Reduced order model-inspired system identification of geometrically nonlinear structures

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Reduced order model-inspired system identification of geometrically nonlinear structures

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Abstract
In the field of structural dynamics, system identification usually refers to building mathematical models from an experimentally-obtained data set. To build reliable models using the measurement data, the mathematical model must be representative of the structure. In this work, attention is given to robust identification of nonlinear structures. We draw inspiration from reduced order modelling to determine a suitable model for the system identification. There are large similarities between reduced order modelling and system identification fields, i.e. both are used to replicate the dynamics of a system using a mathematical model with low complexity. Reduced Order Models (ROMs) can accurately capture the physics of a system with a low number of degrees of freedom; thus, in system identification, a model based on the form of a ROM is potentially more robust. Nonlinear system identification of a structure is presented, where inspiration is taken from a novel ROM to form the model. A finite-element model of the structure is built to simulate an experiment and the identification is performed. It is shown how the ROM-inspired model in the system identification improves the accuracy of the predicted response, in comparison to a standard nonlinear model. As the data is gathered from simulations, system identification is first demonstrated on the high fidelity data, then the fidelity of data is reduced to represent a more realistic experiment. A good response agreement is achieved when using the ROM-inspired model, which accounts for the kinetic energy of unmodelled modes. The estimated parameters of this model are also demonstrated to be more robust and rely on the underlying physics of the system.

Keywords: System identification, Structural dynamics, Nonlinear dynamics, Model uncertainty, Reduced order modelling

1 Introduction
The trend for aesthetic engineering structures, together with improvements in the design methodologies, has resulted in more slender, flexible and lightweight structures. These structures might satisfy the static loading resilience criteria but they often fail under certain dynamic loads, as dynamic loads can cause high amplitude vibrations which push the structures to behave beyond their linear regime [1]. This brings the need for nonlinear dynamic analysis to accurately represent the behaviour of such structures [2].

For nonlinear dynamic analysis of a system, a reliable mathematical model is required. Nonlinear system identification methods can be used
to build a model of the physical structure using data measured experimentally. The discipline of nonlinear data-driven modelling has attracted attention from a wide range of research fields; here we consider this from structural dynamics perspective. One of the earliest works on nonlinear system identification was presented by Masri [3]. Throughout the years the field has witnessed considerable growth with development of new methodologies and the rise of numerous challenges. Mainly, nonlinear system identification can be divided into following main steps: data collection and testing, nonlinearity detection, characterisation, and parameter estimation [4–6]. In the data collection step, data is gathered from the system, typically from physical experiments; in detection, the behaviour of the system is examined to identify whether the system behaves nonlinearly; while in the characterisation step, the functional form of nonlinearity is determined and a model is selected for the system; and in parameter estimation, the parameters of the selected model are estimated using the data [4]. All these phases have undergone some development, whilst, parameter estimation phase has gained significant attention as researchers have progressively found the importance of parameters’ robustness [4]. One of the main challenges of parameter estimation in nonlinear system identification is the individualistic nature of nonlinear systems compared to linear systems which have a well-defined functional form [6]. This can add uncertainty in the parameters of the selected model, and typically uncertainty quantification techniques are required to identify the confidence bound and distribution of parameters. Physics-based parametric approaches are used where a functional form is available or synthesised for the system and parameters are estimated [7–9]. Meanwhile, non-parametric approaches such as Bayesian probabilistic framework and Markov Chain Monte Carlo have been put forward to tackle the model uncertainty of nonlinear systems in identification [10–13]. Similarly, deep learning and neural networks have also opened paths to nonlinear system identification [14, 15]. These approaches can facilitate an optimum model to be selected from a set of candidate models [16], or allow representation of complex nonlinear dynamical functions through finding their intrinsic features [15]. In [17], Quaranta and co-workers present a range of computational techniques which are generally used in nonlinear system identification. Genetic programming and other artificial intelligence algorithms such as particle swarm optimisation are applied on a range of nonlinear dynamical systems [17]. These approaches have imposed a change of paradigm in nonlinear system identification in the recent years [4].

Most recently, in [18], equation discovery and parameter estimation for identification of nonlinear systems is investigated. In this work, a Bayesian framework is applied to discover relevant models from a dictionary of candidate models and the parameters are estimated. The models included in the design matrix are assumed based on a prior knowledge and the size of the matrix can be large enough to include many assumed models [18]. As the dynamics of structures get more complex, and the Degrees of Freedom (DoFs) of the model increase, so does the number of nonlinear parameters and assuming candidate models may require expert domain knowledge. This can further increase uncertainty in the estimated parameters. Models would be more reliable when there is a meaningful relation among the nonlinear parameters and physical parameters of a system.

Reduced Order Models (ROMs) for nonlinear structural dynamics were proposed as a means to improve the computational efficiency for nonlinear systems [19]. They are used as a smaller sized representative model of a complex structure through a set of second order differential equations [20]. Similarly, in system identification, the dynamics of a system are represented by a simplified model. In this paper, we use this observation and use ROM-inspired models in nonlinear system identification. Specifically, we note that the form of the model used in identification must have the same form as the reduced-order model. This not only make the identification process more reliable, but will also make the response simulation computationally cheaper than an identified model that relies on a large number of DoFs to achieve good accuracy. Also, there is a consequential relationship between these nonlinear terms and underlying physics of the system [21].

In this paper we aim to reduce the uncertainty associated with the model, hence, dealing with the characterisation step in the identification. We consider nonlinear system identification of an example
geometrically nonlinear type structure. The lower frequency bending mode is coupled with high frequency axial modes which can be captured by a single DoF ROM. This coupling occurs due to the membrane stretching in structures such as a cantilever beam [20]. System identification is applied using ROM-inspired models and their results are compared to the case where standard nonlinear stiffness model is used. As the shift in the frequency of the system versus amplitude or energy of the system, backbone curve measurements are used as the data set in identification and to compare the results [22–24]. Numerous techniques are available in the literature on measuring the backbone curves of nonlinear systems, [25–28]. Here, the backbone curve of the system is measured using the response decay of the full FE model [25].

In section 2, the structural system and its FE model is described, followed by the identification using a standard nonlinear stiffness model in section 3. Section 4 shows system identification using the ROM-inspired model with its results discussed. Section 5, demonstrates the ROM-inspired identification on pseudo-experimental data. Finally, conclusions are made in section 6.

