High-order analytical solutions of bounded relative motions for Coulomb formation flying

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High-order analytical solutions of bounded relative motions for Coulomb formation flying

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Abstract Close-proximity Coulomb formation flying offers attractive prospects in multiple astronautical missions. An analytical method of constructing the series solution up to an arbitrary order for the relative motion near the equilibrium of Coulomb formation systems is proposed to facilitate the design of Coulomb formations based on a Lindstedt-Poincaré method. The details of the series expansion and coefficient solution for Lissajous orbits and arbitrary $m:n$-period orbits are discussed. To verify the effectiveness of the Lindstedt-Poincaré method in constructing series solutions, the practical convergence domain of series solutions for various bounded orbits is computed by comparison with the corresponding exact numerical solutions. Given the accuracy requirements of practical formation missions, the configuration design for the Coulomb formation can be carried out conveniently and quickly by employing the proposed series solutions.

Keywords Bounded analytical solution · Coulomb formation flying · Formation configuration design · $m:n$-period orbits · Practical convergence domain

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1 Introduction

Close-proximity spacecraft formation has huge potential in telescope-occulter pairs, space-borne interferometry, and Earth imaging [1]. The fast throttle ability and avoidance of plume impingement issues of Coulomb control make it get attracted sustained attention in close-proximity formation mission design [2]. By actively charging spacecraft, repulsive and attractive forces can be generated to execute Coulomb control over a close formation. Several space missions, including SCATHA [3], ATS [4], and CLUSTER [5], have preliminary explored the feasibility of active spacecraft potential control [6].

Unlike traditional formation flying, Coulombe force control can simply generate more types of static configurations, in which spacecraft remain stationary with respect to the rotating center-of-mass frame [7]. On the contrary, the dynamical configurations of Coulomb formation are more complicated due to its high nonlinearity and strong-coupling feature. Therefore, early researches focused more on the dynamics and control of static Coulomb formations. Nataraajan et al. [8] investigated a particular static formation shape of a two-craft Coulomb formation, where the spacecraft aligned the orbit nadir direction. Then a charge feedback control law is firstly introduced to stabilize the formation. Hogan et al. [9] discovered families of multiple invariant shape solution for a collinear three-craft Coulomb formation in the absence of relevant gravitational forces. More generally, an analytical method for computing the required charge to maintain static N-craft Coulomb formations is proposed by Berryman and Schaub in Ref. [10]. Inampudi et al. [11] analyzed the orbit and attitude stability of a two-craft virtual Coulomb structure at Earth–Moon libration points. Then the gravity gradient torques are utilized to stabilize the formation in orbit radial direction. Alikhani et al. [12] studied a static triangular three-craft Coulomb formation and presented a fault-tolerance control law to maintain the desired formation.

Another topic of particular interest is the design, maintenance, and reconfiguration of the dynamic Coulomb formations. Coulomb thrust leads to the high nonlinearity of the formation dynamics, which makes the dynamic formations have rich but very complicated dynamic properties. Jones [13] extended the static Coulomb formation research and derived the periodic relative orbit of two-craft Coulomb formation under the action of open-loop charge for the first time by using the dynamic system theory. In Ref [14], he further studied three-craft collinear equilibria in the presence of a central body gravity field and discussed the possibility of exploiting invariant manifolds to achieve minimum-fuel reconfigurations. Wang [15] developed circular transfer trajectory and the general patched conic section trajectory programming algorithms for the maneuver mission of two-craft formation. Aslanov [16] studied the attitude dynamics of a defunct satellite with flexible appendages under the action of Coulomb forces and found the chaotic motion in Coulomb formation for the first time. Lin et al. [17] also revealed the intermittent chaos generated from a tangent bifurcation point in the Coulomb formation. Ref. [18] studied two different two-craft Coulomb formations in near-circular reference orbits. The
Udwadia–Kalaba formulation is implemented to achieve and maintain the formation along the nadir direction and the formation with a constant separation distance.

The traditional spacecraft formation in a circular or near-circular orbit can be designed by solving the CW equation and determining the parameters of its linear solution [19]. However, the high nonlinearity feature of the dynamical Coulomb formation introduces a variety of dynamical complexities, making a linear solution unable to describe the relative motion well. Thus, tools for analytical descriptions of the nonlinear relative motion are necessary to study the dynamics of the Coulomb formation. The Lindstedt-Poincaré method, an analytical technology in perturbation theory, is first developed to compute periodic orbits of weak nonlinear equations based on the solution of the corresponding linearized equation [20]. It has been successfully extended to studies on stronger nonlinear problems [21, 22], including the computations of periodic and quasi-periodic orbits (center manifolds) [23, 24], hyperbolic manifolds [25], homoclinic and heteroclinic connection [26], and even their bifurcation predictions [27]. In this paper, the Lindstedt-Poincaré method is used to describe the various relative motions of Coulomb formation flying for the formation design, maintenance, and reconfiguration. The high-order analytical solution of general $m:n$-period orbits around equilibrium in the Coulomb formation system is first developed, which can be extended to general nonlinear systems.

The remainder of this paper is organized as follows. First, the dynamical model of a typical eight-charge Coulomb formation [17, 28] is introduced in Section 2. Then, section 3 constructs high-order series solutions of Lissajous orbits and $m:m$-period orbits around equilibrium points in Coulomb formation system by employing the Lindstedt-Poincaré method. In Section 4, the constructed high-order analytical solutions are applied to derive planar and vertical Lyapunov orbits, general Lissajous orbits, and Halo orbits as well as general $m:n$-period orbits. Furthermore, the high-order analytical solutions are compared with the corresponding exact numerical solutions to verify their effectiveness. Finally, a practical convergence domain of the analytical solutions is given for applying the analytical solutions to practical formation mission design. Finally, Section 5 makes some concluding remarks.

