N-fold generalized Darboux transformation and asymptotic analysis of the degenerate solitons for the Sasa-Satsuma equation in the dispersive nonlinear media

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N-fold generalized Darboux transformation and asymptotic analysis of the degenerate solitons for the Sasa-Satsuma equation in the dispersive nonlinear media

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Abstract

In this paper, the Sasa-Satsuma equation, which is applied to the dynamics of the deep water waves, and the pulse propagation in the optical fibers and generally in the dispersive nonlinear media, is investigated. Starting from the first-order Darboux transformation, we construct an N-fold generalized Darboux transformation (GDT) for the Sasa-Satsuma equation, where N is a positive integer. Through the obtained N-fold GDT, we derive three kinds of the semirational solutions, which describe the second-order degenerate solitons, the third-order degenerate solitons, and the interaction between the second-order degenerate solitons and one soliton, respectively. We graphically illustrate the above three kinds of semirational solutions and investigate them through the asymptotic analysis, from which we find that the characteristic lines of the semirational solutions are composed of the straight lines and curves. Expressions of the characteristic lines, positions, amplitudes, slopes, positions and phase shifts of the asymptotic solitons are presented through the asymptotic analysis.

Keywords: Sasa-Satsuma equation; Dispersive nonlinear media; Generalized Darboux transformation; Asymptotic analysis; Degenerate solitons

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1. Introduction

Among the nonlinear evolution equations, the nonlinear Schrödinger (NLS) equation, which describes the slowly varying wave envelopes in the dispersive nonlinear media, has attracted the researchers’ attention [1–5]. Through adding certain higher-order nonlinear terms in the fixed proportions, i.e., the third-order dispersion, self-frequency shift and self-steepening, to the NLS equation, researchers have extended the NLS equation to the Sasa-Satsuma equation [6]. With the higher-order nonlinear terms concluded, the Sasa-Satsuma equation has been considered to have the applications in the dynamics of the deep water waves [7, 8], and the pulse propagation in the optical fibers [9, 10] and generally in the dispersive nonlinear media [11].

Sasa-Satsuma equation has been represented as [12–22]

\[
ig_t + \frac{1}{2}q_{xx} + |q|^2 q + i\sigma \left[q_{xxx} + 3 (|q|^2)_x q + 6|q|^2 q_x \right] = 0, \tag{1}
\]

where \(q = q(x, t)\) denotes the complex envelop of the wave field, \(x\) and \(t\) are the independent variables depending on the related physical contexts, the positive real parameter \(\sigma\) represents the integrable perturbations of the NLS equation, \(i = \sqrt{-1}\) and the subscripts denote the partial differentiations.

For Eq. (1), a Darboux transformation (DT) and the shape-changing phenomena between the solitons and breathers have been studied [12]; a binary DT and the quasi-grammians solutions have been derived [13]; a generalized Darboux transformation (GDT), the high-order rogue waves and solitons have been exhibited [14]; Liouville integrability and gauge equivalence have been investigated [15]; anti-dark solitons, Mexican-hat solitons have been presented [16]; rational W-shaped solitons on a continuous-wave background have been studied [17]; W-shaped solitons generated from a weak modulation have been discussed [18]; rogue waves have been derived [19]; solitons on a background and their limiting cases have been exhibited [20]; twisted rogue-wave pairs have been investigated [21]; interactions between the bright two solitons have been discussed [22]. A Lax pair for Eq. (1) has been offered as [12]

\[
\Phi_x = U\Phi, \quad U = \lambda U_1 + U_0, \\
\Phi_t = V\Phi, \quad V = \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + V_0, \\
U_1 = \frac{1}{6\sigma} \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad U_0 = \begin{pmatrix} 0 & -e^{-i\theta} q & -e^{i\theta} q^* \\ e^{i\theta} q^* & 0 & 0 \\ e^{-i\theta} q & 0 & 0 \end{pmatrix}, \\
V_3 = \frac{1}{4\sigma} U_1, \quad V_2 = \frac{1}{4\sigma} U_0, \quad V_1 = -\frac{1}{12\sigma} U_1 + \sigma[(U_0)_x, U_1] + \sigma U_0[U_0, U_1], \\
V_0 = -\frac{1}{12\sigma} U_0 + \sigma[(U_0)_x, U_0] - \sigma (U_0)_{xx} + 2\sigma U_0^3, \quad \theta = \frac{x}{6\sigma} - \frac{t}{108\sigma^2}, \tag{2}
\]

where \(\lambda\) is a complex spectral parameter, \(U\) and \(V\) are two \(3 \times 3\) matrices, \(\Phi = (f, g, h)^T\) is a vector eigenfunction, \([U, V]\) represents \(UV - VU\), \(f\), \(g\) and \(h\) are the complex functions of \(x\).
and $t$, and the superscript “$T$” denotes the transpose of the matrix. From the zero curvature equation $U_t - V_x + [U, V] = 0$, Eq. (1) has been obtained [12].

However, to our knowledge, for Eq. (1), $N$-fold GDT, which is different from the GDT in Ref. [14], degenerate solitons and corresponding asymptotic analysis have not been studied, where $N$ is a positive integer. In Section 2, starting from the first-order DT, we will construct an $N$-fold GDT (with multiple spectral parameters involved) for Eq. (1). In Section 3, three kinds of the semirational solutions, which describe the second-order degenerate solitons, the third-order degenerate solitons, and the interaction among the second-order degenerate solitons and one soliton, respectively, will be derived and graphically illustrated. Furthermore, we will perform the asymptotic analysis for the above three kinds of semirational solutions. In Section 4, our conclusions will be offered.

2. $N$-fold GDT for Eq. (1)

To construct an $N$-fold GDT (with multiple spectral parameters involved) for Eq. (1), we firstly investigate the first-order DT. Motivated by Ref. [23], we construct the first-order DT for Eq. (1) as

$$
\Phi[1] = R[1] \Phi = (\lambda I - S) \Phi, \quad S = \Theta J \Theta^{-1},
$$

where $I$ is the third-order identity matrix, $R$ is the first-order DT matrix, the superscript “$-1$” denotes the inverse matrix, the superscript “$*$” denotes the complex conjugate, “[1]” denotes the first-order iteration, $\lambda_1$ is a complex spectral parameter and $(f_1, g_1, h_1)^T$ is a solution of Lax Pair (2) at $\lambda = \lambda_1$.

