non-zero element \(p_1\) strictly below 1; otherwise, 1 would be an atom. Because \(p_1\) is not zero, there must be a non-zero element \(p_2\) strictly below \(p_1\); otherwise, \(p_1\) decreasing sequence of elements \(1 > p_1 > p_2 > \cdots\).

Atomless boolean algebra: we have the interval algebra of rational number. The interval algebra of the rational number is atomless.

Lemma 1. We have in every Boolean algebra:
\[p \subset P, q \subset Q, P \cap Q = \emptyset, P \cdot Q = 0 \implies p + q + Q\]

Proof: \(p + q \subseteq P + Q\)

To show:
\[p + q \neq P + Q\]
We assume \(p + q = P + Q\) so.
\[(P + Q) \cdot \bar{p} = P \cdot \bar{p} + Q \cdot \bar{p}\]
Because \(Q \cap p = \emptyset\) we have \(P \cdot \bar{p} + Q = (p + q) \cdot \bar{p}\)
\[p + q \bar{p} = q\]
\[(q \cap p) = \emptyset; q \bar{p} = 0\]
\[\iff P \cdot \bar{q} + Q \cdot \bar{p} = 0\]
Which contradicts with the assumption

Lemma 2. The following formulas are equivalent:

\[\exists x([x \cap s] = 0 \land \bigwedge_{i=1}^{m} u_i x \neq 0 \land \bigwedge_{j=1}^{n} v_j x \neq 0)\]
\[r s = 0 \land \exists y([\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \land \bigwedge_{j=1}^{n} v_j \bar{s} y \neq 0])\]

Proof:
\[\iff\]
If there is \(x\) such that,
\[r x = 0 \land s x = 0 \land \bigwedge_{i=1}^{m} u_i x \neq 0 \land \bigwedge_{j=1}^{n} v_j x \neq 0\]
\[r x = 0 \land rs \bar{x} = 0 \implies \forall s x = 0 \implies rs(1) = 0 \implies rs = 0\]
\[u_i x \neq 0 \implies u_i x(r + \bar{r}) \neq 0\]
\[u_i x \bar{r} \neq 0\]
\[\implies \exists x([\bigwedge_{i=1}^{m} u_i \bar{r} x \neq 0 \land \bigwedge_{j=1}^{n} v_j \bar{s} x \neq 0])\]
\[\iff\]
Suppose \(rs = 0\) there is \(y\) such that,
\[\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \land \bigwedge_{j=1}^{n} v_j \bar{s} y \neq 0\]

We put,
\[x = r \cdot (s + y)\]
\[\bar{x} = r + \bar{y}\]
\[s \cdot (r + \bar{y})\]

We show,
\[\bigwedge_{i=1}^{m} u_i x \neq 0 \land \bigwedge_{j=1}^{n} v_j x \neq 0\]
\[u_i x = u_i \bar{r} (s + y) = u_i \bar{r} s + u_i \bar{r} y \geq u_i \bar{r} y \neq 0\]
\[v_j x = v_j \bar{s} (r + \bar{y}) = v_j \bar{s} r + v_j \bar{s} y \geq v_j \bar{s} y \neq 0\]

Theorem 1. The theory of atomless Boolean algebra in the language \(L = \{0, 1, \land, \lor, \neg, =\}\) accepts the quantifier elimination.

Proof:
\[F = \{A | F_1 = F_2 | | F_3 \lor F_2 | \neg F | \exists F | \forall F\}\]
\[A = \{t_1 = t_2\}\]
\[T = \{x | 0 \lor t_2 \lor t_1 \land t_2 \land t_2\}\]
we have:
\[t = s\] and \(t \neq s\) so
Theorem: For any formula $\varphi$, the quantifier-elimination is decidable.

Proof: Let $\varphi$ be a formula. We assume that $\varphi$ is decidable and that it contains no quantifiers. Then there is a finite sequence of formulas $\varphi_0, \varphi_1, \ldots, \varphi_n$, where $\varphi_n$ is a tautology, such that $\varphi_0 = \varphi$ and $\varphi_{i+1} = \varphi_i \rightarrow \psi_i$, where $\psi_i$ is quantifier-free. We define a sequence of sets $C_0, C_1, \ldots, C_n$, where $C_0 = \emptyset$, $C_n = \emptyset$, and $C_{i+1} = C_i \cup \{\psi_i\}$ for $i < n$. We claim that $\varphi$ is true if and only if $\varphi_n$ is true.