2 Dynamical model of Coulomb formation

For the eight-charge Coulomb formation, the chief spacecraft has an independent Coulomb propulsion system, where eight controllable charged spheres are evenly distributed on eight vertices of a cube around the chief as shown in Fig. 1. The mass of the chief is far larger than that of the deputy, where the deputy is modeled as an ideal charged sphere. The chief can manipulate the relative motion of the deputy by controlling the eight charged spheres. The dynamics of Coulomb formation is generally derived based on Clohessy-Wiltshire (CW) equations. The relative motion of the deputy with respect to the chief spacecraft is described in the Hill frame. As shown in Fig. 1, the origin of the Hill
Fig. 1: The eight-charge Coulomb formation.

frame is at the centroid of the chief. The $x$-axis is directed from the chief radially outward, the $z$-axis is normal to the orbit plane and positive in the direction of the angular momentum vector, and the $y$-axis completes a right-handed triad. $r = [x \ y \ z]^T$ denotes the position of deputy, $R_i = [X_i \ Y_i \ Z_i]^T$ denotes the constant position of the $i$-th charged sphere of the chief ($i = 1, 2 \ldots, M$). $M = 8$ is the number of controllable charged spheres. Then the relative motion of the deputy with respect to the chief can be formulated as

$$
\ddot{x} - 2\omega \dot{y} - 3\omega^2 x = \frac{k_c}{m} \sum_{i=1}^{M} \frac{1+\frac{r_i}{\lambda_d}}{r_i^{3+\frac{1}{\lambda_d}}} q_i x_i
$$

$$
\ddot{y} + 2\omega \dot{x} = \frac{k_c}{m} \sum_{i=1}^{M} \frac{1+\frac{r_i}{\lambda_d}}{r_i^{3+\frac{1}{\lambda_d}}} q_i y_i
$$

$$
\ddot{z} + \omega^2 z = \frac{k_c}{m} \sum_{i=1}^{M} \frac{1+\frac{r_i}{\lambda_d}}{r_i^{3+\frac{1}{\lambda_d}}} q_i z_i
$$

where $r_i = [x_i \ y_i \ z_i]^T = r - R_i$ is the position to deputy from the $i$-th charged sphere, $\omega$ is the constant orbital rate of the chief, $m$ is the deputy mass, $k_c = 8.99 \times 10^9 \text{Nm}^2/\text{C}^2$ is the Coulomb’s constant, $\lambda_d$ is the Debye length over which the charge carriers are electrically screened, $q$ is the deputy charge and $q_i$ is the $i$-th active charge of the chief. It is denoted as

$$
\dot{Q}_i = \frac{k_c q q_i}{m c^2} d\tau = \omega \frac{dX}{dt} = \frac{1}{2} \frac{dX^2}{dt}
$$

where $\tau$ is time-like nondimensional variable; $X$ represents the state variables of (1). It is noticed that $\frac{1+\frac{r_i}{\lambda_d}}{r_i^{\frac{1}{\lambda_d}}} = \frac{\partial}{\partial r_i} \left( \frac{1}{r_i^{\frac{1}{\lambda_d}}} \right)$. Then, (1) can be normal-
ized as
\[
\begin{align*}
x'' - 2y' - 3x &= -\sum_i M \frac{\partial}{\partial x_i} \frac{\tilde{Q}_i}{r_i e_i / \lambda} \\
y'' + 2x' &= -\sum_i M \frac{\partial}{\partial y_i} \frac{\tilde{Q}_i}{r_i e_i / \lambda} \\
z'' + z &= -\sum_i M \frac{\partial}{\partial z_i} \frac{\tilde{Q}_i}{r_i e_i / \lambda}.
\end{align*}
\] (3)

Compared with the concept of effective potential in the Restricted Three-Body Problem [29], the effective potential of a Coulomb formation \( \tilde{\phi} \) is given by
\[
\tilde{\phi}(x, y, z) = -\frac{1}{2} (3x^2 - z^2) - \sum_{i=1}^{M} \frac{\tilde{Q}_i}{r_i e_i / \lambda}.
\] (4)

Then (3) can be simplified as
\[
\begin{align*}
x'' - 2y' &= -\tilde{\phi}_x \\
y'' + 2x' &= -\tilde{\phi}_y \\
z'' &= -\tilde{\phi}_z
\end{align*}
\] (5)

Thus, we have
\[
\frac{d}{dt} C = \frac{d}{dt} \left[ -\left(x'^2 + y'^2 + z'^2\right) - 2\tilde{\phi} \right] = 0
\] (6)

where \( C \) is called the Jacobi integral and thus (5) represents a conservative system. Defining momenta as \( p_x = x' - y, p_y = y' + x, p_z = z' \), we can formulate the dynamical equations in a Hamiltonian form as
\[
\begin{align*}
H(x, y, z, p_x, p_y, p_z) &= \frac{1}{2} \left(p_x^2 + p_y^2 + p_z^2\right) + yp_x \\
&\quad - xp_y + \frac{1}{2} \left(x'^2 + y'^2\right) + \tilde{\phi}.
\end{align*}
\] (7)

### 3 Lindstedt-Poincaré method

In this section, the high-order expansions for a particular eight-charge Coulomb formation are investigated by means of the Lindstedt–Poincaré method. As shown in Fig. 2, eight identical charged spheres are located on eight vertices of a cube with a side length \( L = 5m \). The electrostatic potential of the eight charged spheres varies are 20 kV. Other parameters of this formation are defined according to Ref. [13], as shown in Table. 1.

Based on the given parameters, equilibrium points of (3) can be derived by
\[
\begin{align*}
Q \sum_i \frac{1+r_i/\lambda_d}{r_i e_i / \lambda} x_i - 3x &= 0 \\
Q \sum_i \frac{1+r_i/\lambda_d}{r_i e_i / \lambda} y_i &= 0 \\
Q \sum_i \frac{1+r_i/\lambda_d}{r_i e_i / \lambda} z_i + z &= 0
\end{align*}
\] (8)
Table 1: Parameters used in the Coulomb formation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>50</td>
<td>kg</td>
</tr>
<tr>
<td>kc</td>
<td>8.99×10^9</td>
<td>Nm^2/C^2</td>
</tr>
<tr>
<td>φ</td>
<td>-20</td>
<td>kV</td>
</tr>
<tr>
<td>ω</td>
<td>7.2593×10^{-5}</td>
<td>rad/s</td>
</tr>
<tr>
<td>λ_d</td>
<td>180</td>
<td>m</td>
</tr>
<tr>
<td>φ_i(i = 1 , M)</td>
<td>20</td>
<td>kV</td>
</tr>
<tr>
<td>r_n</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>L</td>
<td>5</td>
<td>m</td>
</tr>
</tbody>
</table>

Equilibrium points on the X-axis and Y-axis can be obtained, as shown in Table. 2 [17].

Table 2: The equilibrium points on the X-axis and Y-axis and the type of their linearized vector field

<table>
<thead>
<tr>
<th>Equilibrium points</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>Linear behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>L_1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Center × focus × focus</td>
</tr>
<tr>
<td>L_2, L_3</td>
<td>±0.5610689</td>
<td>0</td>
<td>0</td>
<td>Center × saddle × saddle</td>
</tr>
<tr>
<td>L_4, L_5</td>
<td>±47.777433</td>
<td>0</td>
<td>0</td>
<td>Center × center × saddle</td>
</tr>
<tr>
<td>L_6, L_7</td>
<td>0</td>
<td>±0.188348</td>
<td>0</td>
<td>Center × center × saddle</td>
</tr>
</tbody>
</table>

where \( Q_i = \frac{k_i q_i q_j}{m} \) (\( i = 1, 2, ..., M \)). The equilibrium points on the X-axis and Y-axis can be obtained, as shown in Table. 2 [17].