Through DT (3), we convert Lax Pair (2) into a new one as

$$
\Phi_{[1]} = U_{[1]} \Phi_{[1]}, \quad U_{[1]} = U_1 + U_0[1],
$$

$$
\Phi_{[1]} = V_{[1]} \Phi_{[1]}, \quad V_{[1]} = V_3 \lambda^2 + V_2 \lambda^2 + V_1 \lambda + V_0[1],
$$

in which the forms of $U_{[1]}$ and $V_{[1]}$ are the same as $U$ and $V$, respectively, except that $q$ is replaced with $q_{[1]}$. We find that Lax Pair (4) possesses the same matrix form as Lax Pair (2). Combining DT (3), Lax Pairs (2) with (4), we obtain

$$
U_{[1]} R_{[1]} - R_{[1]} U_{[1]} = 0,
$$

$$
V_{[1]} R_{[1]} - R_{[1]} V_{[1]} = 0.
$$

Afterwards, we expand Eqs. (5) and set the coefficients of each subterm of $\lambda$ to be zero, then,
we have
\[ U_1^{[1]} = U_1, \]
\[ U_0 - U_0^{[1]} + U_1^{[1]} S - SU_1 = 0, \]
\[ S_n + SU_0 - U_0^{[1]} S = 0, \]
\[ V_3^{[1]} = V_3, \]
\[ V_2 - V_2^{[1]} + V_3^{[1]} S - S^{[1]} V_3 = 0, \]
\[ V_1 - V_1^{[1]} + V_2^{[1]} S - S V_2 = 0, \]
\[ V_0 - V_0^{[1]} + V_1^{[1]} S - S V_1 = 0, \]
\[ -S_i^{[1]} + V_0^{[1]} S - S V_0 = 0. \]
(6)

After simplifying and calculating Eqs. (6), we find that the equivalent condition of Eqs. (6) is
\[ S_{1,2} = S_{3,1} = -S_{1,3} = -S_{2,1}, \]
(7)
and the relationship between the first-order solution \( q^{[1]} \) and the seed solution \( q \) is
\[ S_{1,2} = \frac{2\sigma i(q - q^{[1]})}{e^{i\theta}} = \frac{(\lambda_1 - \bar{\lambda}_1^*) f g^*}{ff^* + gg^* + hh^*}, \]
(8)
where \( S_{\xi,\iota} \) denotes the elements in the \( \xi \) row and the \( \iota \) column of \( S \), and \( \xi, \iota = 1, 2, 3 \). Combining Equivalent Condition (7) with DT (3), we obtain \( f g^* = f^* h \). That is to say, we must carefully select the values of the spectral parameter \( \lambda_1 \) and eigenfunction \( \Phi \) to ensure that Equivalent Condition (7), i.e., \( f g^* = f^* h \), is met.

Next, motivated by Refs. [24–26], \( n \) spectral parameters are used to construct an \( N \)-fold GDT for Eq. (1), where \( n \) is a positive integer and \( n \leq N \). Among the \( n \) spectral parameters, the \( k \)th spectral parameter \( \lambda_k \) will be iterated \( m_k \) times, where \( \sum_{k=1}^{n} m_k = N \) and \( k = 1, 2, 3, \ldots, n \).

Afterwards, \( N \)-fold DT matrix for Eq. (1) is constructed as
\[ R^{[N]} = R^{(m_n)}_n \cdots R^{(1)}_{n-1} \cdots R^{(m_2)}_2 \cdots R^{(1)}_1 \cdots R^{(m_1)}_1 \cdots R^{(1)}_1, \]
(9)
where
\[ R^{(j)}_k = (\lambda - \lambda_k^*) I - (\lambda_k - \lambda_k^*) \Xi^{(j)}_k \Xi^{(j)\dagger}_k, \]
\[ \Xi^{(j)}_k = \begin{pmatrix} f_{k,j-1} \\ g_{k,j-1} \\ h_{k,j-1} \end{pmatrix} = \sum_{\gamma=0}^{j-1} \Gamma_{j-1,\gamma} \frac{d}{d\mu^\gamma} \begin{pmatrix} f_k \\ g_k \\ h_k \end{pmatrix} \bigg|_{\mu=0}, \]
\[ \Gamma_{\gamma} = \sum_{\sum \delta_k^\gamma = \gamma} K^{(j-1)}_k \cdots K^{(1)}_k \cdots K^{(m_1)}_1 \cdots K^{(1)}_1, \quad K^{(b)}_k = \begin{cases} R^{(b)}_k, & \delta_k^b = 0, \\ I, & \delta_k^b = 1, \end{cases} \]
\[ \delta \] is the Kronecker symbol, the superscript “ \( \dagger \) ” denotes the complex conjugate of the matrix, the superscript “ \([N]\)” denotes the \( N \)th-order iteration, \( R^{(j)}_k \) and \( K^{(j)}_k \) denote the \( j \)th iteration of
the matrix $T$ and $S$ with the spectral parameter as $\lambda_k$, $(f_k, g_k, h_k)^T$ is an eigenfunction for Lax Pair (2) with the spectral parameter as $\lambda_k + \mu$, $\mu$ is a small parameter, $f_k$’s, $g_k$’s and $h_k$’s are the complex functions of $x$ and $t$, $j = 1, 2, \ldots$, $\max(m_k), b = 1, 2, \ldots$, $\max(m_k, j - 1)$ and $\gamma$ is a nonnegative integer.

Combining Lax Pair (2) with $N$-Fold Darboux Matrix (9), we derive the $N$-fold solutions for Eq. (1) as

$$q^{[N]} = q - \frac{i}{2\sigma} \frac{\text{Det} (\mathcal{F})}{\text{Det} (\mathcal{G})} e^{i\theta},$$

$$\mathcal{G} = \left( \begin{array}{ccc} G_1^T & \frac{\text{d}}{\text{d} \lambda_1} G_1^T & \frac{\text{d}}{\text{d} \lambda_1} \ldots \frac{\text{d}^{m_1-1}}{\text{d} \lambda_1^{m_1-1}} G_1^T \\ \vdots & \vdots & \vdots \\ G_n^T & \frac{\text{d}}{\text{d} \lambda_n} G_n^T & \frac{\text{d}}{\text{d} \lambda_n} \ldots \frac{\text{d}^{m_n-1}}{\text{d} \lambda_n^{m_n-1}} G_n^T \end{array} \right)^T,$$

$$\mathcal{F} = \left( \begin{array}{ccc} F_1^T & \frac{\text{d}}{\text{d} \lambda_1} F_1^T & \frac{\text{d}}{\text{d} \lambda_1} \ldots \frac{\text{d}^{m_1-1}}{\text{d} \lambda_1^{m_1-1}} F_1^T \\ \vdots & \vdots & \vdots \\ F_n^T & \frac{\text{d}}{\text{d} \lambda_n} F_n^T & \frac{\text{d}}{\text{d} \lambda_n} \ldots \frac{\text{d}^{m_n-1}}{\text{d} \lambda_n^{m_n-1}} F_n^T \end{array} \right)^T,$$

$$G_k = \left( \begin{array}{cccc} \lambda_k^{N-1} f_k & \lambda_k^{N-1} g_k & \lambda_k^{N-1} h_k & \ldots \\ - (\lambda_k^{N-1} g_k) & (\lambda_k^{N-1} f_k)^* & 0 & \ldots \\ - (\lambda_k^{N-1} h_k) & 0 & (\lambda_k^{N-1} f_k)^* & \ldots \\ \cdots & \cdots & \cdots & \cdots \end{array} \right),$$

$$F_k = \left( \begin{array}{cccc} \lambda_k^{N-1} f_k & - \lambda_k^{N} f_k & \lambda_k^{N-1} h_k & \ldots \\ - (\lambda_k^{N-1} g_k) & (\lambda_k^{N} g_k)^* & 0 & \ldots \\ - (\lambda_k^{N-1} h_k) & 0 & (\lambda_k^{N} g_k)^* & \ldots \\ \cdots & \cdots & \cdots & \cdots \end{array} \right) \tag{10},$$

where Det denotes the determinant of the matrix.