2 Cantilever beam system

In this section the structure on which the system identification will be demonstrated is described. The system is a cantilever type structure with a massless spring attached axially at its free end, as shown in Fig. 1. This spring is unstretched when the system is at equilibrium, and gravity is not included. The corresponding parameters of the system are as summarised in Table 1. The system is modelled in FE package Abaqus [29] and meshed using Timoshenko beam elements (three-node quadratic beam, B32). A total of 120 beam elements are used, resulting in 1440 DoFs. Additionally, to allow resonant decay to be used to measure the backbone curves, the mass and stiffness proportional Rayleigh damping coefficients are applied as $\alpha = 0.5782$ and $\beta = 5 \times 10^{-6}$, respectively. These values are selected to obtain around 0.5% damping ratio for the first mode and having relatively slight increase in damping ratios for higher modes.

The generalised physical coordinates are given in the $n \times u$ matrix from the FE model, which we call $X$, $n$ representing the DoFs and $u$ being the number of time steps during the decay responses. As we will consider models expressed in modal coordinates, we must perform a linear modal transform. By applying a modal analysis in Abaqus FE package we get the eigenvalues matrix ($\Lambda$) as an $n \times n$ matrix containing $n^\text{th}$ squared natural frequency ($\omega_n^2$) in the $n^\text{th}$ element along its diagonal. Similarly, the mass-normalised modeshape matrix ($\Phi$) is obtained as an $n \times n$ matrix. In a physical experiment, these can be obtained through an experimental modal analysis. We apply $X = \Phi q$ for the modal transformation, where $q$ is $n \times u$ matrix of modal coordinates. The modeshapes for the first two bending modes and the first two axial modes of the FE model are shown in Fig. 2.

To demonstrate the coupling between the modes of the system, the steady state undamped modal displacement responses of the system, over a period of one second, are shown in Fig. 3(a). A static modal force is applied to the first bending mode of the system to get a tip displacement of around $1/3$rd of the beam length. The resulting dynamic response, after the static load is released, is dominated by the first mode, whilst the second highest amplitude response is from the $27^\text{th}$ mode, corresponding to the first axial mode. Due to membrane stretching, there exists some significant coupling between these two modes, as

![Fig. 1: Schematic plot of the cantilever beam system](image)

Table 1: The parameter values for the cantilever beam system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>0.3 m</td>
</tr>
<tr>
<td>$L_{sp}$</td>
<td>0.1 m</td>
</tr>
<tr>
<td>$K_{sp}$</td>
<td>20 Nm$^{-1}$</td>
</tr>
<tr>
<td>$A$(width× thickness)</td>
<td>0.025 m × 0.001 m</td>
</tr>
<tr>
<td>$E$(Young’s Modulus)</td>
<td>205 GPa</td>
</tr>
<tr>
<td>$v$ (Poisson’s ratio)</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho$ (Density)</td>
<td>7800 kg m$^{-3}$</td>
</tr>
</tbody>
</table>
Fig. 2: Modeshapes and natural frequencies of the first two bending and first two axial modes. a Mode-1 (bending) 57.8 rad s\(^{-1}\); b Mode-2 (bending) 362.3 rad s\(^{-1}\); c Mode-27 (axial) 26842.8 rad s\(^{-1}\); d Mode-59 (axial) 80528.5 rad s\(^{-1}\)

demonstrated in Fig. 3(b), which shows half a cycles of the steady state responses plotted against each other. Additionally, as the coupling between the first mode and other modes in the system is symmetric, this may indicate that the system is symmetric.

We assume that, when lightly damped, the decaying response of the system follows its backbone curve, therefore, the backbone curve of the system can be constructed using its decay response [25]. As the dynamics of the structure is mainly governed by the first bending mode with \(\omega_1 = 57.8\text{rad s}^{-1}\), the resonant decay is performed by applying a tip force, which is easiest to apply on a real system, on the damped structure and is displaced to around \(1/3rd\) of the beam length, \(L\). The force is released to capture the decaying response of the first mode over a period of 20 seconds, as shown in Fig. 4. Using this decay response the backbone curve for the first mode of the system is constructed. The time at every other zero-crossing is found using interpolation to identify exact zero-crossings, as \(T_0, T_1, \ldots, T_Z\), with \(T_0\) as the time of the initial zero-crossing and \(T_Z\) being the time at last zero-crossing point in the decay response. Note that additional steps are required when the response is gathered from a physical experiment, such as filtering, to remove multiple zero-crossings around the crossing points in noisy data. These steps are necessary to reduce the error in the frequency assessment procedure [25]. The maximum amplitudes are measured over the window of each cycle \((T_{z-1} \leq t \leq T_z)\), and the time period of the cycle \(z\) is approximated using zero-crossing times (i.e \(T_z - T_{z-1}\)). The response frequencies of each periodic cycle is then mapped with its corresponding maximum amplitude as the backbone curve of the system shown in Fig. 5.

This synthetic data-set will be used to perform system identification and are referred to as full FE measurements in the rest of this paper. In the following section, system identification is illustrated using a standard nonlinear stiffness model for the system.

3 System identification using a standard nonlinear stiffness model

In this section, system identification is performed using a standard nonlinear stiffness model. As the response of the system is dominated by the first mode we select a single mode model given by:

\[
\ddot{q}_r + \omega_r^2 q_r + \sum_{m=2}^{M} \gamma_m q_r^m = 0
\]  

where \(\ddot{q}_r\), \(\omega_r\) and \(q_r\) are the acceleration, natural frequency and displacement of mode \(r\), respectively. As is common in the literature, we approximate the geometric stiffness nonlinearity using a polynomial function of up to \(M^{th}\) order with \(\gamma_m\)
Fig. 3: Free steady state modal responses of the undamped system following an initial tip displacement of around 1/3rd of the beam length. a Modal amplitude responses against time; b Modal response coupling represented by the amplitude of the mode $n$ vs the amplitude of first mode

Fig. 4: Amplitude of the first mode during the resonant decay

being $m^{th}$ nonlinear parameter. Therefore, this equation is referred to as the standard stiffness nonlinear model.