Without loss of generality, the Taylor expansion of the Hamiltonian function in (7) around the chosen equilibrium point is given as:

\[
H = \frac{1}{2} (p_x^2 + p_y^2) + yp_y - xp_y + \frac{a}{2} x^2 + \frac{b}{2} y^2 + cxy + \frac{1}{2} p_x z^2 + \frac{1}{2} y z^2 + \sum_{n \geq 3} H_n. \tag{9}
\]

Lindstedt–Poincaré method is an iterative calculation technology. The main principle is to observe the fact that the nonlinear terms of the system affect the
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basic frequency of the corresponding linear system, and then the relationship between the frequency and amplitude of the system is given. Starting from the linear basic solution of the system, it continuously adjusts the relationship between the frequency and amplitude by iterating the known low-order solution step by step to obtain a higher-order series solution.

In order to perform the computations using the Lindstedt–Poincaré method, the motion equations of Coulomb formation are expanded in power series as

\[
\begin{align*}
  x'' - 2y' - ax - cy &= \sum_{N\geq 2, i+j+k=N} c_{x,ijk} x^i y^j z^k \\
y'' + 2x' - cx - by &= \sum_{N\geq 2, i+j+k=N} c_{y,ijk} x^i y^j z^k \\
z'' + dz &= \sum_{N\geq 2, i+j+k=N} c_{z,ijk} x^i y^j z^k
\end{align*}
\]  

(10)

where \( c_{x,ijk} \), \( c_{y,ijk} \), and \( c_{z,ijk} \) are the coefficients of \((i, j, k)\)-th item of the power series of \(-\tilde{\phi}_x\), \(-\tilde{\phi}_y\) and \(-\tilde{\phi}_z\), respectively.

3.1 Lissajous orbits

In this subsection, the analytical solution of Lissajous orbits is derived. The first step is to find the linear solution. If the higher order terms on right side in (10) is ignored, the linearized equation of the system is

\[
\begin{align*}
  x'' - 2y' - ax - cy &= 0 \\
y'' + 2x' - cx - by &= 0 \\
z'' + dz &= 0
\end{align*}
\]  

(11)

The part of solution of center manifolds for (11) can be obtained as

\[
\begin{align*}
x &= \alpha \cos (\omega_0 t + \phi_1) \\
y &= \kappa \alpha \sin (\omega_0 t + \phi_1) \\
z &= \beta \cos (\nu_0 t + \phi_2)
\end{align*}
\]  

(12)

where \( \omega_0 = \sqrt{\frac{(a+b+2) + \sqrt{(a-b)^2 + 4(2a+2b+c^2)}}{2}} \), \( \nu_0 = \sqrt{d} \), \( \kappa = \frac{-\omega_0^2 + a}{2\omega_0} \), \( \alpha \) and \( \beta \) are the amplitude in \( XY\)-plane and \( Z\) axis, respectively. \( \phi_1 \) and \( \phi_2 \) are the corresponding phases, respectively. When the nonlinear terms are considered, the analytical solution around the equilibrium can be expanded in the power series about \( \alpha \) and \( \beta \).

\[
\begin{align*}
x(t) &= \sum_{i+j \geq 1} \left( \sum_{|k| \leq i, |m| \leq j} x_{ijkm} \cos (k\theta_1 + m\theta_2) \right) \alpha^i \beta^j \\
y(t) &= \sum_{i+j \geq 1} \left( \sum_{|k| \leq i, |m| \leq j} y_{ijkm} \sin (k\theta_1 + m\theta_2) \right) \alpha^i \beta^j \\
z(t) &= \sum_{i+j \geq 1} \left( \sum_{|k| \leq i, |m| \leq j} z_{ijkm} \cos (k\theta_1 + m\theta_2) \right) \alpha^i \beta^j
\end{align*}
\]  

(13)
where $\theta_1 = \omega t + \phi_1$ and $\theta_2 = \nu t + \phi_2$. Due to the nonlinear terms, the frequency of the system is no longer constant, but should also be expanded in the power series about $\alpha$ and $\beta$, i.e.,

$$
\omega = \sum_{i,j \geq 0} \omega_{ij} \alpha^i \beta^j,
$$

$$
v = \sum_{i,j \geq 0} v_{ij} \alpha^i \beta^j.
$$

In (13) and (14), $i$ and $j \in \mathbb{N}$, $k$ and $m \in \mathbb{Z}$. Due to the symmetry of the system, $k$ and $m$ have the same parity as $i$ and $j$, respectively. Besides, due to the symmetry of sine and cosine functions, it can be assumed that $k \geq 0$, and $m \geq 0$ when $k = 0$. The series of $\omega$ and $v$ only include those even items.

Our goal is to compute the coefficients $x_{ijkm}$, $y_{ijkm}$, $z_{ijkm}$, $\omega_{ij}$, $v_{ij}$ in (13) and (14) up to finite order $N$. The Lindstedt–Poincaré method calculates these coefficients following an iterative scheme from the linear term. Compared to the solution of the linear part (12), we can know $x_{1010} = 1$, $y_{1010} = \kappa$, $z_{0101} = 1$, $\omega_{00} = \omega_0$, $v_{00} = v_0$. By substituting this linear solution into (10), the coefficients of the second-order solution can be derived. Similarly, when the coefficients up to order $N - 1$ are obtained, i.e., $x(t)$, $y(t)$, and $z(t)$ are determined up to order $N - 1$, $\omega$ and $v$ are determined up to order $N - 2$. Substituting them into the right side of (10), we can obtain three power series up to $N$ order, denoted by $p$, $q$, and $r$. Here, what we are interested in is those $N$-order terms. Without losing generality, the $N$-order terms of $p$, $q$, and $r$ are denoted by $p_{ijkm}$, $q_{ijkm}$, and $r_{ijkm}$ ($i + j = N$) respectively, and the $N$-order terms of $\omega$ and $v$ are denoted by $\omega_{ij}$ and $v_{ij}$. Next, we analyze the composition of $N$-order terms on the left side of (10). According to (13), the derivatives of variable $x$ can be given as

$$
\begin{align*}
\frac{dx'}{dt} &= \omega x' + \sum_{i,j \geq 0} \omega_{ij} \alpha^i \beta^j, \\
\frac{dx''}{dt} &= \omega^2 \frac{\partial^2 x}{\partial \theta_1^2} + 2\omega v \frac{\partial^2 x}{\partial \theta_1 \partial \theta_2} + v^2 \frac{\partial^2 x}{\partial \theta_2^2}.
\end{align*}
$$

Similarly, we can obtain the derivatives of $y$ and $z$. Let $fg$ denote the first-order derivative term, where $f$ represents the power series of the frequencies ($\omega_{ij}$ and $v_{ij}$), and $g$ represents the coordinate variables ($x$, $y$, and $z$). $i_f$ and $i_g$ denote their corresponding order. Then, the $fg$ satisfying $i_f + i_g = N$ is the $N$-order terms we need. When $i_f = 0$ or $N - 1$, the corresponding $i_g$ is $N$ or 1 and $fg$ is an unknown term that needs to be solved. When $i_f = 1, 2, ..., N - 2$, $fg$ is a known term, which needs to be moved to the right hand of (10). Table 3 summarizes the unknown and known terms of the first derivatives, where $\delta_{ij}$ denotes Kronecker function.