Through setting the values of $N$, $n$ and $m_k$ in Solutions (10), we can obtain different types of the semirational solutions for Eq. (1). When $m_k \geq 2$, i.e., the complex spectral parameter $\lambda_k$ is iterated more than twice, $\lambda_k$ corresponds to the $m_k$th-order degenerate solitons, degenerate breathers or rogue waves in Solutions (10); when $m_k = 1$, $\lambda_k$ corresponds to the one soliton or one breather in Solutions (10). However, in this paper, all the next discussions are only about the solitons on the zero background. In sum, $N$-Fold Darboux Matrix (9) and Solutions (10) form an $N$-Fold GDT for Eq. (1).

3. Degenerate solitons for Eq. (1) and corresponding asymptotic analysis

In order to construct the degenerate solitons for Eq. (1), we take the seed solutions for Eq. (1) as $q[0] = 0$. Therefore, eigenfunction $\Phi (\lambda_k + \mu)$ with $\lambda = \lambda_k + \mu$ for Lax Pair (2) can be presented as

$$\Phi (\lambda_k + \mu) = \left( \begin{array}{c} f_k \\ g_k \\ h_k \end{array} \right) = \left( \begin{array}{c} e^{- \frac{(\lambda_k + \mu)}{3\sigma} x} - \frac{\gamma}{24\sigma^2} e^{\frac{(\lambda_k + \mu)}{12\sigma^2} t} \\ e^{- \frac{(\lambda_k + \mu)}{6\sigma} x} - \frac{\gamma}{24\sigma^2} e^{\frac{(\lambda_k + \mu)}{12\sigma^2} t} \\ e^{- \frac{(\lambda_k + \mu)}{6\sigma} x} - \frac{\gamma}{24\sigma^2} e^{\frac{(\lambda_k + \mu)}{12\sigma^2} t} \end{array} \right) \tag{11}.$$

5
Next, we expand eigenfunction $\Phi(\lambda_k + \mu)$ with a small parameter $\mu$ as follows:

$$\begin{pmatrix} f_k \\ g_k \\ h_k \end{pmatrix} = \begin{pmatrix} f_{k,0} \\ g_{k,0} \\ h_{k,0} \end{pmatrix} + \begin{pmatrix} f_{k,1} \\ g_{k,1} \\ h_{k,1} \end{pmatrix} \mu + \begin{pmatrix} f_{k,2} \\ g_{k,2} \\ h_{k,2} \end{pmatrix} \mu^2 + \cdots,$$

$$\begin{pmatrix} f_{k,0} \\ g_{k,0} \\ h_{k,0} \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\lambda_k}{6}x} \left( \frac{\lambda_k}{36\sigma^2} - \frac{\lambda_k^3}{12\sigma^4} \right) t \\ e^{\frac{i\lambda_k}{6}x} \left( \frac{\lambda_k}{72\sigma^2} - \frac{\lambda_k^3}{24\sigma^4} \right) t \\ e^{\frac{i\lambda_k}{6}x} \left( \frac{\lambda_k}{72\sigma^2} - \frac{\lambda_k^3}{24\sigma^4} \right) t \end{pmatrix}, \quad \begin{pmatrix} f_{k,1} \\ g_{k,1} \\ h_{k,1} \end{pmatrix} = \begin{pmatrix} -i\left( \frac{2x}{3\sigma} + \frac{9\lambda_k^2}{36\sigma^2} - \frac{x}{2\sigma} \right) f_{k,0} \\ i\left( \frac{x}{6\sigma} + \frac{9\lambda_k^2}{72\sigma^2} - \frac{x}{2\sigma} \right) g_{k,0} \\ i\left( \frac{x}{6\sigma} + \frac{9\lambda_k^2}{72\sigma^2} - \frac{x}{2\sigma} \right) h_{k,0} \end{pmatrix}, \quad (12)$$

$$\begin{pmatrix} f_{k,2} \\ g_{k,2} \\ h_{k,2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \left[ \frac{i\lambda_k}{\sigma^2} + \frac{(12x + 3\lambda_k^2 t - t^2)}{648\sigma^2} \right] f_{k,0} \\ \frac{1}{4} \left[ \frac{i\lambda_k}{2\sigma^2} + \frac{(12x + 3\lambda_k^2 t - t^2)}{2592\sigma^2} \right] g_{k,0} \\ \frac{1}{4} \left[ \frac{i\lambda_k}{2\sigma^2} + \frac{(12x + 3\lambda_k^2 t - t^2)}{2592\sigma^2} \right] h_{k,0} \end{pmatrix}, \cdots$$

We find that $f_{k,1}$, $g_{k,1}$, $h_{k,1}$, $f_{k,2}$, $g_{k,2}$, $h_{k,2}$ and so on are the mixtures of polynomials and exponential functions. Therefore, Solutions (10), which contain both the polynomials and exponential functions, are called the semirational solutions.

### 3.1 The second-order degenerate solitons for Eq. (1)

As we set $n = 1$ and $m_1 = N = 2$ in Solutions (10), i.e., only one complex spectral parameter $\lambda_1$ is iterated two times, the second-order degenerate solitons for Eq. (1) can be derived.

It should be noted that in order to ensure the establishment of Condition (7), we set $\lambda_1 = i\rho$ as a pure imaginary number, where $\rho$ is the non-zero real number. Combining $\lambda_1 = i\rho$, Solutions (10) with Eigenfunctions (12), we obtain the second-order degenerate soliton solutions for Eq. (1) as

$$q^{[1]} = 6\sigma pe^{i\theta} \left( \frac{2(\beta - 24\sigma^2)e^\alpha - (\beta + 24\sigma^2)e^{-\alpha}}{288\sigma^4 e^{2\alpha} + 72\sigma^4 e^{-2\alpha} + 288\sigma^4 + \beta^2} \right),$$

$$\alpha = \rho (t + 3\rho^2 t - 12\sigma x) \quad \frac{24\sigma^2}{24\sigma^2}, \quad \beta = \rho t(1 + 9\rho^2) - 12\sigma \beta x, \quad \theta = \frac{x}{6\sigma} - \frac{t}{108\sigma^2}. \quad (13)$$

Figs. 1 show the second-order degenerate solitons under three kinds of parameter conditions. From Figs. 1, we find that as $\sigma$ increases from $\frac{2}{3}$ to 1, then to $\frac{3}{2}$, the velocities of the second-order degenerate solitons decrease and the widths of them increase.
Figs. 1. The second-order degenerate solitons via Solution (13) with $\rho = 2$, (a) $\sigma = \frac{2}{3}$; (b) $\sigma = 1$; (c) $\sigma = \frac{3}{2}$.

Next, we will analyze the properties of Solutions (13). Motivated by Refs. [27, 28], we will perform the following asymptotic analysis procedure to investigate the asymptotic behaviors of Solutions (13).

It can be seen from Figs. 1 that characteristic lines of the second-order degenerate solitons are not the straight lines, which will be proved as follows:

We consider an arbitrary line $L$: $c_1 t - \frac{\rho x}{2\sigma} = c_2$, where $c_1$ and $c_2$ are the arbitrary real numbers. Since $\beta = 24\sigma^2\alpha + 6\rho^3 t$, Solutions (13) are dependent only on the variables $\alpha$ and $t$. Thus, it is necessary to investigate the behavior of $\alpha$ alone $L$ as $|t| \to \infty$. In view of

$$\alpha - (c_1 t - \frac{\rho x}{2\sigma}) = \left(\frac{\rho}{24\sigma^2} + \frac{\rho^3}{8\sigma^2} - c_1\right) t,$$

as $|t| \to \infty$, the value of $\alpha$ is

$$\alpha = \begin{cases} 
-\infty, & c_1 > \frac{\rho}{24\sigma^2} + \frac{\rho^3}{8\sigma^2}, \\
O(1), & c_1 = \frac{\rho}{24\sigma^2} + \frac{\rho^3}{8\sigma^2}, \\
+\infty, & c_1 < \frac{\rho}{24\sigma^2} + \frac{\rho^3}{8\sigma^2},
\end{cases}$$

(14)

where $O(1)$ denotes that the two quantities are of the same order, i.e., the ratio limit of two quantities tends to a nonzero constant.