For the example considered here, we select nonlinearity of up to 5th order as the base model ($M = 5$) and considering a single mode model representing the first mode ($r = 1$). Eq. (1) can be written as below:

$$\ddot{q}_1 + \omega_1^2 q_1 + \gamma_2 q_1^2 + \gamma_3 q_1^3 + \gamma_4 q_1^4 + \gamma_5 q_1^5 = 0$$  (2)

In system identification, the model is fitted into the measured data to identify the nonlinear parameters $\gamma_m$. Considering Eq. (2), we know the acceleration ($\ddot{q}_1$) and displacement responses of the first mode ($q_1$) from the backbone measurements. The remaining parameters are treated as unknown. Note that, although the linear natural frequency of the system ($\omega_1$) is known from the linear modal analysis, it is still included in the estimation. This is to avoid propagation of error from linear natural frequency to the nonlinear terms.
To identify the unknowns in Eq. (2), we propose a method based on using the Fourier information of the measurement signal. We include certain coefficients of the complex exponential form of the response measurements. Li and colleagues [15] have also demonstrated that using the frequency content of the measurements remove some sensitivity associated with the time domain measurements in the identification of different nonlinear systems.

As the backbone measurement is constructed using the decay response, each point on the curve can be assumed to be a periodic cycle of the decay. A moving window is used to extract the Fourier components of each cycle z. The lengths of moving windows are defined after the zero-crossing times (\(T_z\)) of each cycle z, whilst the Fourier components are achieved using Fast Fourier Transform (FFT) algorithm in Matlab [30], which gives the two-sided spectral components. Applying the Fourier transformation, the time domain Eq. (2) can be written as below for the initial periodic cycle:

\[
\{ F_H[\bar{q}_1(T_0 \leq t < T_1)]\} + \omega_1^2 \{ F_H[q_1(T_0 \leq t < T_1)]\} \\
+ \gamma_2 \{ F_H[q_1^2(T_0 \leq t < T_1)]\} \\
+ \gamma_3 \{ F_H[q_1^3(T_0 \leq t < T_1)]\} \\
+ \gamma_4 \{ F_H[q_1^4(T_0 \leq t < T_1)]\} \\
+ \gamma_5 \{ F_H[q_1^5(T_0 \leq t < T_1)]\} = 0
\]  

(3)

where, \(T_0 \leq t < T_1\) shows the time length of the window for the first periodic cycle. \(T_0\) is the time of the beginning and \(T_1\) the time of the end zero-crossing of the cycle 1, as described previously. \(F_H\) indicates the column vector of complex Fourier components of a general term \((w(t))\) and can be defined as below with \(T\) being its period:

\[
F_H[w(t)] = \begin{cases} \\
\frac{1}{T} \int_{0}^{T} w(t)e^{-j\omega t} dt, & h = 0 \\
\frac{1}{T} \int_{0}^{T} w(t)e^{-jh\omega t} dt, & h = 1, 2, 3, \ldots
\end{cases}
\]  

(4)

In the above, \(H = h\omega, h\) indicates the Fourier component and \(\omega\) is the fundamental frequency of the system. Once the column vector of Fourier components for each periodic cycle is extracted, the real and imaginary parts of certain components (i.e. \(h = [h_{\text{min}} : h_{\text{max}}]\)) of all data points are stacked on top of each other as a single column vector. Now Eq. (2) can be written as the following matrices:

\[
P = [\omega_1^2, \gamma_2, \gamma_3, \gamma_4, \gamma_5]^T
\]

\[
\begin{bmatrix}
\Re F_H[q_1(T_0 \leq t < T_1)] & \Re F_H[q_1^2(T_0 \leq t < T_1)] \\
\Re F_H[q_1(T_1 \leq t < T_2)] & \Re F_H[q_1^2(T_1 \leq t < T_2)] \\
\vdots & \vdots \\
\Re F_H[q_1(T_{z-1} \leq t < T_z)] & \Re F_H[q_1^2(T_{z-1} \leq t < T_z)] \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Im F_H[q_1(T_0 \leq t < T_1)] & \Im F_H[q_1^2(T_0 \leq t < T_1)] \\
\Im F_H[q_1(T_1 \leq t < T_2)] & \Im F_H[q_1^2(T_1 \leq t < T_2)] \\
\vdots & \vdots \\
\Im F_H[q_1(T_{z-1} \leq t < T_z)] & \Im F_H[q_1^2(T_{z-1} \leq t < T_z)] \\
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
\Re F_H[q_1^3(T_0 \leq t < T_1)] & \Re F_H[q_1^4(T_0 \leq t < T_1)] \\
\Re F_H[q_1^3(T_1 \leq t < T_2)] & \Re F_H[q_1^4(T_1 \leq t < T_2)] \\
\vdots & \vdots \\
\Re F_H[q_1^3(T_{z-1} \leq t < T_z)] & \Re F_H[q_1^4(T_{z-1} \leq t < T_z)] \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Im F_H[q_1^3(T_0 \leq t < T_1)] & \Im F_H[q_1^4(T_0 \leq t < T_1)] \\
\Im F_H[q_1^3(T_1 \leq t < T_2)] & \Im F_H[q_1^4(T_1 \leq t < T_2)] \\
\vdots & \vdots \\
\Im F_H[q_1^3(T_{z-1} \leq t < T_z)] & \Im F_H[q_1^4(T_{z-1} \leq t < T_z)] \\
\end{bmatrix}
\]

\[
d = \begin{bmatrix}
\Re F_H[\bar{q}_1(T_0 \leq t < T_1)] \\
\Re F_H[\bar{q}_1(T_1 \leq t < T_2)] \\
\vdots \\
\Re F_H[\bar{q}_1(T_{z-1} \leq t < T_z)] \\
\end{bmatrix}
\]

(5)

where, \(G\) is the design matrix including all the known terms in the model, \(d\) is the vector of known variables and \(P\) is the vector of unknowns.
parameters. These matrices can be represented by relation below:

\[ GP = d \]  \hspace{1cm} (6)

To identify the unknown parameters \((P)\), Eq. (6) can be applied in a least squares regression form. For this example, we use up to the \(10^{th}\) complex Fourier components including the 0 Hz (DC) component (DC component has only real part) in the identification as higher harmonics of each data point are negligible so \(h = [0; 1; 2; 3; \ldots; 10]\) in Eq. (5). This allows the fitting process to reach high levels of accuracy as the measurements are simulated and there are no much error associated with the data. We note that, such high harmonics may not be reliable for a real, physical experiment subject to noise, this will be considered in Section 5.