For the second derivatives of $x$, it can be similarly summarized as shown in Table 4. Then we move all the known terms to the right hand of (10) to add them to $p_{ijkm}$, $q_{ijkm}$, and $r_{ijkm}$, which are re-denoted by $\bar{p}_{ijkm}$, $\bar{q}_{ijkm}$, and $\bar{r}_{ijkm}$. Besides, we should further discuss the calculation of unknown term $f_{i-1j}$ and $f_{ij-1}$. In fact, they are composed of the unknown term $2\omega_{00} \omega_{i-1j} (2v_{00} v_{i-1j})$ and the remaining known terms, i.e., $f_{i-1j} = 2\omega_{00} \omega_{i-1j} + \sum C_2^{1-\delta_{i1} \delta_{i2}} x_{i1j1} v_{i1j2} v_{i2j2}$ and $f_{ij-1} = 2v_{00} v_{i-1j} + \sum C_2^{1-\delta_{i1} \delta_{i2}} x_{i1j1} v_{i1j2} v_{i2j2}$ ($i_1 + j_1 + i_2 + j_2 = N - 1$). For
It is known that the system frequencies vary with the amplitudes \( \alpha \) and \( \beta \), and thus, the frequencies will be resonant when \( \omega / v = m/n \) \( (m,n \in \mathbb{Z}^+) \) at

<table>
<thead>
<tr>
<th>( f )</th>
<th>( g )</th>
<th>( \omega(\partial x / \partial \theta_1) )</th>
<th>( v(\partial x / \partial \theta_2) )</th>
<th>( \omega(\partial y / \partial \theta_1) )</th>
<th>( v(\partial y / \partial \theta_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unknown term</td>
<td>0</td>
<td>0</td>
<td>(-\omega_0 k x_{ijkm})</td>
<td>(-v_0 m x_{ijkm})</td>
<td>(-\omega_0 k y_{ijkm})</td>
</tr>
<tr>
<td>( N - 1 )</td>
<td>1</td>
<td>(-\omega_{i-1,j} \delta_{1k} \delta_{0m})</td>
<td>0</td>
<td>(\nu \omega_{i-1,j} \delta_{1k} \delta_{0m})</td>
<td>0</td>
</tr>
<tr>
<td>Known term</td>
<td>(1,2,..., N - 2)</td>
<td>( N - i_f )</td>
<td>(-\omega_{i-1,j} k x_{ijkm})</td>
<td>(-v_{i-1,j} m x_{ijkm})</td>
<td>(-\omega_{i-1,j} k y_{ijkm})</td>
</tr>
</tbody>
</table>

Table 3: Derivatives of \( x \) and \( y \) with respect to time

<table>
<thead>
<tr>
<th>( f )</th>
<th>( \partial^2 / \partial \theta_1^2 )</th>
<th>( \omega^2(\partial^2 x / \partial \theta_1^2) )</th>
<th>(-2\omega v(\partial^2 x / \partial \theta_1 \partial \theta_2) )</th>
<th>( v^2(\partial^2 x / \partial \theta_2^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unknown term</td>
<td>0</td>
<td>0</td>
<td>(-\omega_0^2 k^2 x_{ijkm})</td>
<td>(-2\omega_0 v_0 k m x_{ijkm})</td>
</tr>
<tr>
<td>( N - 1 )</td>
<td>1</td>
<td>(-f_{ij-1} x_{1010} \delta_{1k} \delta_{0m})</td>
<td>0</td>
<td>(-f_{ij-1} x_{1010} \delta_{1k} \delta_{1m})</td>
</tr>
<tr>
<td>Known term</td>
<td>(1,2,..., N - 2)</td>
<td>( N - i_f )</td>
<td>(-\omega_0^2 k^2 x_{ijkm})</td>
<td>(-\omega_{i,j-1} k m x_{ijkm})</td>
</tr>
</tbody>
</table>

Table 4: Second derivatives of \( x \), \( y \) and \( z \) with respect to time

the second derivatives of \( y \) and \( z \), similar results can be obtained. In summary, the linear equation of \( N \)-order unknown terms is yielded

\[
\begin{align*}
&\left[-(\tilde{\omega}^2_{km} + \alpha) x_{ijkm} - (2\tilde{\omega}_{km} + c) y_{ijkm} - 2(\omega_0 + \kappa) \omega_{i-1,j} \delta_{1k} \delta_{0m} \right] = \bar{p}_{ijkm} \\
&\left[-(2\tilde{\omega}_{km} + c) x_{ijkm} - (\tilde{\omega}^2_{km} + \beta) y_{ijkm} - 2(\omega_0 + 1) \omega_{i-1,j} \delta_{1k} \delta_{0m} \right] = \bar{q}_{ijkm} \\
&\left[ (d - \tilde{\omega}^2_{km}) z_{ijkm} - 2v_0 v_{i,j-1} \delta_{1k} \delta_{1m} \right] = \bar{r}_{ijkm}
\end{align*}
\]

where \( \tilde{\omega} = \omega_0 + m v_0 \) and

\[
\begin{align*}
\bar{p}_{ijkm} &= \bar{p}_{ijkm} + \delta_{1k} \delta_{0m} \sum C_1^{i_1,i_2} \delta_{i_1,i_2} \omega_{i_1,j_{1}i_{12}j_{2}} \\
\bar{q}_{ijkm} &= \bar{q}_{ijkm} + \kappa \delta_{1k} \delta_{0m} \sum C_1^{i_1,i_2} \delta_{i_1,i_2} \omega_{i_{1},j_{1}i_{12}j_{2}} \\
\bar{r}_{ijkm} &= \bar{r}_{ijkm} + \delta_{1k} \delta_{0m} \sum C_1^{i_1,i_2} \delta_{i_1,i_2} v_{i_{1},j_{1}i_{12}j_{2}} \\
& (i_1,j_{1},i_{2},j_{2}) \in \{1,1+j_{1}+i_{2}+j_{2} = N - 1\}.
\end{align*}
\]