As shown in Expressions (14), value of $\alpha$ can be $+\infty$, $-\infty$ or $O(1)$ at infinity on line $L$. Hence, we can calculate the dominant behaviors of Solutions (13) corresponding to the above three cases of $\alpha$ as

$$q^{[1]} = \begin{cases} 
-p\rho e^{i\theta} \frac{4\sigma^2\alpha + 4\sigma^2 + \rho^3 t}{2\sigma^3 e^{-\alpha}}, & \alpha \to -\infty, \\
\sigma\rho e^{i\theta} \frac{(2\sqrt{2}\rho^3 t + 8\sqrt{2}\sigma^2\alpha) \sinh (\alpha + \frac{1}{2}\ln 2) - 16\sqrt{2}\cosh (\alpha + \frac{1}{2}\ln 2)}{\rho^3 t^2 + 8\rho^3\sigma^2\alpha t + 16\sigma^4\alpha^2 + 16\sigma^4\cosh^2 (\alpha + \frac{1}{2}\ln 2)}, & \alpha = O(1), \\
\rho e^{i\theta} \frac{4\sigma^2\alpha - 4\sigma^2 + \rho^3 t}{4\sigma^3 e^\alpha}, & \alpha \to \infty.
\end{cases}$$

(15)
We know that $e^{-\alpha} \gg |4\sigma^2\alpha + 4\sigma^2 + \rho^3 t|$ as $\alpha \to -\infty$, $e^{\alpha} \gg |4\sigma^2\alpha - 4\sigma^2 + \rho^3 t|$ as $\alpha \to \infty$, and $\rho^6 t^2 \gg |(2\sqrt{2}\rho^3 t + 8\sqrt{2}\sigma^2)\sinh (\alpha + \frac{1}{2}\ln 2)|$ as $\alpha = O(1)$, $t \to \pm \infty$. That is to say, no matter which of the three cases in Expressions (14), $q^{[1]}$ will approach 0 as $|t| \to \infty$ alone the line $L$. In summary, characteristic lines of Solutions (13) are not the straight lines.

Next, we assume that the characteristic lines of Solutions (13) are curve $C$ in the $x - t$ plane. Along the curve $C$ to infinity, $e^\alpha$ and $\beta$ approach infinity. Thus, we consider the balance between $e^\alpha$ and $\beta$ as

$$\frac{\beta}{e^{p\alpha}} \sim O(1), \quad |t| \to +\infty,$$

where $p$ is a real variable constant depending on the values of $e^\alpha$ and $\beta$. According to the relationship between $p$ and $\pm 1$, we classify and obtain the following six dominant behaviors of Solutions (13) as

$$q^{[1]} = \begin{cases} 
12\sigma \rho e^{i\theta} \frac{e^\alpha}{\beta}, & p > 1, \alpha \to \infty, \\
12\sigma \rho e^{i\theta} \frac{\beta e^\alpha}{288\sigma^4 e^{2\alpha} + \beta^2}, & p = 1, \alpha \to \infty, \\
\rho e^{i\theta} \frac{\beta}{24\sigma^3 e^\alpha}, & 0 < p < 1, \alpha \to \infty, \\
-\rho e^{i\theta} \frac{\beta}{12\sigma^3 e^{-\alpha}}, & -1 < p < 0, \alpha \to -\infty, \\
-6\sigma \rho e^{i\theta} \frac{\beta e^{-\alpha}}{72\sigma^4 e^{-2\alpha} + \beta^2}, & p = -1, \alpha \to -\infty, \\
-6\sigma \rho e^{i\theta} \frac{e^{-\alpha}}{\beta}, & p < -1, \alpha \to -\infty.
\end{cases} \quad (16)$$

From Eqs. (16), we find that Solutions (13) behave as the solitons with the stable amplitudes only when $p = \pm 1$. Whether $p = 1$ or $-1$, $t$ may approach to $+\infty$ or $-\infty$. Therefore, we calculate the following four asymptotic solitons as

$$q^{[1]} \to q^{[1]+}_{n,1} = \frac{\rho}{2\sqrt{2}\sigma} e^{i\theta} \text{sech} \left( \alpha + \ln \frac{12\sqrt{2}\sigma^2}{\beta} \right), \quad \beta e^{-\alpha} = O(1), \alpha \to +\infty, t \to +\infty, \beta > 0,$$

$$q^{[1]} \to q^{[1]-}_{n,1} = \frac{\rho}{2\sqrt{2}\sigma} e^{i\theta} \text{sech} \left( \alpha + \ln \frac{-\beta}{6\sqrt{2}\sigma^2} \right), \quad \beta e^\alpha = O(1), \alpha \to -\infty, t \to -\infty, \beta < 0,$$

$$q^{[1]} \to q^{[1]+}_{n,2} = -\frac{\rho}{2\sqrt{2}\sigma} e^{i\theta} \text{sech} \left( \alpha + \ln \frac{\beta}{6\sqrt{2}\sigma^2} \right), \quad \beta e^\alpha = O(1), \alpha \to -\infty, t \to +\infty, \beta > 0,$$

$$q^{[1]} \to q^{[1]-}_{n,2} = -\frac{\rho}{2\sqrt{2}\sigma} e^{i\theta} \text{sech} \left( \alpha + \ln \frac{12\sqrt{2}\sigma^2}{-\beta} \right), \quad \beta e^{-\alpha} = O(1), \alpha \to +\infty, t \to -\infty, \beta < 0. \quad (17)$$

3D figure and contour figure of the second-order degenerate solitons via Solutions (13) are shown in Figs. 2.
From Asymptotic Solitons (17), we can get the following properties of the four bright-type asymptotic solitons $q_{n,1}^\pm$ and $q_{n,2}^\pm$ as follows:

(a) Amplitudes:

$$A(q_{n,1}^\pm) = A(q_{n,2}^\pm) = \frac{|\rho|}{2\sqrt{2}\sigma}.$$ 

(b) Slopes:

$$S(q_{n,1}^\pm) = \frac{1 + 3\rho^2}{12\sigma} + \frac{12\sigma \rho^2}{24\sigma^2 - |\beta|}; \quad S(q_{n,2}^\pm) = \frac{1 + 3\rho^2}{12\sigma} + \frac{12\sigma \rho^2}{24\sigma^2 + |\beta|};$$

(c) Characteristic lines:

$$q_{n,1}^+ : 12\sqrt{2}\sigma^2 e^\alpha - \beta = 0; \quad q_{n,1}^- : 6\sqrt{2}\sigma^2 e^{-\alpha} + \beta = 0;$$

$$q_{n,2}^+ : 6\sqrt{2}\sigma^2 e^{-\alpha} - \beta = 0; \quad q_{n,2}^- : 12\sqrt{2}\sigma^2 e^\alpha + \beta = 0.$$

(d) Phase shifts $P(\chi)$ between $q_{n,\chi}^+$ and $q_{n,\chi}^-$ ($\chi = 1, 2)$:

$$P(1) = -P(2) = 2 \ln \frac{12\sigma^2}{|\beta|}.$$ 

We find that the four asymptotic solitons own the same amplitude. In view of $P(1)$ and $P(2)$ are opposite under the same $|\beta|$, and the value of $S(q_{n,\chi}^+)$ at $\beta$ is equal to that of $S(q_{n,\chi}^-)$ at $-\beta$, we can infer that the interaction between $q_{n,1}$ and $q_{n,2}$ is elastic.