The 10 real (ℜ) parts, 10 imaginary (ℑ) parts and the 1 DC component of the complex exponential Fourier series of periodic cycles of data points on the backbone curve are stacked as column vectors. Such that the matrix \(G\) has \(21 \times Z\) rows (\(Z\) being number of periodic cycles). The model is fitted to the full FE measurement data, using data across a pre-determined amplitude range to which the backbone curve fit is required (and later plotted). The resulting parameters are summarised in Table 2.

Using the estimated parameters, the response of identified model from the Eq. (2) is simulated and compared with the Full FE measurements backbone curve. The known linear natural frequency is used for the response simulation as the estimated one was added to account for the error and is assumed to be less accurate. The backbone curve is computed using the Matlab-based continuation toolbox - Continuation Core (CoCo) [31]. This comparison is shown as Fig. 6.

From the backbone curves in Fig. 6, it is clear that the identified standard model is a poor representation of the full measurements, particularly at high amplitudes, despite the near-perfect data used for fitting. The estimated parameters in Table 2 cannot be relied upon as they fail to produce the correct backbone curve. To verify that this poor representation is not due to an insufficient order of the nonlinearity in the stiffness model, we also include different order terms in the standard model - nonlinearities of up to \(11^{th}\) are considered and the identified models responses are compared with the full FE measurements in Fig. 7.

Fig. 6: Full FE measurements backbone curve with the \(5^{th}\) order identified standard nonlinear model backbone curve

Fig. 7: Full FE measurements backbone curve with \(7^{th}\), \(9^{th}\) and \(11^{th}\) order identified standard nonlinear models

Fig. 7 shows that even the higher order standard models cannot capture the true response of the system and still diverge from the true measurements. This implies that the estimated parameters may not capture the true physics-based information about the system. Note that even-order models (\(6^{th}\), \(8^{th}\) and \(10^{th}\)) showed minimal differences to the odd-order models (\(5^{th}\), \(7^{th}\) and \(9^{th}\)) and so are not shown. This indicates that the even-order terms had no contribution to the response of the system.
The poor fit is likely to be due to the fitting limitation imposed by the choice of mathematical model for this structure - i.e. Eq. (1). Although polynomial terms of different order could represent stiffness-based nonlinearity, the response of cantilever type systems is dominated by longitudinal inertia and large curvature [28, 32]. Adding, conservative terms were found to improve the response of the model relative to the experimental results in [32], whilst, Urasaki and Yabuno [28], accurately identified the backbone curves of a cantilever type system by including quadratic velocity terms in a mathematical model derived through Hamilton’s principle. Similarly, here we consider a ROM structure in line with [33]. This ROM has additional terms accounting for the kinetic energy of un-modelled modes. This model with its identification is described in the next section.

4 System identification using a ROM-inspired model

The previous section demonstrated that the standard nonlinear stiffness model is insufficient for the system identification of the cantilever type system. From the ROM work in [33] it is shown that additional inertial-based terms are necessary to account for the kinetic energy of un-modelled modes of a cantilever system. Note that, whilst [33] also considers a cantilever system, the form of the ROM is general, and provides a good description of any system, reducible to a single mode, with large inertia. We use this model to inspire the form of the equations used in the system identification of a cantilever type system, in this section. Note that these equations are general, they ensure kinetic energy is captured while requiring no underlying knowledge of the system - i.e. no system-specific model is required.

The main mathematical structure of the ROM, which is referred to as Inertially-Compensated ROM (IC-ROM), can be expressed as below:

\[
\ddot{q}_r + \left[\begin{array}{c}
\frac{\partial q_1}{\partial q_r} \\
\frac{\partial q_2}{\partial q_r} \\
\vdots \\
\frac{\partial q_s}{\partial q_r}
\end{array}\right]^T \left[\begin{array}{c}
\frac{\partial^2 q_1}{\partial q_r^2} \\
\frac{\partial^2 q_2}{\partial q_r^2} \\
\vdots \\
\frac{\partial^2 q_s}{\partial q_r^2}
\end{array}\right] q_r^2
+ \omega^2 q_r + \sum_{m=2}^{M} \gamma_m q_r^m = 0
\]

\[g_i = \sum_{m=2}^{M} \beta_{i,m} q_r^m\]  

(8)

Compared to the standard model, two additional terms appear in Eq. (7) where, the partial derivatives of a function \(g_i\) with respect to \(q_r\) are included for each unmodelled mode \(i\). The function \(g_i\) represents the coupling between the reduced mode \(q_r\) and mode \(i\) (which includes all modes other than the reduced mode). \(s\) is equal to the total number of modes subtracted by the number of reduced modes (here, 1). \(\beta_{i,m}\) represents the \(m^{th}\) order coupling function coefficient relating to the \(i^{th}\) un-modelled coupled mode.

We have a large number of parameters \(\beta_{i,m}\) to estimate. To reduce the number of independent parameters we first write the function \(g_i\) in the following vector form:

\[g = \begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_s
\end{bmatrix} = \beta q = \begin{bmatrix}
\beta_{1,2} \beta_{1,3} \cdots \beta_{1,M} \\
\beta_{2,2} \beta_{2,3} \cdots \beta_{2,M} \\
\vdots \\
\beta_{s,2} \beta_{s,3} \cdots \beta_{s,M}
\end{bmatrix} \begin{bmatrix}
q_r^2 \\
q_r^3 \\
\vdots \\
q_r^M
\end{bmatrix}\]

(9)

Table 2: Estimated parameters of the standard nonlinear model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(\omega_1^2)</th>
<th>(\gamma_2)</th>
<th>(\gamma_3)</th>
<th>(\gamma_4)</th>
<th>(\gamma_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated value</td>
<td>(3.4113 \times 10^4)</td>
<td>(3.7000 \times 10^2)</td>
<td>(7.4409 \times 10^6)</td>
<td>(-2.1804 \times 10^6)</td>
<td>(-3.1435 \times 10^{10})</td>
</tr>
</tbody>
</table>
where $\beta$ is $s \times K$ (with $K = M - 1$) coefficients matrix and $\bar{q}$ is $K$ length vector, derivatives of which with respect to $q_r$ may be written as:

$$
\dot{q}' = \begin{bmatrix}
2q_r \\
3q_r^2 \\
\vdots \\
(K+1)q_r^K
\end{bmatrix} \quad \text{and} \\
\ddot{q}'' = \begin{bmatrix}
2 \\
6q_r \\
\vdots \\
(K^2 + K)q_r^{K-1}
\end{bmatrix}
$$