When \( (k,m) \neq (1,0), (k,m) \neq (0,1) \), and the system frequencies are non-resonant, the coefficients \( x_{ijkm}, y_{ijkm}, \) and \( z_{ijkm} \) can be solved immediately. When \( (k,m) = (1,0) \), \( x_{ijkm} \) and \( y_{ijkm} \) are coupled. \( x_{ijkm} \) can be set zero, and then \( y_{ijkm} \) and \( \omega_{i-1,j} \) are solved by the first two equations of (16). \( z_{ijkm} \) is obtained by \( (d - \tilde{\omega}^2_{km}) z_{ijkm} = \bar{r}_{ijkm} \). When \( (k,m) = (0,1) \), \( x_{ijkm} \) and \( y_{ijkm} \) are solved by the first two equations of (16). The coefficient of \( z_{ijkm} \) in (16) is zero, and thus \( z_{ijkm} \) is set zero. \( v_{i,j-1} \) is obtained by \(-2 \omega_0 v_{i,j-1} \delta_{1k} \delta_{1m} = \bar{r}_{ijkm} \).

3.2 \( m : n \)-periodic orbits

It is known that the system frequencies vary with the amplitudes \( \alpha \) and \( \beta \), and thus, the frequencies will be resonant when \( \omega / v = m/n \) \( (m,n \in \mathbb{Z}^+) \) at
a particular point. In this case, the relative trajectories of deputy spacecraft will become $m:n$-period orbits. In this subsection, we discuss the analytic solution of $m:n$-period orbits in the case that the frequencies $\omega$ and $\nu$ are resonant. Since $m:n$-period orbits appear as a result of the effect of nonlinear frequencies, they cannot be obtained in the linear equation (11). To apply Lindstedt–Poincaré method to the high-order solution of $m:n$-period orbits, the form of (10) should be adjusted by adding a linear correction term $\Delta$ about $z$. Then the equation becomes

$$
\begin{align}
    x'' - 2y' - ax - cy &= \sum_{N \geq 2, i+j+k=N} c_{x,ijk}x^iy^jz^k \\
y'' + 2x' - cx - by &= \sum_{N \geq 2, i+j+k=N} c_{y,ijk}x^iy^jz^k \\
z'' + dz &= \sum_{N \geq 2, i+j+k=N} c_{z,ijk}x^iy^jz^k + \Delta z. 
\end{align}
$$

(18)

Our goal is to find one-dimensional torus ($m:n$-period orbits) while the condition $\Delta = 0$ is satisfied. In the iterative process up to finite order using the Lindstedt–Poincaré method, the correction term coefficient $\Delta$ is also expanded into a power series about $\alpha$ and $\beta$, i.e.,

$$\Delta = \sum_{i,j \geq 0} d_{ij} \alpha^i \beta^j,$$

where its coefficients are also obtained by iterative computation. Analogously to the procedure to find the analytical solution of Lissajous orbits, the iteration starts from the linear solution of the system

$$
\begin{align}
x'' - 2y' - ax - cy &= 0 \\
y'' + 2x' - cx - by &= 0 \\
z'' + dz &= d_{00}z,
\end{align}
$$

(19)

and the linear solution is

$$
\begin{align}
x &= \alpha \cos (\omega_0 t + \phi) \\
y &= \kappa \alpha \sin (\omega_0 t + \phi) \\
z &= \beta \cos (\nu_0 t + \phi)
\end{align}
$$

(20)

where $\omega_0 : \nu_0 = m : n$, $\phi$ is an arbitrary phase, $d_{00} = d - \frac{n^2}{m^2} \omega_0^2$. It is known that the amplitudes $\alpha$ and $\beta$ are independent for Lissajous orbits. However, the $m:n$-period orbits, as one-dimensional torus, are determined by only $\alpha$ or $\beta$. $\alpha$ and $\beta$ are coupled and their relationship is constrained by the condition $\Delta = 0$, i.e., $\alpha = \alpha(\beta)$. According to $\Delta = 0$, we have $d_{00} = 0$ for linear solution (20). However, this does not meet the necessary condition for one-dimensional torus, which means that the $m:n$-period orbit has no linear solution. Further considering the nonlinear term in (18) and the symmetry of sine and cosine functions, the analytical solution around the equilibrium can be expanded in
the power series as

\[
x(t) = \sum_{i+j \geq 1} \left( \sum_{0 \leq k \leq i+j} x_{ijk} \cos(k \theta) \right) \alpha^i \beta^j
\]

\[
y(t) = \sum_{i+j \geq 1} \left( \sum_{0 \leq k \leq i+j} y_{ijk} \sin(k \theta) \right) \alpha^i \beta^j
\]

\[
z(t) = \sum_{i+j \geq 1} \left( \sum_{0 \leq k \leq i+j} z_{ijk} \cos(k \theta) \right) \alpha^i \beta^j
\]

where \( \theta = \Omega t + \phi, \frac{\omega}{m} = \frac{\nu}{\bar{n}} = \Omega, \Omega = \sum_{i,j \geq 0} \Omega_{ij} \alpha^i \beta^j, \Delta = \sum_{i,j \geq 0} d_{ij} \alpha^i \beta^j = 0. \)

Then, the \( N \)-order analytical solution can be obtained by calculating the coefficients \( x_{ijk}, y_{ijk}, z_{ijk}, \Omega_{ij}, \) and \( d_{ij} \) up to \( N \) order. Similarly, \( k \) has the same parity as \( (i + j) \) due to the symmetry of the system. The series of \( \Omega \) and \( \Delta \) only include those even items.