We have

$$S(q_{n,1}^\pm) < S(\alpha) = \frac{1 + 3\rho^2}{12\sigma}, \quad \text{when } |\beta| > 24\sigma^2;$$

$$S(\alpha) = \frac{1 + 3\rho^2}{12\sigma} < S(q_{n,2}^\pm) < S(\beta) = \frac{1 + 9\rho^2}{12\sigma}. \quad (18)$$

Therefore, asymptotic solitons $q_{n,2}^\pm$ are located between the two straight lines $l_1: \alpha = 0$ and $l_2: \beta = 0$, and asymptotic solitons $q_{n,1}^\pm$ are located outside of the straight line $l_1$, as shown in
Figs. 2. However, since the characteristic lines of $q_{n,1}^+$ and $q_{n,2}^+$ are the transcendental equations, we cannot get the expressions of the distances between the asymptotic solitons $q_{n,1}^+$ and $q_{n,2}^+$ (or $q_{n,1}^-$ and $q_{n,2}^-$).

3.2 The third-order degenerate solitons for Eq. (1)

As we set $n = 1$ and $m_1 = N = 3$ in Solutions (10), i.e., only one complex spectral parameter $\lambda_1$ is iterated three times, the third-order degenerate solitons for Eq. (1) can be derived. Combining $\lambda_1 = i \rho$, Solutions (10) with Eigenfunctions (12), we obtain the third-order degenerate soliton solutions for Eq. (1) as

$$q^{[1]} = -\rho e^{i\theta} \frac{\Upsilon_1 e^{2\alpha} + \Upsilon_2 e^{-2\alpha} + \Upsilon_3}{41472 \sigma^9 (8e^{3\alpha} + e^{-3\alpha}) + \Pi_1 e^{-\alpha} + \Pi_2 e^{\alpha}},$$

$$\alpha = \rho (t + 3\rho^2 t - 12\sigma x), \quad \beta = \rho t (1 + 9\rho^2) - 12\sigma \rho x,$$

$$\Upsilon_1 = 576 \sigma^4 \left[ (\beta - 36\sigma^2)^2 - 432 (\sigma^4 + t \sigma^2 \rho^3) \right],$$

$$\Upsilon_2 = 144 \sigma^4 \left[ (\beta + 36\sigma^2)^2 - 432 (\sigma^4 - t \sigma^2 \rho^3) \right],$$

$$\Upsilon_3 = 20736 \sigma^4 (40 \sigma^4 + 9t^2 \rho^6 - t \rho^3 \beta) - (\beta^2 + 576 \sigma^4)^2,$$

$$\Pi_1 = \sigma \left[ \beta^4 + 48 \sigma^2 \beta^3 + 20736 t \sigma^4 \rho^3 \beta + 186624 t^2 \sigma^4 \rho^6 + 864 (2 \sigma^4 + t \sigma^2 \rho^3) \beta^2 + 248832 \sigma^8 \right],$$

$$\Pi_2 = 2 \sigma \left[ \beta^4 - 48 \sigma^2 \beta^3 + 20736 t \sigma^4 \rho^3 \beta + 186624 t^2 \sigma^4 \rho^6 + 864 (2 \sigma^4 - t \sigma^2 \rho^3) \beta^2 + 248832 \sigma^8 \right].$$

Figs. 3 show the third-order degenerate solitons under three different parameter conditions. Similar to the second-order degenerate solitons shown in Figs. 1, we find that as $\sigma$ increases from $\frac{2}{3}$ to 1, then to $\frac{3}{2}$, the velocities of the third-order degenerate solitons decrease and the widths of them increase. However, different from Figs. 1, the third-order degenerate solitons contain three branches.

Similar to Section 3.1, we will perform the following asymptotic analysis procedure to inves-
tigate the asymptotic behaviors of Solutions (19). As can be seen from Figs. 3, characteristic lines of the third-order degenerate solitons consist of a straight line and two curves. Along the straight line \( \alpha = \frac{\rho (t + 3 \rho^2 - 12 \sigma^2 t)}{24 \sigma^2} \) = constant to \( t \to \pm \infty \), we only keep the highest power of \( t \) in the numerator and denominator of Solutions (19). Then, the following two asymptotic solitons are derived as
\[
q^{[1]} \to q^{\pm}_{n,1} = -\frac{\rho e^{i\theta}}{2\sqrt{2} \sigma} \text{sech}(\alpha + \frac{1}{2} \ln 2), \quad t \to \pm \infty.
\]
(20)

From Asymptotic Solitons (20), we know that \( q^{\pm}_{n,1} \) are the two bright solitons and have the same characteristic line, amplitude, velocity and width.

Since \( \beta = 24 \sigma^2 \alpha + 6 \rho^3 t \), we also consider that Solutions (19) are dependent only on the two variables \( \alpha \) and \( t \). Next, we assume the balance between \( e^\alpha \) and \( \beta \) as
\[
\frac{\beta}{e^{p\alpha}} \sim O(1), \quad |t| \to +\infty,
\]
where \( p \) is a real variable constant depending on the values of \( e^\alpha \) and \( \beta \). According to the relationship between \( p \) and \( \pm \frac{1}{2} \), we classify and obtain the following six dominant behaviors of Solutions (19):

\[
q^{[1]} = \begin{cases}
-\frac{\rho e^{i\theta} 576 \sigma^4 \beta^2 e^{2\alpha} - \beta^4}{2 \sigma \beta^4 e^{\alpha}}, & p > \frac{1}{2}, \alpha \to \infty, \\
-\frac{\rho e^{i\theta} 288 \sigma^3 \beta^2 e^{2\alpha}}{16588 \sigma^8 e^{3\alpha} + \beta^4 e^{\alpha}}, & p = \frac{1}{2}, \alpha \to \infty, \\
-\frac{\rho e^{i\theta} \beta^2}{576 \sigma^5 e^{\alpha}}, & 0 < p < \frac{1}{2}, \alpha \to \infty, \\
-\frac{\rho e^{i\theta} 288 \sigma^5 e^{-\alpha}}, & -\frac{1}{2} < p < 0, \alpha \to -\infty, \\
-\frac{\rho e^{i\theta} 144 \sigma^3 \beta^2 e^{-2\alpha}}{41472 \sigma^8 e^{-3\alpha} + \beta^4 e^{-\alpha}}, & p = -\frac{1}{2}, \alpha \to -\infty, \\
-\frac{\rho e^{i\theta} 144 \sigma^4 \beta^2 e^{-2\alpha} - \beta^4}{\sigma \beta^4 e^{-\alpha}}, & p < -\frac{1}{2}, \alpha \to -\infty.
\end{cases}
\]
(21)