By substituting these into the inertial compensation terms of Eq. (7), we get:

\[
\begin{bmatrix}
\frac{\partial r}{\partial q_1} \\
\frac{\partial r}{\partial q_2} \\
\vdots \\
\frac{\partial r}{\partial q_s}
\end{bmatrix}^T \begin{bmatrix}
(\frac{\partial q_1}{\partial q_1})_r \\
(\frac{\partial q_2}{\partial q_1})_r \\
\vdots \\
(\frac{\partial q_s}{\partial q_1})_r
\end{bmatrix} \ddot{q}_r + \begin{bmatrix}
(\frac{\partial q_1}{\partial q_1})_r \\
(\frac{\partial q_2}{\partial q_1})_r \\
\vdots \\
(\frac{\partial q_s}{\partial q_1})_r
\end{bmatrix}^T \begin{bmatrix}
(\frac{\partial q_1}{\partial q_2})_r \\
(\frac{\partial q_2}{\partial q_2})_r \\
\vdots \\
(\frac{\partial q_s}{\partial q_2})_r
\end{bmatrix} \ddot{q}'_r
\]

\[
= (\beta \dot{q}'^T \beta \dot{q}') \ddot{q}_r + (\beta \ddot{q}''^T \beta \dddot{q}') \dddot{q}_r^2
\]

In (11), $\beta^T \beta$ will end up in a $K \times K$ matrix with $\beta$ terms for each coupled mode (i.e. $i = 2, 3, \ldots, 1400$ for the example considered here) in each element of the matrix. As this will be a symmetric matrix, for simplification the $\beta^T \beta = \bar{\beta}$ matrix may be written as below with number of $\beta$ terms dramatically reduced from 5756 parameters to $K \times K$ parameters:

$$
\bar{\beta} = \beta^T \beta = \begin{bmatrix}
\bar{\beta}_{1,1} & \bar{\beta}_{1,2} & \cdots & \bar{\beta}_{1,K} \\
\bar{\beta}_{1,2} & \bar{\beta}_{2,2} & \cdots & \bar{\beta}_{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{\beta}_{1,K} & \bar{\beta}_{2,K} & \cdots & \bar{\beta}_{K,K}
\end{bmatrix}
$$

now using $\bar{\beta}$, (11) gives:

\[
(\beta \dot{q}'^T \beta \dot{q}') \ddot{q}_r + (\beta \ddot{q}'^T \beta \ddot{q}') \dddot{q}_r^2
\]

\[
= (\dot{q}'^T \bar{\beta} \dot{q}') \ddot{q}_r + (\ddot{q}'^T \bar{\beta} \ddot{q}') \dddot{q}_r^2
\]

We select an IC-ROM of up to $5^{th}$ order for both the coupling function ($g_i$) and nonlinear terms. Whilst using (13) & (12), the inertial compensation terms are derived as below with $K = 4$:

\[
(\dot{q}'^T \bar{\beta} \dot{q}') \ddot{q}_1 + (\ddot{q}'^T \bar{\beta} \ddot{q}') \dddot{q}_1^2
\]

\[
= (4q_1^2q_1 + 4q_1^2q_1^3) \bar{\beta}_{1,1} + (12q_1^2q_1^3 + 18q_1^3q_1^2) \bar{\beta}_{1,2} + (16q_1^4 + 32q_1^3q_1^3) \bar{\beta}_{3,1} + (20q_1^5 + 50q_1^2q_1^4) \bar{\beta}_{1,4} + (9q_1^6 + 18q_1^2q_1^5) \bar{\beta}_{2,2} + (24q_1^7 + 60q_1^2q_1^6) \bar{\beta}_{2,3} + (30q_1^6 + 90q_1^2q_1^5) \bar{\beta}_{2,4} + (16q_1^6 + 48q_1^2q_1^5) \bar{\beta}_{3,3} + (40q_1^7 + 140q_1^2q_1^6) \bar{\beta}_{3,4} + (25q_1^8 + 100q_1^2q_1^7) \bar{\beta}_{4,4}
\]

\[
\sum_{j=3}^{9} \left(4q_1 q_1^3 + 4q_1^2 q_1^3 \right) Y_j + \left(12q_1^2 q_1^3 + 18q_1^3 q_1^2 \right) Y_4 + \left(4q_1^2 q_1^3 + 2q_1^2 q_1^3 \right) Y_5 + \left(4q_1 q_1^3 + 10q_1^2 q_1^3 \right) Y_6 + \left(2q_1^2 q_1^3 + 6q_1^2 q_1^5 \right) Y_7 + \left(40q_1^7 + 140q_1^2 q_1^6 \right) Y_8 + \left(25q_1^8 + 100q_1^2 q_1^7 \right) Y_9
\]

\[
Y_j \text{ represents the } j^{th} \text{ order (the total order of the term) inertial compensation parameter which relates to } \beta \text{ parameters in Eq. (14) as below:}
\]

\[
Y_3 = \bar{\beta}_{1,1}, \quad Y_4 = \bar{\beta}_{1,2}, \quad Y_5 = 16\bar{\beta}_{1,3} + 9\bar{\beta}_{2,2}, \\
Y_6 = 5\bar{\beta}_{1,4} + 6\bar{\beta}_{2,3}, \quad Y_7 = 15\bar{\beta}_{2,4} + 8\bar{\beta}_{3,3}, \quad Y_8 = \bar{\beta}_{3,4}, \quad Y_9 = \bar{\beta}_{4,4}
\]