For the coefficients of linear part in (21), compared to (20), we can know \( x_{101} = 1, y_{101} = \kappa, y_{011} = 1, \Omega_{000} = \omega_0/m, d_{000} = d - \frac{v^2}{m^2} \omega_0^2. \) By substituting this linear solution into (18), the coefficients of the second-order solution can be solved using the Lindstedt–Poincaré method. Similarly, when the coefficients up to order \( N - 1 \) are obtained, i.e., \( x(t), y(t), \) and \( z(t) \) are determined up to order \( N - 1, \) \( \omega \) and \( \Delta \) are determined up to order \( N - 2. \) Substituting them into the right side of (18), we can obtain four power series up to \( N \) order, denoted by \( p, q, r, \) and \( \Delta z. \) Similarly, we are interested in the \( N \)-order terms of \( p, q, r, \) and \( \Delta z, \) which are denoted by \( p_{ijkm}, q_{ijkm}, r_{ijkm}, \) and \( d_{ij}(i+j = N) \) respectively, and the \( N \)-order terms of \( \Omega \) denoted by \( \Omega_{ij}. \)

To compute the \( N \)-order power series of \( m : n \)-period orbits, we should also analyze the composition of the \( N \)-order term on the left side of (18). Different from Lissajous orbits, the \( N \)-order unknown term of \( \Delta z \) appears in \( m : n \)-period orbits. The unknown and known terms of the first derivatives for \( m : n \)-period orbits are summarized in Table. 5. The known terms are added to the corresponding series \( p_{ijk}, q_{ijk}, \) and \( r_{ijk}, \) and re-denoted by \( \bar{p}_{ijk}, \bar{q}_{ijk}, \) and \( \bar{r}_{ijk}. \)

The second derivatives can also be similarly summarized in Table. 6, where \( f \) represents the series corresponding to \( \Omega^2. \) \( 2v/2 \) represents the second derivatives of \( x, y, \) and \( z. \) \( f_{1-i-j} = 2m^2 \Omega_{00} \Omega_{1-1-j} + m^2 \sum C_2^1 \delta_{jk} \delta_{i1} \delta_{i2} \Omega_{1i1j} \Omega_{2i2j}, \)

Finally, the linear equation of \( N \)-order unknown term is yielded

\[
\begin{bmatrix}
-(k^2 m^2 \Omega_{00}^2 + a)x_{ijk} - (2km \Omega_{00} + c)y_{xjk} \\
-2(m \Omega_{00} + \kappa)m \Omega_{1-1-j} \delta_{jk} \\
-(2km \Omega_{00} + c)x_{ijk} - (k^2 m^2 \Omega_{00}^2 + b)y_{yjk}
\end{bmatrix}
= \begin{bmatrix}
\bar{p}_{ijk} \\
\bar{q}_{ijk} \\
\bar{r}_{ijk}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2(m \Omega_{00} + 1)m \Omega_{1-1-j} \delta_{jk}
\end{bmatrix}
= \begin{bmatrix}
-(d - k^2 m^2 \Omega_{00}^2 - d_{0})z_{ijk} \\
- (d_{ij-1} + 2m^2 \Omega_{00} \Omega_{1-1-j}) \delta_{jk}
\end{bmatrix}
\]

(22)
Table 5: Derivatives of $x$ and $y$ with respect to time for $m: n$-period orbits

<table>
<thead>
<tr>
<th>$f$</th>
<th>$g$</th>
<th>$\Omega(\partial x/\partial \theta_1)$</th>
<th>$\Omega(\partial y/\partial \theta_1)$</th>
<th>$\Delta z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unknown term</td>
<td>0</td>
<td>$-\Omega_{0k} k m x_{ijk}$</td>
<td>$-\Omega_{0k} k m y_{ijk}$</td>
<td>$d_{00} z_{ijk}$</td>
</tr>
<tr>
<td>$N - 1$</td>
<td>1</td>
<td>$-\Omega_{i-1j} m \delta_{jk}$</td>
<td>$\kappa \Omega_{i-1j} m \delta_{jk}$</td>
<td>$d_{ij-1} \delta_{jk}$</td>
</tr>
<tr>
<td>Known term</td>
<td>$1, 2, \ldots, N - 2$</td>
<td>$N - i_j$</td>
<td>$-\Omega_{i-1j} k m x_{ijk}$</td>
<td>$-\Omega_{i-1j} k m y_{ijk}$</td>
</tr>
</tbody>
</table>

Table 6: Second derivatives of $x$, $y$ and $z$ with respect to time for $m: n$-period orbits

<table>
<thead>
<tr>
<th>$f$</th>
<th>$\partial^2 x/\partial \theta^2$</th>
<th>$\Omega^2 (\partial^2 x/\partial \theta^2)$</th>
<th>$\Omega^2 (\partial^2 y/\partial \theta^2)$</th>
<th>$\Omega^2 (\partial^2 z/\partial \theta^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unknown term</td>
<td>0</td>
<td>$-\Omega_{0k}^2 k^2 m^2 x_{ijk}$</td>
<td>$-\Omega_{0k}^2 k^2 m^2 y_{ijk}$</td>
<td>$-\Omega_{0k}^2 k^2 n z_{ijk}$</td>
</tr>
<tr>
<td>$N - 1$</td>
<td>1</td>
<td>$-\Omega_{i-1j} k^2 m^2 x_{ijk}$</td>
<td>$-\Omega_{i-1j} k^2 m^2 y_{ijk}$</td>
<td>$-\Omega_{i-1j}^2 k^2 n^2 z_{ijk}$</td>
</tr>
<tr>
<td>Known term</td>
<td>$1, 2, \ldots, N - 2$</td>
<td>$N - i_j$</td>
<td>$-\Omega_{i-1j}^2 k^2 m^2 x_{ijk}$</td>
<td>$-(\Omega_{i-1j} i j_{i-1}) k^2 m^2 y_{ijk}$</td>
</tr>
</tbody>
</table>

where

$$\bar{p}_{ijk} = \bar{p}_{ijk} + \delta_{jk} m^2 \sum C^1_{i} \Omega_{i1} \Omega_{i2}$$
$$\bar{q}_{ijk} = \bar{q}_{ijk} + \kappa \delta_{jk} m^2 \sum C^1_{i} \Omega_{i1} \Omega_{i2}$$
$$\bar{r}_{ijk} = \bar{r}_{ijk} + \delta_{ijk} \kappa \kappa \Omega_{i1} \Omega_{i2}$$

(23)

When $k \neq 1$, the coefficients $x_{ijk}$, $y_{ijk}$, and $z_{ijk}$ can be solved immediately. When $k = 1$, $x_{ijk}$, $y_{ijk}$ are coupled, and the coefficient of $z_{ijk}$ is zero. Thus, $x_{ijk}$ and $z_{ijk}$ can be set zero, and then $y_{ijk}$ and $\Omega_{i-1j}$ are solved by the first two equations of (22). $d_{ij-1}$ are solved by the last equation of (22). Finally, it is noted that $\Delta = \sum_{i,j \geq 0} d_{ij} \alpha^i \beta^j = 0$ also gives the approximate analytical condition of a bifurcation of $m: n$-period orbits from planar or vertical Lyapunov periodic orbit. The higher the order of series, the higher the accuracy of the bifurcation point. Based on this analytical condition, the bifurcation point and bifurcation direction can be quickly calculated in the continuation of periodic orbit families.