Expressions (21) show that Solutions (19) behave as the solitons with the stable amplitudes only when \( p = \pm \frac{1}{2} \). Whether \( p = \frac{1}{2} \) or \( -\frac{1}{2} \), \( t \) may approach to \( +\infty \) or \( -\infty \). Therefore, we calculate the following four asymptotic solitons as
\[
q^{[1]} \to q^{[1]+}_{n,2} = -\frac{\rho}{2\sqrt{2} \sigma} e^{i\theta} \text{sech} \left( \alpha + \ln 288 \sqrt{2} + 2 \ln \frac{\sigma^2}{\beta} \right), \quad \beta e^{-\alpha} = O(1), \quad \alpha \to +\infty, \quad t \to +\infty,
\]
(22)

\[
q^{[1]} \to q^{[1]−}_{n,2} = -\frac{\rho}{2\sqrt{2} \sigma} e^{i\theta} \text{sech} \left( \alpha - \ln 144 \sqrt{2} + 2 \ln \frac{-\beta}{\sigma^2} \right), \quad \beta e^{\alpha} = O(1), \quad \alpha \to -\infty, \quad t \to -\infty,
\]

\[
q^{[1]} \to q^{[1]+}_{n,3} = -\frac{\rho}{2\sqrt{2} \sigma} e^{i\theta} \text{sech} \left( \alpha - \ln 144 \sqrt{2} + 2 \ln \frac{\beta}{\sigma^2} \right), \quad \beta e^{\alpha} = O(1), \quad \alpha \to -\infty, \quad t \to +\infty,
\]

\[
q^{[1]} \to q^{[1]−}_{n,3} = -\frac{\rho}{2\sqrt{2} \sigma} e^{i\theta} \text{sech} \left( \alpha + \ln 288 \sqrt{2} + 2 \ln \frac{\sigma^2}{-\beta} \right), \quad \beta e^{-\alpha} = O(1), \quad \alpha \to +\infty, \quad t \to -\infty.
\]
3D figure and contour figure of the third-order degenerate solitons via Solutions (19) are shown in Figs. 4.

![3D figure](image1)

![Contour figure](image2)

(a) 3D figure; (b) Contour figure of the third-order degenerate solitons via Solutions (13) with $\rho = 2$, $\sigma = 1$.

From Asymptotic Solitons (22), we can get the following properties of the four bright-type asymptotic solitons $q_{n,2}^\pm$ and $q_{n,3}^\pm$ as follows:

(a) Amplitudes:

$$A(q_{n,2}^\pm) = A(q_{n,3}^\pm) = \frac{|\rho|}{2\sqrt{2}\sigma}.$$

(b) Slopes:

$$S(q_{n,2}^\pm) = \frac{1 + 3\rho^2}{12\sigma} + \frac{24\rho^2\sigma}{48\sigma^2 - |\beta|}; \quad S(q_{n,3}^\pm) = \frac{1 + 3\rho^2}{12\sigma} + \frac{24\rho^2\sigma}{48\sigma^2 + |\beta|}.$$

(c) Characteristic lines:

$$q_{n,2}^\pm, q_{n,3}^- : 288\sqrt{2}\sigma^4 e^\alpha - \beta^2 = 0; \quad q_{n,2}^-, q_{n,3}^+ : 144\sqrt{2}\sigma^4 e^{-\alpha} - \beta^2 = 0.$$

(d) Phase shifts $P(\epsilon)$ between $q_{n,\epsilon}^+$ and $q_{n,\epsilon}^-$ ($\epsilon = 2, 3$):

$$P(2) = -P(3) = 4 \ln \frac{12\sqrt{2}\sigma^2}{|\beta|}.$$  

We find that the four asymptotic solitons own the same amplitudes. In view of $P(2)$ and $P(3)$ are opposite under the same $\beta$, and the value of $S(q_{n,\epsilon}^+)$ at $\beta$ is equal to that of $S(q_{n,\epsilon}^-)$ at $-\beta$, we can infer that the interaction between $q_{n,1}$ and $q_{n,2}$ is elastic.

We have

$$S(q_{n,2}^\pm) < S(\alpha) = \frac{1 + 3\rho^2}{12\sigma}, \quad \text{when } |\beta| > 48\sigma^2;$$

$$S(\alpha) = \frac{1 + 9\rho^2}{12\sigma} < S(q_{n,3}^\pm) < S(\beta) = \frac{1 + 9\rho^2}{12\sigma}. \quad (23)$$
Therefore, asymptotic solitons $q_{n,3}^\pm$ are located between the two straight lines $l_1 : \alpha = 0$ and $l_2 : \beta = 0$, and asymptotic solitons $q_{n,2}^\pm$ are located outside of the straight line $l_1$, as shown in Figs. 4. However, since $288\sqrt{2}\sigma^4 e^\alpha - \beta^2 = 0$ and $\beta^2 144\sqrt{2}\sigma^4 e^\alpha = 0$ are the transcendental equations, we cannot get the expressions of the distances between the asymptotic solitons $q_{n,2}^+$ and $q_{n,3}^+$ (or $q_{n,2}^-$ and $q_{n,3}^-$).

In short, the third-order degenerate solitons can be seen as a combination of the one soliton and second-order degenerate solitons. One soliton component has no phase shift, while the second-order degenerate solitons component have phase a shift proportional to $\ln|\beta|$.

### 3.3 Interaction among the one soliton and second-order degenerate solitons for Eq. (1)

As we set $n = m_1 = 2, m_2 = 1$ and $N = 3$ in Solutions (10), i.e., two complex spectral parameter $\lambda_1$ and $\lambda_2$ are used, Solutions (10) describe the interaction between the second-order degenerate solitons and one soliton for Eq. (1). Combining $\lambda_1 = i\rho, \lambda_2 = i\omega$, Solutions (10) with Eigenfunctions (12), where $\omega$ is the non-zero real number, we obtain the mixed solutions for Eq. (1) as

$$q^{[1]} = \frac{e^{\theta}}{\sigma} \left[ \frac{\Lambda_1 e^{-2\alpha} + \Lambda_2 e^{2\alpha} + \Lambda_3 e^{\alpha+\eta} + \Lambda_4 e^{-\alpha-\eta} + \Lambda_5 e^{\alpha-\eta} + \Lambda_6 e^{-\alpha+\eta} + \Lambda_7}{\Omega_1 e^{-2\alpha-n} + \Omega_2 e^{2\alpha+n} + \Omega_3 e^{2\alpha-n} + \Omega_4 e^{-2\alpha+\eta} + \Omega_5 e^{-\alpha} + \Omega_6 e^{\alpha} + \Omega_7 e^{-\eta} + \Omega_8 e^{\eta}} \right],$$

$$\alpha = \frac{\rho (t + 3\rho^2 t - 12\sigma x)}{24\sigma^2}, \quad \beta = \rho t + 9\rho^3 t - 12\sigma \rho x, \quad \eta = \frac{\omega (t + 3\omega^2 t - 12\sigma x)}{24\sigma^2},$$

$$\Lambda_1 = 72\omega\sigma^4(\omega^2 - \rho^2)^2, \quad \Lambda_2 = 288\omega\sigma^4(\omega^2 - \rho^2)^2,$$

$$\Lambda_3 = 24\rho\sigma^2(\omega - \rho)^2 \left[ (\rho^2 - \omega^2)(24\sigma^2 - \beta) - 48\sigma^2 \rho \omega \right],$$