From the geometry of the specific example of the cantilever beam, we can assume that it is symmetric. In symmetric systems the non-linearity is deemed to be dominated by odd order terms [34]. Also, as noticed in the previous section, even order terms had least affect on the improvement of the backbone curves for the identified standard stiffness nonlinear model. So the even order terms in Eq. (15) are removed, and as we only have up to $5^{th}$ order stiffness nonlinearity we only include third and fifth order IC terms (relating to $Y_3$ and $Y_5$) into (7) with terms relating to $\gamma_3$ and $\gamma_5$. This will construct a fifth order IC-ROM as the model of the structural system as below:

\[
\ddot{q}_1 + Y_3 (4\dot{q}_1 q_1^3 + 4q_1^2 q_1^3) + Y_5 (\dot{q}_1 q_1^4 + 2q_1^2 q_1^3) + \omega_1^2 q_1 + \gamma_3 q_1^3 + \gamma_5 q_1^5 = 0
\]}
This model can now be fitted to the simulated measurements to estimate the unknown parameters. The acceleration, velocity and displacement ($\dot{q}_1$, $\dot{q}_1$ and $q_1$) are known and the remaining parameters are treated as unknown. As previously, linear natural frequency is included in the estimation, to account for any associated error propagating into nonlinear terms. Note that the simulated data contains no error, however the identification using data which is closer to a real experiment, will be demonstrated later in the paper.

Similar to the standard model fitting, we use the complex exponential Fourier components ($\mathcal{F}_H$) of the measurements in the estimation so the identical data set as previous section is used. $\mathcal{F}_H$ is computed for each term in the Eq. (16) after Eq. (4). By applying the transformation the time domain Eq. (16) can be written in matrix form as:

$$
\mathbf{\tilde{G}} = \\
\begin{bmatrix}
\Re\mathcal{F}_H[(4\ddot{q}_1q_1^2 + 4q_1^2q_1)(T_0 \leq t < T_1)] \\
\Re\mathcal{F}_H[(4\ddot{q}_1q_1^2 + 4q_1^2q_1)(T_1 \leq t < T_2)] \\
\vdots \\
\Im\mathcal{F}_H[(4\ddot{q}_1q_1^2 + 4q_1^2q_1)(T_{Z-1} \leq t < T_Z)] \\
\Im\mathcal{F}_H[(4\ddot{q}_1q_1^2 + 4q_1^2q_1)(T_{T-1} \leq t < T_2)] \\
\vdots \\
\Im\mathcal{F}_H[(4\ddot{q}_1q_1^2 + 4q_1^2q_1)(T_{Z-1} \leq t < T_Z)] \\
\end{bmatrix}
$$

(17)

$$
\mathbf{\tilde{P}} = [\mathbf{\tilde{G}}^T, \mathbf{\tilde{P}}^T] \\
$$

where each column of the matrix $\mathbf{\tilde{G}}$ corresponds the terms in (16), in the same order as unknown parameters in vector $\mathbf{\tilde{P}}$, and the known variables vector $\mathbf{d}$ remain the same as standard model fitting described in Eq. (5). The DC, 1st, 3rd and 5th Fourier components of measurements are added in fitting process ($h = [0; 1; 3; 5]$), as the odd components are dominant in symmetric systems and are due to odd order terms in the model [1, 35]. As such, 3 real ($\Re$), 3 imaginary ($\Im$) and 1 DC components of complex Fourier components ($\mathcal{F}_H$) of each periodic cycles are included in the identification. This generates the $\mathbf{\tilde{G}}$ matrix with $7 \times Z$ rows. To identify the parameters $\mathbf{\tilde{P}}$ in $\mathbf{G}\mathbf{\tilde{P}} = \mathbf{d}$, we apply the problem in a least squares form same as previous section.

The model is fitted to the simulated data at all levels of amplitude of the backbone curve, whilst the identification algorithm remains similar to the previous section. The parameters of the model are estimated as summarised in Table 3. The parameters from direct reduced order modelling [33] are also included as the true parameters. The existence of an FE model of the structural system has allowed the true parameters to be obtained, however, in system identification cases, only a physical system is available, so direct reduced order modelling cannot be applied.

In the Table 3 the relative error ($RE$) between the true ($\mathbf{\tilde{P}}_{true}$) and estimated ($\mathbf{\tilde{P}}$) parameters are calculated using the relation below:

$$
RE = \|\mathbf{\tilde{P}}_{true} - \mathbf{\tilde{P}}\| / \|\mathbf{\tilde{P}}_{true}\| \times 100\% 
$$

The relative errors for most parameters appear small, however, the error in $\mathbf{\tilde{P}}_5$ is significantly bigger than the others and will be discussed later in this section.

Now using the estimated parameters we reconstruct the model in Eq. (16). The estimated natural frequency is not included in the identified model, as it was added in the estimation to avoid error propagation into nonlinear terms. The response of the identified model is simulated in Matlab based continuation toolbox CoCo, and compared with the full FE measurements as shown in Fig. 8.

From the backbone comparison, a good match is shown between the Full FE and the identified ROM-inspired model. Now the estimated parameters which were defined based on the physics of the system can be reliable.

As was demonstrated, identifying the cantilever type structural system using a standard model which contains different order nonlinear
Table 3: Estimated parameters of the ROM-inspired model compared with the true parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\bar{\Upsilon}_3$</th>
<th>$\bar{\Upsilon}_5$</th>
<th>$\omega_1^2$</th>
<th>$\gamma_3$</th>
<th>$\gamma_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated ($\bar{P}$)</td>
<td>$2.237 \times 10^5$</td>
<td>$5.0444 \times 10^5$</td>
<td>$3.3491 \times 10^4$</td>
<td>$9.6142 \times 10^6$</td>
<td>$-1.7800 \times 10^{10}$</td>
</tr>
<tr>
<td>True ($P_{true}$)</td>
<td>$2.1710 \times 10^5$</td>
<td>$6.9109 \times 10^5$</td>
<td>$3.3426 \times 10^5$</td>
<td>$9.8202 \times 10^6$</td>
<td>$-1.7966 \times 10^{10}$</td>
</tr>
<tr>
<td>RE (%)</td>
<td>3.06</td>
<td>27</td>
<td>0.20</td>
<td>2.10</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Fig. 8: Full FE measurements backbone curve with 5th order identified ROM-inspired model backbone curve

terms does not give good match with the measurements of the system. From the results of the standard nonlinear stiffness model, adding nonlinear terms of up to 11th order showed that the identified responses diverge from the true measurements at all levels of amplitudes. This is mainly due to the model not being able to capture the nonlinearity of the beam and indicates that identifying a structure using a model with different candidate model terms can give a set of parameters which does not represent the structure. Adding some additional terms containing displacement-acceleration and displacement-velocity terms as inertial compensation for the kinetic energy of unmodelled modes has resulted in better response match with the measurements. Fig. 8, shows that the 5th order ROM based model gives good match with the measurements at all levels. However, the number of parameters in the ROM based model is similar to the standard model, the second with the same number of parameters was unable to capture the true dynamics of the system.