4 Simulation results

4.1 Lissajous orbits

The series expansion of Lissajous orbits around $L_4$ in the Coulomb formation system is computed in this subsection. Table 7 gives their coefficients up to order 3. When $\alpha = 0$ and $\beta \neq 0$, the series solution (13) corresponds to a
Table 7: The coefficient of the series expansion of Lissajous orbits about $L_4$ up to order 3

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$\omega_{ij}$</th>
<th>$v_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2.06367242050136</td>
<td>1.99689569339432</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>5.06561129136032</td>
<td>-0.652763403182743</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>4.06252300819152</td>
<td>0.935894801432859</td>
</tr>
</tbody>
</table>

vertical Lyapunov orbit near the equilibrium. Fig. 3 shows a family of vertical Lyapunov orbits around with the order $N = 30$ and the amplitude $\beta \in [0.001, 1.0]$. Taking the series solution as the initial value, the exact numerical solution with the corresponding amplitude can be obtained by using the differential correction [29]. Fig. 4 gives the comparison of the time-varying positions of the linear analytical solution, the 10-order analytical solution, and the accurate numerical solution in an orbit period. It can be observed that the linear solution ignores the motion in the $X$ and $Y$ directions, and thus, it can only approximate the vertical Lyapunov when the amplitude is very small. However, the 10-order analytical solution almost coincides with the exact numerical solution. Finally, Fig. 5 shows the error distribution of the analytical solution of the vertical Lyapunov orbits with the order from 1 to 20 and of the amplitude from $[0.001, 0.7]$. The higher-order analytical solution can also be well approximated to the vertical Lyapunov with large amplitude.

When $\alpha \neq 0$ and $\beta = 0$, the series solution (13) corresponds to a planar Lyapunov orbit near the equilibrium. Fig. 6 shows a family of planar Lyapunov orbits with the order $N = 30$ and the amplitude $\alpha \in [0.001, 0.35]$. The time-varying positions of the linear analytical solution, the 10-order analytical solution, and the exact numerical solution in an orbit period are compared as shown in Fig. 7. It is shown that the error of the linear solution is very large, while the 10-order analytical solution almost coincides with the accurate numerical result, and its error magnitude is $10^{-3}$ m. Although the analytical solution of planar orbits has high accuracy, it is nearly 3 orders of magnitude lower than the analytical solution of vertical Lyapunov orbit ($10^{-6}$ m). Fig. 8 shows the error distribution of the analytical solution of the planar Lyapunov orbits with the order from 1 to 20 and of the amplitude from $[0.001, 0.4]$. Comparing Fig. 5 with Fig. 8, we can conclude that for the Coulomb formation system, the accuracy of the vertical Lyapunov orbit is much higher than that of the
Fig. 3: Family of vertical Lyapunov orbits about $L_4$ in Coulomb formation system. ($N = 30$, $\alpha = 0$, $\beta \in [0.001, 1.0]$)

Fig. 4: Comparison of analytical solutions of different orders for vertical Lyapunov orbits. black dash-dotted: linear solution; red dash: 10-order, and blue solid: exact numerical solution ($\alpha = 0$, $\beta = 0.1$).

planar Lyapunov orbit in the case of the same order of the analytical solution and the same amplitude. This is because, for an unstable periodic orbit, the deviation between the series solution and the numerical solution will enlarge rapidly with time. On the contrary, the vertical Lyapunov orbit is equivalent to a simple harmonic motion in the $Z$ direction, and its stability is high, so its analytical solution has the highest accuracy.
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Fig. 5: Position error variation between the analytical solution and exact numerical solution. ($N \in [1, 20], \alpha = 0, \beta \in [0.001, 0.7]$)

Fig. 6: Family of the planar Lyapunov orbit in Coulomb formation system. ($N = 30, \beta \in [0.001, 0.35], \beta = 0$)

When $\alpha \neq 0$ and $\beta \neq 0$, (13) corresponds to a general Lissajous orbit near the equilibrium. Fig. 9 shows a Lissajous orbit near $L_4$ with the order $N = 20$, and the amplitudes $\alpha = 0.08, \beta = 0.1$. The comparison with the corresponding exact numerical solution and its position error in a time interval of dimensionless length $\pi$ are shown as in Fig. 10. It should be noted that the Lissajous orbit of the exact numerical solution is very difficult to calculate due to the hyperbolic characteristics of $L_4$. The numerical solution here is obtained by taking the initial value provided by the analytical solution with...
Fig. 7: Comparison of analytical solutions of different orders for the planar Lyapunov orbit. \( N = 20, \alpha = 0.1, \beta = 0 \)

Fig. 8: Position error variation of the analytical solution and exact numerical solution for the planer Lyapunov orbits. \( N \in [1, 20], \alpha \in [0.001, 0.4], \beta = 0 \)

\( N = 10 \) as the initial condition, and then numerically integrating the Coulomb formation system equation. Fig. 10(b) indicates that even if the initial error of the analytical solution is very small, it will increase exponentially with the evolution of the orbit, and the error expansion speed in the \( X \) and \( Y \) directions is much greater than that in the \( Z \) direction.

Finally, by comparing the higher-order analytical solution with the exact numerical solution, the convergence region of the higher-order analytical solution can be given. Specifically, for an analytical solution with the given order,
its initial condition is obtained after the amplitudes $\alpha$ and $\beta$ are given, and then the dynamic equation is numerically integrated with the normalized integration time $T = \pi$ (about one orbital period). By comparing the position of analytical solution and exact numerical solution at the final time $T_f = \pi$, the analytical solution up to order $N$ is regarded as practically convergent if the Euclidean norm of position error is less than $10^{-3}$m. It should be noted that due to the small denominator problem, the analytical solution obtained by the Lindstedt-Poincaré method is not convergent in fact. "Practical convergence" refers to that the error of analytical solution during one orbital period is less than a certain threshold, which is used to describe the degree of the high-order analytical solution approaching the exact numerical solution. The reason why the integration time of one orbital period is selected here is that the orbit of the equilibrium is unstable, and even the accurate numerical solution diverges rapidly after several periods. Therefore, comparing the error during one period is enough to illustrate whether the analytical solution converges practically. For the selection of the threshold value, other values can also obtain similar results. Fig. 11 shows the practical convergence domain of the analytical solution of Lissajous orbits up to different orders concerning the amplitudes $\alpha$ and $\beta$. The results indicate that the convergence region of the series solution obtained by the Lindstedt-Poincaré method enlarges with the increasing order. On the one hand, the convergence domain shows the validity of the construction of the high-order analytical solution. On the other hand, it also provides a reference benchmark for when to use the analytical solutions. If the required accuracy is less than or equal to the specified accuracy of the convergence domain, the orbit can be approximated by the analytical solution in the convergence domain. For the case outside the convergence domain, the
analytical solution cannot be directly applied, but at this time, it can still provide a good initial value to obtain the exact numerical solution by employing the differential correction.