$$\Lambda_4 = 6\rho\sigma^2(\omega - \rho)^2 \left[ (\rho^2 - \omega^2)(24\sigma^2 + \beta) - 48\sigma^2 \rho \omega \right],$$

$$\Lambda_5 = 12\rho\sigma^2(\omega + \rho)^2 \left[ (\rho^2 - \omega^2)(24\sigma^2 - \beta) + 48\sigma^2 \rho \omega \right],$$

$$\Lambda_6 = 12\rho\sigma^2(\omega + \rho)^2 \left[ (\rho^2 - \omega^2)(24\sigma^2 + \beta) + 48\sigma^2 \rho \omega \right],$$

$$\Lambda_7 = \omega \left[ (\beta^2 + 288\sigma^4)(\rho^2 - \omega^2)^2 - 4608\sigma^4 \rho^4 \right],$$

$$\Omega_1 = 72\omega^4(\omega - \rho)^4, \quad \Omega_2 = 576\omega^4(\omega - \rho)^4, \quad \Omega_3 = 288\omega^4(\omega + \rho)^4, \quad \Omega_4 = 144\omega^4(\omega + \rho)^4,$$

$$\Omega_5 = 96\omega^2 \rho \omega \left[ 24\sigma^2 (\rho^2 + \omega^2) - \beta(\rho^2 - \omega^2) \right], \quad \Omega_6 = -192\omega^2 \rho \omega \left[ 24\sigma^2 (\rho^2 + \omega^2) + \beta(\rho^2 - \omega^2) \right],$$

$$\Omega_7 = \beta^2 (\rho^2 - \omega^2)^2 - 96\omega^2 \rho \omega (\rho^2 - \omega^2) + 288\sigma^4 \left[ (\rho^2 + \omega^2)^2 + 4\rho^2 \omega^2 \right],$$

$$\Omega_8 = 2 \left\{ \beta^2 (\rho^2 - \omega^2)^2 + 96\omega^2 \beta \rho \omega (\rho^2 - \omega^2) + 288\sigma^4 \left[ (\rho^2 + \omega^2)^2 + 4\rho^2 \omega^2 \right] \right\}.$$

(24)
Figs. 5. Interaction among the second-order degenerate solitons and one soliton via Solutions (24) with \( \rho = 2, \omega = -\frac{3}{2} \); (a) \( \sigma = \frac{3}{4} \); (b) \( \sigma = 1 \); (c) \( \sigma = \frac{4}{3} \).

Figs. 5 show the interactions among the second-order degenerate solitons and one soliton under three different parameter conditions. We find that as \( \sigma \) increases from \( \frac{3}{4} \) to 1, then to \( \frac{4}{3} \), the velocities of the one soliton and second-order degenerate solitons decrease and the widths of them increase. As can be seen from Figs. 5, characteristic lines of the one soliton and second-order degenerate solitons consist of a straight line and two curves.

Without loss of generality, we assume that \( 0 < |\omega| < \rho \). For Solutions (24), if \( \eta \) is fixed, \( \alpha \) can be expressed as

\[
\alpha = \frac{\rho}{\omega} \eta + \frac{3\rho(\rho^2 - \omega^2)}{24\sigma^2} t.
\]

Thus, alone the straight line \( \eta = \frac{\omega(t + 3\omega^2 t - 12\sigma x)}{24\sigma^2} \) = constant, when \( t \to +\infty \), we have \( \alpha \to +\infty \). Then, we only keep the highest infinite term \( e^{2\alpha} \) in the numerator and denominator of Solutions (24), and get

\[
q^{[1]} \to q_{n,1}^{+} = -\frac{\omega e^{i\theta}}{2\sqrt{2\sigma}} \text{sech} \left[ \eta + \ln \frac{\sqrt{2}(\omega - \rho)^2}{(\omega + \rho)^2} \right], \quad t \to +\infty; \quad (25)
\]

When \( t \to -\infty \), we have \( \alpha \to -\infty \). Then, we only keep the highest infinite term \( e^{-2\alpha} \) in the numerator and denominator of Solutions (24), and get

\[
q^{[1]} \to q_{n,1}^{-} = -\frac{\omega e^{i\theta}}{2\sqrt{2\sigma}} \text{sech} \left[ \eta + \ln \frac{\sqrt{2}(\omega + \rho)^2}{(\omega - \rho)^2} \right], \quad t \to -\infty. \quad (26)
\]

From Asymptotic Solitons (25) and (26), we know that \( q_{n,1}^{\pm} \) are the two bright solitons and have the same amplitude, velocity and width. Before and after the interaction, \( q_{n,1} \) has a phase shift \( 4\ln \frac{|\omega - \rho|}{|\omega + \rho|} \).

Next, we assume the balance between \( e^{\alpha} \) and \( \beta \) as

\[
\frac{\beta}{e^{\alpha}} \sim O(1), \quad |t| \to +\infty,
\]
where $p$ is a real variable constant depending on the values of $e^\alpha$ and $\beta$. Similar to Solutions (13), Solutions (24) behave as the solitons with the stable amplitudes only when $p = 1$. Whether $p = 1$ or $-1$, $t$ may approach to $+\infty$ or $-\infty$. Therefore, we calculate the following four asymptotic solitons as

$$q^{[1]} \rightarrow q^{[1]}_{n,2} = \frac{\rho}{2\sqrt{2}\sigma} e^{i\eta} \text{sech} \left[ \alpha + \ln \frac{12\sqrt{2}\sigma^2 (\rho + \omega)}{\beta (\rho - \omega)} \right], \beta e^{-\alpha} = O(1), \alpha, t \rightarrow +\infty, \eta \rightarrow -\infty, \beta > 0,$$

$$q^{[1]} \rightarrow q^{[1]}_{n,2} = \frac{\rho}{2\sqrt{2}\sigma} e^{i\eta} \text{sech} \left[ \alpha + \ln \frac{-\beta (\rho - \omega)}{6\sqrt{2}\sigma^2 (\rho + \omega)} \right], \beta e^\alpha = O(1), \alpha, t \rightarrow -\infty, \eta \rightarrow +\infty, \beta < 0,$$

$$q^{[1]} \rightarrow q^{[1]}_{n,3} = -\frac{\rho}{2\sqrt{2}\sigma} e^{i\eta} \text{sech} \left[ \alpha + \ln \frac{\beta (\rho + \omega)}{6\sqrt{2}\sigma^2 (\rho - \omega)} \right], \beta e^\alpha = O(1), \alpha, \eta \rightarrow -\infty, t \rightarrow +\infty, \beta > 0,$$

$$q^{[1]} \rightarrow q^{[1]}_{n,3} = -\frac{\rho}{2\sqrt{2}\sigma} e^{i\eta} \text{sech} \left[ \alpha + \ln \frac{12\sqrt{2}\sigma^2 (\rho - \omega)}{-\beta (\rho + \omega)} \right], \beta e^{-\alpha} = O(1), \alpha, \eta \rightarrow +\infty, t \rightarrow -\infty, \beta < 0.$$

(27)

3D figure and contour figure of the interaction among the one soliton and second-order degenerate solitons via Solutions (24) are shown in Figs. 6.

![3D and contour figures](image)

Figs. 6. (a) 3D figure; (b) Contour figure of the interaction among the one soliton and second-order degenerate solitons via Solutions (24) with $\rho = 2$, $\omega = -\frac{1}{2}$, $\sigma = 1$.