To better understand the contribution of additional terms in the ROM-based model, the significance of the nonlinear terms are shown in Fig. 9. The absolute magnitude of the nonlinear terms (i.e. the estimated parameter times its corresponding response term) are shown as bars at highest, middle, lower backbone response amplitude levels. It is shown that the contribution of third order stiffness nonlinear term ($\gamma_3 q_3^3$), is significant across different levels of amplitude. Similarly, the significance of the fifth order term ($\gamma_5 q_5^5$) is considerable at higher amplitude, slightly reduces at mid-level and fluctuates at lower level. This demonstrates that the coupling between the lower frequency bending and high frequency axial modes increase in higher amplitudes, thus, higher order terms ($\gamma_5 q_5^5$ here), are needed to capture the dynamic coupling. This also demonstrates that higher order nonlinearity in ROM lead to better accuracy in the response. Although, the coupling is less in lower levels of amplitude, hence, the higher order term has least significance.

Fig. 9: Full FE measurements backbone curve with nonlinear parameters sizes at different amplitude levels

In Fig. 9, the absolute relative magnitudes of inertial compensation terms ($\Upsilon_3(4q_3 q_1^4 + 4q_1^2 q_3)$ and $\Upsilon_5(q_1 q_1^4 + 2q_3^2 q_3)$) are shown at high, mid and lower backbone response amplitudes. Whilst, the inclusion of higher order terms is deemed to
be necessary to capture for the high inertial non-linearity in high amplitude levels. At the highest amplitude on the backbone curve, the contribution of third order IC-term \( \Upsilon_3(4\ddot{q}_1q_2^2 + 4\ddot{q}_2^2q_1) \) is significant compared to the fifth order IC-term \( \Upsilon_5(\dot{q}_1q_1^4 + 2\dot{q}_2^2q_1^2) \). They diminish at mid-level whilst the only considerable magnitude of IC-terms goes to third order at a lower response amplitude - meaning that inertial compensation is always needed for this system. This shows that including higher order inertial compensation terms are necessary to capture the response of the structural system at higher levels of amplitude. Also, from Table 3 the estimation of \( \Upsilon_5 \) is associated with more error. Given its low relative magnitude at all the levels, the amount of error has a negligible influence on the response.

The estimated parameters summarised in Table 3, can be more robust and carry information about the underlying physics of the structural system. Also, these terms are derived based on the physics of the structural system hence, the uncertainty in the model of the structural system has been tackled throughout this work and the estimated parameters can be more reliable.

Considering that data was perfectly clean out of the FE model and not close to reality, in the next section we demonstrate the methodology while using a data set that is more representative of a real physical test.

5 ROM-inspired identification using a simulated experiment

The data used in previous sections consisted of 240 node measurements, and each node had 3 translational and 3 rotational movements. This resulted in 1440 measurements in physical coordinates. In this section, to replicate a more realistic physical experimental scenario, the measurement set is dramatically reduced, such that measurements are taken from only 5 points in two DoFs along the length of the beam as shown as circular dots in Fig. 10. These measurement points are located at 20mm, 90mm, 160mm, 230mm and 300mm from the fixed boundary on the left of the beam.

The structure is given an initial deflection similar to the previous section and for each measurement point, decay response measurements are extracted in vertical and axial coordinates of each point, as a \( n \times u \) matrix denoted \( \bar{X} \) (\( u \) represents number of time steps and \( n = 10 \) as the total number of coordinates). Note that acceleration, velocity and displacement responses are all directly extracted from the model. These displacements are transformed into modal coordinates using a reduced modeshape matrix \( \bar{\Phi} \), consisting of "measured" data. \( \bar{\Phi} \) is a \( 10 \times 3 \) matrix which captures the three initial bending modes of the system, as these would be the easiest to measure in a physical test. To make the data more representative of a physical test, we also add white Gaussian noise (\( \epsilon \)) to both modeshapes and physical coordinates with signal to noise ratio of 30 decibels. Now we can estimate the modal displacements responses using \( \bar{\tilde{q}} = (\bar{\Phi} + \epsilon_k)(\bar{X} + \epsilon_X) \) where \( \epsilon \) is the noise. The first mode decay response is shown for the first 12 seconds in Fig. 11, which is similar to the decay response using 240 node measurements in Fig. 4. Note that lower amplitude responses were cut from this signal as they were significantly affected by noise. The first mode response is then filtered using a 4th order Butterworth filter with a cut-off frequency of around ten times of the first mode natural frequency (\( f_c = 100Hz \)), to remove the noise in the signal. The filter is applied to the signal using Matlab command \texttt{filtfilt} which filters the data forward and backwards to avoid a phase shift in the signal [30]. The filtered response of the first mode is as shown in the Fig. 11.

The backbone curve of the first mode is constructed using the filtered decay response of the first mode, following the procedure described previously. The backbone curve is shown in Fig. 12, where there are some errors associated with the approximation of the frequency of low-amplitude responses due to the noise in the data. This
ROM-inspired system identification

Fig. 11: Noisy and filtered first mode decay response using fewer physical measurement points; simulated experiment. a full time history; b shorter time history (zoomed-in plot of a)

resulted in a non-smooth backbone curve - especially at lower amplitudes where noise becomes more significant.

Fig. 12: First mode backbone curve with fewer physical measurement points; simulated experiment

The ROM-inspired model in Eq. (16) is now fitted to the simulated experimental data following the methodology proposed in the previous section. Again, we use the exponential Fourier components of the measurements for estimating the unknown parameters. Given that the data now contains errors, high-frequency Fourier components are unlikely to be reliable. The 1st, 3rd and 5th Fourier coefficients of measurements are added including the DC component, as the odd harmonics are dominant in symmetric systems and are due to odd order terms in the model [1, 35]. The estimated parameters are given in Table 4 which are compared to the parameters derived based on direct reduced order modelling as true parameters. The relative error between the two sets are also shown in table using Eq. (18).