4.2 $m : n$-periodic orbits

In this subsection, the series expansion of $m : n$-period around $L_4$ in the Coulomb formation system is studied. First, when $(m, n) = (1, 1)$, the analyt-
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Fig. 11: Domain of practical convergence of analytical solution with $N = 3, 5, 10, 40$.

Fig. 12: Family of Halo orbits around $L_4$ in Coulomb formation system and their projections on the $xy$, $xz$, and $yz$ planes. ($N = 20$, $\alpha \in [\alpha_{\text{min}}, 0.3]$)

The analytical solution (21) represents a typical Halo orbit. Fig. 12 shows the family of Halo orbits with $N = 20$ and $\alpha \in [\alpha_{\text{min}}, 0.3]$, where $\alpha_{\text{min}} = 0.137099841192699$. $\alpha_{\text{min}}$ can be obtained by solving the equation $\Delta = \sum_{i,j \geq 0} d_{ij} \alpha^i \beta^j = 0$ with $\beta = 0$. It corresponds to the planar Lyapunov orbit near the equilibrium which the Halo orbit bifurcate from as shown in Fig. 13 (red solid line). Besides, $\Delta = \sum_{i,j \geq 0} d_{ij} \alpha^i \beta^j = 0$ can also analytically describe the relationship between $\alpha$ and $\beta$ of Halo orbits as shown in Fig. 13 when $N = 20$. The convergence of
the series solution of Halo orbits is also investigated. Fixed amplitude $\alpha = 0.2$, when the order $N$ is from 3 to 20, the corresponding position error of the series solution of the Halo orbit is shown in Fig. 14. It is seen that when the order is small, the position error is large and reaches the magnitude of 1 m, while the error is less than 0.1 mm when the order is $N = 20$, which can fully meet the requirements for the practical formation mission. To further evaluate the accuracy of the analytical solution, the initial residual acceleration (IRA) of Halo orbits is computed as follows. First, the spacecraft moves along the orbit determined by the series solution in a given time interval, and the corresponding analytical acceleration can be obtained by second-order derivation of the series solution; Then, the numerical acceleration is obtained by substituting the state of the spacecraft into (5). Finally, the average residual acceleration is the average value of the deviation between the numerical acceleration and the analytical acceleration of the spacecraft during this time. Fig. 14 shows that the higher the order of the series solution, the smaller the average residual acceleration, which further demonstrates the validity of the high-order analytical solution.

Finally, the case of general $m : n$-period orbits is investigated. Fig. 15 shows the relationship between the amplitudes $\alpha$ and $\beta$ of $m : n$-period orbits around $L_4$ in Coulomb formation system under different resonance ratios. It can be observed that whether the resonance ratio is greater than or less than 1, there exists a periodic orbit with the minimum amplitudes. The red curves are those cases that the resonance ratio $m : n > 1 : 1$. The minimum $\beta$ indicates that $m : n$-periodic orbits bifurcate from a vertical Lyapunov orbit. At this time, the amplitude $\alpha$ can be arbitrarily small. On the contrary, the blue curves are those cases that the resonance ratio $m : n < 1 : 1$, where the
Fig. 14: the variation of errors with the order of series solution. Blue solid (left longitudinal axis): position error; red dash (right longitudinal axis): IRA error ($\alpha = 0.2$).

Fig. 15: Relationships between the amplitudes $\alpha$ and $\beta$ of $m:n$-period orbits around $L_4$ ($N = 20$)

$m:n$-periodic orbits bifurcate from the planar Lyapunov orbit. Besides, we can see that when the basic frequency ratio is close to 1:1, only a very large $\alpha$ can produce $m:n$-periodic orbits, and the greater the resonance ratio $m:n$, the smaller the minimum amplitude $\alpha_{\text{min}}$ where the periodic orbit bifurcates.

Fig. 16 gives the periodic orbits with resonance ratios of $m:n = 4:3, 5:4, 4:5, \text{ and } 5:6$ respectively. It shows that the projection of $m:n$-periodic orbits in $XY$ plane is actually a planar Lyapunov periodic orbit, and has the
Fig. 16: $m : n$-period orbits and their projections on the $xy$, $xz$, and $yz$ planes ($N = 10$). (a) $m : n = 4 : 3$; (b) $m : n = 5 : 4$; (c) $m : n = 4 : 5$; (d) $m : n = 5 : 6$.

properties of Lissajous orbit in the direction of $Z$ axis. The resonance ratios in Fig. 16 (a) and (b) are greater than 1:1. In these cases, the amplitudes of the orbits in $XY$ plane are very small, and the amplitudes in $Z$ axis are up to 50 m. When the resonance ratios in Fig. 16 (c) and (d) are less than 1:1, the projection in $XY$ plane is a typical large-amplitude planar Lyapunov orbit, and the projection in the $XZ$ plane is no longer symmetrical.

5 Conclusion

This paper proposes the analytical series solutions up to arbitrary finite order for Lissajous orbits and $m:n$-period orbits near the equilibrium of Coulomb formation system based on the Lindstedt-Poincaré method. The computation of the series expansion and its coefficient for the various bounded orbits are discussed in detail. The analytical expressions up to finite order for Lyapunov periodic orbit, Lissajous orbit, halo orbit, and arbitrary $m : n$-periodic orbit near the equilibrium $L_4$ of Coulomb formation system are given. Furthermore,
the series solutions also give the approximate analytical condition of a bifurcation of $m : n$-period orbits from planar or vertical Lyapunov periodic orbit. Finally, to verify the effectiveness of the series solution constructed by the Lindstedt-Poincaré method, the practical convergence regions of the series solutions of those bounded orbits are numerically calculated by using the 7-order Runge Kutta integrator. For the actual Coulomb formation mission with the given accuracy requirements, the formation configuration can be designed conveniently and quickly by using the proposed analytical series solutions.

In the further work, the proposed high-order analytical method can be extended to describe the more complicated relative motions of Coulomb formation systems, such as formation reconfiguration or time-dependent cases with elliptical reference orbits.

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References


Statements & Declarations

Competing Interests

The authors declare that they have no conflict of interest.
Author Contributions

Mingpei Lin contributed to the study conception and Methodology, and wrote the first draft of the manuscript. Jinxiu Zhang contributed to the study conception and carried out project administration. Investigation is performed by Ming Xu. Xiao Pan provided the funding and the support in writing, review and editing. All authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Data Availability

The datasets generated during and analysed during the current study are available from the corresponding author on reasonable request.