From Asymptotic Solitons (27), we can get the following properties of the four bright-type asymptotic solitons $q^{\pm}_{n,2}$ and $q^{\pm}_{n,3}$ as follows:

(a) Amplitudes:

$$A(q^{\pm}_{n,2}) = A(q^{\pm}_{n,3}) = \frac{|\rho|}{2\sqrt{2}\sigma}.$$

(b) Slopes:

$$S(q^{\pm}_{n,2}) = \frac{1 + 3\rho^2}{12\sigma} + \frac{12\rho^2}{24\sigma^2 - |\beta|} \quad \text{and} \quad S(q^{\pm}_{n,3}) = \frac{1 + 3\rho^2}{12\sigma} + \frac{12\rho^2}{24\sigma^2 + |\beta|}.$$
(c) Characteristic lines:

\[ q_{n,2}^+ : \frac{\rho + \omega}{\rho - \omega} 12\sqrt{2}\sigma^2 e^{\alpha} - \beta = 0; \quad q_{n,2}^- : \frac{\rho + \omega}{\rho - \omega} 6\sqrt{2}\sigma^2 e^{\alpha} + \beta = 0. \]

\[ q_{n,3}^+ : \frac{\rho - \omega}{\rho + \omega} 6\sqrt{2}\sigma^2 e^{\alpha} - \beta = 0; \quad q_{n,3}^- : \frac{\rho - \omega}{\rho + \omega} 12\sqrt{2}\sigma^2 e^{\alpha} + \beta = 0. \]

(d) Phase shifts \( P(\tau) \) between \( q_{n,2}^+ \) and \( q_{n,2}^- \) (\( \tau = 2, 3 \)):

\[ P(2) = -P(3) = 2 \ln \frac{12\sigma^2(\rho + \omega)}{|\beta|(\rho - \omega)}. \]

We find that the four asymptotic solitons own the same amplitude. In view of \( P(2) \) and \( P(3) \) are opposite under the same \( \beta \), and the value of \( S(q_{n,\tau}^+) \) at \( \beta \) is equal to that of \( S(q_{n,\tau}^-) \) at \( -\beta \), we can infer that the interaction between \( q_{n,2} \) and \( q_{n,3} \) is elastic.

Comparing the properties of the four asymptotic solitons \( q_{n,2}^\pm \) and \( q_{n,3}^\pm \) with them in Chapter 3.1, we find that they have the same soliton amplitudes, slopes. Therefore, similar to Figs. 2, asymptotic solitons \( q_{n,3}^\pm \) are located between the two lines \( \alpha = 0 \) and \( \beta = 0 \), and asymptotic solitons \( q_{n,2}^\pm \) are located outside of the line \( \alpha = 0 \), as shown in Figs. 4. However, in the discussion in this section, since the one soliton component \( q_{n,1}^\pm \) is introduced into Solutions (24), characteristic lines and phase shifts of the four asymptotic solitons \( q_{n,2}^\pm \) and \( q_{n,3}^\pm \) are changed (related to \( \omega \)).

Although both Solutions (19) and (24) describe the one soliton and second-order degenerate solitons, main difference between Solutions (19) and (24) is that the one soliton component in Solutions (19) has no phase shift, the one soliton component in Solutions (24) has a phase shift as \( 4 \ln \frac{\rho - \rho}{\rho + \rho} \) before and after the interaction.

4. Conclusion

In this paper, the Sasa-Satsuma equation, i.e., Eq. (1), which has been applied to the dynamics of the deep water waves, and the pulse propagation in the optical fibers and generally in the dispersive nonlinear media, has been investigated.

For Eq. (1), starting from the First-Order DT (3), we have constructed an \( N \)-fold GDT which consists of \( N \)-Fold Darboux Matrix (9) and Solutions (10). Based on Solutions (10), three kinds of the semirational solutions have been derived and analyzed as follows:

- When \( n = 1 \) and \( m_1 = N = 2 \), we have derived Solutions (13) which describe the second-order degenerate solitons. 3D figures and contour figure of the second-order degenerate solitons have been shown in Figs. 1 and 2. Asymptotic analysis on Solutions (13) has been proceeded, from which we have calculated the four asymptotic solitons \( q_{n,1}^\pm \) and \( q_{n,2}^\pm \) as Asymptotic Solitons (17). Amplitudes, slopes, characteristic lines, phase shifts and positions of the four asymptotic solitons have been offered. It has been shown that the characteristic lines of \( q_{n,1}^\pm \) and \( q_{n,2}^\pm \) are the two curves \( 12\sqrt{2}\sigma^2 e^\alpha - \beta = 0 \) and \( 6\sqrt{2}\sigma^2 e^{-\alpha} + \beta = 0 \).
- When $n = 1$ and $m_1 = N = 3$, we have derived Solutions (19) which describe the third-order degenerate solitons. 3D figures and contour figure of the third-order degenerate solitons have been shown in Figs. 3 and 4. Asymptotic analysis on Solutions (19) has been proceeded, from which we have calculated the six asymptotic solitons as Asymptotic Solitons (20) and (22). We have found that the third-order degenerate solitons consists of the one soliton ($q_{n,1}^{\pm}$) and second-order degenerate solitons ($q_{n,2}^{\pm}$ and $q_{n,3}^{\pm}$). We have investigated that the characteristic line of $q_{n,1}^{\pm}$ is a straight line $\alpha + \frac{1}{2}\ln 2 = 0$, and the characteristic lines of $q_{n,2}^{\pm}$ and $q_{n,3}^{\pm}$ are the two curves $288\sqrt{2}\sigma^4 e^\alpha - \beta^2 = 0$ and $144\sqrt{2}\sigma^4 e^{-\alpha} - \beta^2 = 0$.

- When $n = m_1 = 2$, $m_2 = 1$ and $N = 3$, Solutions (24), which describe the interactions among the one soliton and second-order degenerate solitons, have been obtained. 3D figures and contour figure of Solutions (24) have been shown in Figs. 5 and 6. Asymptotic analysis on Solutions (24) has been proceeded, from which we have calculated the six asymptotic solitons as Asymptotic Solitons (25), (26) and (27). It has been shown that the characteristic lines of the one soliton component are $\eta \pm \ln \sqrt{2} \left(\frac{\omega - \rho}{\omega + \rho}\right)^2 = 0$, and the characteristic lines of the second-order degenerate solitons component are $\frac{\omega + \rho}{\rho - \omega} 6\sqrt{2}\sigma^2 e^\alpha - \beta = 0$, $\frac{\omega + \rho}{\rho - \omega} 6\sqrt{2}\sigma^2 e^\alpha + \beta = 0$, $\frac{\rho - \omega}{\rho + \omega} 12\sqrt{2}\sigma^2 e^\alpha - \beta = 0$ and $\frac{\rho - \omega}{\rho + \omega} 12\sqrt{2}\sigma^2 e^\alpha + \beta = 0$. Different from Solutions (19), we have found that the one soliton component in Solutions (19) has a phase shift $4\ln \frac{\omega - \rho}{\omega + \rho}$ before and after the interaction.

Asymptotic analysis on Solutions (24) has shown that the interactions among the second-order degenerate solitons component and one soliton component are elastic. We have also found a common behavior of the above degenerate solitons, which is that as $\sigma$ increases, the velocities of the above degenerate solitons decrease while the widths of them increase.

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**Ethics declarations**

**Conflicts of interest**

The authors declare that they have no conflict of interest.

**Data availability**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
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