Indeed, if $\varphi$ is true, then there is a valuation $\mathcal{V}$ such that $\mathcal{V}(\varphi) = 1$. Then $\mathcal{V}(\varphi_0) = 1$, and since $\varphi_{i+1} = \varphi_{i} \rightarrow \psi_{i}$, we have $\mathcal{V}(\varphi_{i+1}) = 1$ for all $i < n$. Therefore, $\mathcal{V}(\varphi_n) = 1$, and hence $\varphi_n$ is true.

Conversely, if $\varphi_n$ is true, then there is a valuation $\mathcal{V}$ such that $\mathcal{V}(\varphi_n) = 1$. Then $\mathcal{V}(\varphi_{i+1}) = 1$ for all $i < n$, and hence $\mathcal{V}(\varphi_i) = 1$ for all $i < n$. Therefore, $\mathcal{V}(\varphi) = 1$, and hence $\varphi$ is true.

We have shown that the quantifier-elimination is decidable.

The above proof proved the theory of Boolean algebras by the quantifier-elimination is decidable.

The first-order theory of Boolean algebras, established by Alfred Tarski in 1940 (found in 1940 but announced in 1949).

<table>
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<tr>
<th>Theory of</th>
<th>Proved by</th>
<th>A method of proof</th>
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<tr>
<td>Boolean algebras</td>
<td>by Tarski in 1949</td>
<td>model completeness</td>
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</table>

X. Conclusion

This paper we deals with Decidability Quantifier elimination is a very powerful property, as it helps in the proof of decidability. We study the decidability structures of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ over different languages and we present some explicit axiomatization for them. The structures $\langle \mathbb{N}, + \rangle, \langle \mathbb{Z}, + \rangle, \langle \mathbb{Q}, + \rangle, \langle \mathbb{R}, + \rangle, \langle \mathbb{N}, \times \rangle, \langle \mathbb{Z}, \times \rangle, \langle \mathbb{Q}, \times \rangle, \langle \mathbb{R}, \times \rangle$ and $\langle \mathbb{R}, +, \times \rangle$ are decidable and thus axiomatizable by
recursively enumerable sets of sentences. By Godel incompleteness theorem the theory of the structure \( \langle \mathbb{N}, +, \times \rangle \) is not decidable and by the four square theorem of Lagrange the set \( \mathbb{N} \) is definable in \( \langle \mathbb{Z}, +, \times \rangle \). So \( \langle \mathbb{Z}, +, \times \rangle \) is not decidable as well. Also, the theory of structure \( \langle \mathbb{Q}, +, \times \rangle \) is not decidable, since \( \mathbb{Z} \) is definable in it, and we consider the theories of the structures \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) in the language \( \{<, \times\} \). The theory of the structure \( \langle \mathbb{N}, <, \times \rangle \) is not decidable, and so no computability enumerable set of sentences can axiomatize this structure because:

\[
x < y \iff \exists z ((z + z = z) \land (x + z = y))
\]

and \( \leq \) is definable in the structure \( \langle \mathbb{N}, +, \times \rangle \) or \( \langle \mathbb{Z}, +, \times \rangle \). The undecidability of the theory of the structure \( \langle \mathbb{N}, <, \times \rangle \) also implies the undecidability of the theory of the structure \( \langle \mathbb{Z}, <, \times \rangle \). The structure \( \langle \mathbb{R}, <, \times \rangle \) is decidable (results of a theorem of Tarski).

Tarski proved in 1929 (published in 1984) that the theory of real closed fields in the theory \( Th(\mathbb{R}, +, 0, \leq) \) and is decidable. The structure of \( \langle \mathbb{Q}, < \rangle \) is decidable, by quantifier elimination. The theory \( Th(\mathbb{Q}, +) \) admits quantifier elimination. \( Th(\mathbb{Q}, \times) \) is decidable, and \( \langle \mathbb{Q}^+, \leq \rangle \) is \( \langle \mathbb{Q}^+, \times \rangle \) and \( a \sqsubset b \iff \exists x (a \times x = b) \). We review that \( \langle \mathbb{Q}^+, \leq \rangle \) is axiomatizable and decidable. So this theory is complete.

Boolean algebras are mathematical structures important in many branches of mathematics and computer science. Boolean algebras are closely related to Boolean lattices and Boolean rings. We prove here that the theory of atomic Boolean algebras is decidable and furthermore admits elimination of quantifiers down to the language including the Boolean operations and the relations expressing the height or size of an object, \( |x| \geq n \) and \( |x| = n \).

And we prove here that the theory of atomless Boolean algebras is decidable, too.

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