The estimated parameters for simulated experiment shown in Table 4 are comparable to those achieved for the high-fidelity FE data. As previously discussed, the high error percentage in \( \Upsilon_5 \) is likely to be due to its low magnitude at different levels of backbone curve. Similarly, \( \Upsilon_5 \) is more sensitive to the change in data which can be due to its less accuracy. The relative error in other parameters are all below 10%, with lowest (0.2% and 0.03%) for the linear natural frequency estimations. The RE in both set of estimated nonlinear parameters (full FE data and simulated experiment data) can be shown as Fig. 13, where the relative magnitude for each nonlinear parameter is also shown at the mid-level amplitude of the backbone curve. It is prominent that for the parameters with less error, the relative magnitude has higher significance, whilst for parameters with larger error, the significance reduces.

The model is reconstructed using the estimated parameters of simulated experiment data and its backbone curve is computed using CoCo,
Table 4: Estimated parameters of the ROM-inspired model for simulated experiment data compared with the true parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\bar{\Gamma}_3$</th>
<th>$\bar{\Gamma}_5$</th>
<th>$\omega_3^2$</th>
<th>$\gamma_3$</th>
<th>$\gamma_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated ($\bar{P}$)</td>
<td>$2.2831 \times 10^2$</td>
<td>$9.0413 \times 10^5$</td>
<td>$3.3435 \times 10^3$</td>
<td>$1.0179 \times 10^7$</td>
<td>$-1.9503 \times 10^{10}$</td>
</tr>
<tr>
<td>True ($\bar{P}_{\text{true}}$)</td>
<td>$2.1710 \times 10^2$</td>
<td>$6.9109 \times 10^5$</td>
<td>$3.3426 \times 10^3$</td>
<td>$9.8202 \times 10^6$</td>
<td>$-1.7966 \times 10^{10}$</td>
</tr>
<tr>
<td>$RE$ (%)</td>
<td>5.16</td>
<td>30.8</td>
<td>0.03</td>
<td>3.65</td>
<td>8.55</td>
</tr>
</tbody>
</table>

Fig. 13: Relative error in estimated parameters for full FE data and simulated experiment data with their relative magnitudes, which is compared with the simulated experiment in Fig. 14.

Fig. 14: Simulated experiment backbone curve with 5th order, odd order only terms, identified ROM-inspired model backbone curve for fewer physical measurement points data.

From the plot it is clear that the identified ROM-based model backbone curve using the erroneous data with fewer measurement points still matches the simulated experiment measurements very closely. The estimated parameters and identified model backbone curve is similar to the previous section. Also, the assumption of the structure being symmetric has been true from the results.

6 Conclusions

In this paper a method for generating a physically appropriate model, based on ROMs, that can be used in system identification was proposed. The system identification method was demonstrated on a nonlinear cantilever type structure. We highlighted a fundamental link between nonlinear system identification and reduced order modelling.

An FE model of the system was constructed and its decay response was simulated to provide synthetic data to test the proposed identification technique. It was shown that using a nonlinear model with only nonlinear stiffness terms does not lead to an accurate identified model. In contrast, the identification based on the ROM-inspired model showed good match with the measurements taken from the structure at different amplitude levels of the backbone curve. We assume that this is because the set of estimated ROM-inspired model parameters carry more meaningful information about the physics of the structure and ensure appropriate energy balancing across modes can occur. Furthermore, the ROM-based model has been constructed using the physics of the structural system rather than assuming an arbitrary set of models. The results were shown for a set of high fidelity data and also for a set of measurements from fewer points on the structure to represent a more realistic test. It was shown that the methodology also works when the synthetic data is polluted with noise. We conclude that ROM-inspired models can robustly represent a structural system through a set of meaningful terms and are ideally suited for use in system identification. As a future work this method will be taken to real experimental investigations, where
the data measurements can contain more uncertainties (in this work the data measured from the FE model was clean and was artificially polluted in the last section). This work could also be further expanded to more complex models, which account for internal resonances, by continuing to draw inspiration from the reduced order modelling.

Data Availability Data sharing not applicable to this article as there is no additional data.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A ROM-inspired model fitting design matrix

The design matrix $\mathbf{G}$ for the ROM-based model fitting can be shown as below:

$$
\mathbf{G} = \begin{bmatrix}
\Re \mathbf{F}_H [(4 \dot{q}_1 q_1^2 + 4 \dot{q}_2^2 q_1)_{(T_0 \leq t < T_1)}] \\
\Re \mathbf{F}_H [(4 \dot{q}_1 q_1^2 + 4 \dot{q}_2^2 q_1)_{(T_1 \leq t < T_2)}] \\
\vdots \\
\Re \mathbf{F}_H [(4 \dot{q}_1 q_1^2 + 4 \dot{q}_2^2 q_1)_{(T_{z-1} \leq t < T_2)}] \\
\Im \mathbf{F}_H [(4 \dot{q}_1 q_1^2 + 4 \dot{q}_2^2 q_1)_{(T_0 \leq t < T_1)}] \\
\Im \mathbf{F}_H [(4 \dot{q}_1 q_1^2 + 4 \dot{q}_2^2 q_1)_{(T_1 \leq t < T_2)}] \\
\vdots \\
\Im \mathbf{F}_H [(4 \dot{q}_1 q_1^2 + 4 \dot{q}_2^2 q_1)_{(T_{z-1} \leq t < T_2)}] \\
\Re \mathbf{F}_H [q_1(T_0 \leq t < T_1)] \\
\Re \mathbf{F}_H [q_1(T_1 \leq t < T_2)] \\
\vdots \\
\Re \mathbf{F}_H [q_1(T_{z-1} \leq t < T_2)] \\
\Im \mathbf{F}_H [q_1(T_0 \leq t < T_1)] \\
\Im \mathbf{F}_H [q_1(T_1 \leq t < T_2)] \\
\vdots \\
\Im \mathbf{F}_H [q_1(T_{z-1} \leq t < T_2)] \\
\end{bmatrix}
$$

(A1)
References


