A method to obtain fuzzy relations between uncertain objects using similarity measure

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Fuzzy relations between uncertain objects using the 9 intersection matrix

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Abstract

Topological relation between geographical objects is one of the key components of geographical information system (GIS). Considering the objects to be exact with sharp boundaries, 4 intersection and 9 intersection matrices initiated the study of calculating the topological relations. Due to the extensive existence of vague spatial phenomenons, the study of uncertainty object modeling has grown rapidly. Parallelly, the relationship calculating tools have been upgraded to handle the vague concepts. The fuzzy 9 intersection matrix and Egg yolk methods are such popular tools. Although the geographical objects are themselves uncertain, surprisingly, most of these tools generate certain relations between the objects. Thereby, the qualitative nature of these tools overlook the very essence of their uncertainty. To overcome these drawbacks, we propose a quantitative fuzzy valued 9 intersection matrix to obtain a fuzzy relationship between uncertain geographical objects. Then two new similarity computation techniques have been introduced to calculate the membership grade of similarity between the proposed matrix and the known crisp matrices. These similarity computations allow two spatial objects to have partial membership against the eight established topological relations. The quantitative calculations indicate the strength of the relationship. The superiority of the proposed model is established through various numerical examples. Further, certain linguistic variables are linked to the evaluated membership grades to generate an immediate association with the known crisp relations. An example is provided in support of this association.

Keywords: 9 intersection matrix; GIS; Fuzzy modeling; uncertain object modeling.
1 Introduction

Topological relation between geographical objects is one of the key components for a geographical information system (GIS) modeling. Some of the most frequently questioned relations include far, near, left, right, aside, inside, touch, and meet. In GIS, the information about geographical objects is stored. It is important to construct a model that can determine the relationship between any two objects. Otherwise, it would be very time-consuming to keep track of all the relationships between objects. In 1990, Egenhofer et al. [1] defined the 4 intersection matrix as one of the primary algebraic models for determining such relationships. Subsequently, the topic has been vastly studied and applied in GIS related studies. [2–6] are some of the recent studies where the intersection matrix model has been defined. The primary model by Egenhofer et al. [1] was constructed using the point set topological properties of the objects; hence, it comes with an advantage that the relations remain same despite any scale change, translation, rotation. Egenhofer [7] later improved the four-intersection matrix to the nine-intersection model. Additionally, Egenhofer et al. [8] compared the nine-intersection matrix to the four-intersection matrix and discussed the benefits and drawbacks of each method. Both of the mentioned models are elementary models, constructed on the assumption that the items are sufficiently distinct to allow for the drawing of sharp boundaries between them. However, in actuality, the geographical objects are too imprecise to define precise boundaries. This motivated the researchers to develop the concept of an object’s maximal and minimal extents. The minimal extent defines the area within which the object’s existence is certain, whereas the maximal extent includes the object’s uncertain portion. Using this concept, Clementini and Di Felice [9] introduced the concept of broad boundary and upgraded the 9 intersection matrix. The Egg yolk method by Cohn et al. [10] is another way to describe the uncertain relations. In this method, the minimum extent is considered as a yolk and the maximum extent as an egg. Both models describe an object with a known and an unknown portion. So the models completely ignore the gradual changes in the thematic properties of objects, despite the fact that such changes are common in general. Furthermore, neither of the two models takes into account the topological properties of point sets. Thus, the relationships between the two models may not be the same if certain topological changes happen.

Incappability of expressing the uncertain geographical elements as a point set topological objects can be overcome using the fuzzy topology. In 2002, Tang and Kainz [11] introduced a model based on fuzzy topology, and later several studies [12–14] were proposed based on fuzzy topology. Quite contrary to the fuzzy object modeling the resultant relationships obtained from the mentioned fuzzy models are always exact (crisp). In reality, it is not always possible to determine any specific relation between two uncertain objects. The relations would be better realizable as fuzzy valued. Another problem with those fuzzy models is that the entries of the intersection matrix between two geographical objects are considered to be 1, despite any non-zero intersection values, however small it may be. The model by Liu and Shi [12] focuses on determining quantitative relations, but in usability, they also determine the relationships based on complete emptiness or non-emptiness. Further 4 × 4 and 5 × 5 fuzzy topological matrices are also proposed by Tang [11] by upgrading the fuzzy 9 intersection matrix. Though it is possible to find additional relations using these upgraded
matrices, the matrix elements are hard to compute and the relationships are hard to realize in linguistic terminology. The study by Bjørke [15] focused on that particular area, which introduced a method to describe the topological relations between fuzzy regions in verbal terms. His study pointed out the shortcomings of broad boundary and qualitative methods. He proposed a basic model to realize the relationship between the uncertain objects in a linguistic way. Language terms like ‘slightly’, ‘somewhat’, and ‘mostly’ are used to describe the degree of membership in the relationships. The model computes membership by computing similarity between the 4 intersection crisp matrix and the fuzzy valued 4 intersection matrix, which is constructed using the intersection between the interior and boundary of two fuzzy area objects. Throughout the study of intersection matrices [9,11], it has been observed that consideration of the exterior helps realizing additional relationship between two uncertain objects. Due to the absence of an exterior in the four intersection model, many non-identical relations cannot be distinguished using it. Another issue with the model is that it becomes dependent on a specific minimum value regardless of the values of the other parts. To address requirements of all of the aforementioned investigations, we develop a model for determining the topological relationship between uncertain areas in terms of fuzzy membership. It should be noted that the computational examples provided in this article only demonstrate the concepts introduced in the article.

The structure of the paper is followed. The section 2 describes the preliminary concepts and previous studies. Section 3 proposes a model capable of describing the relationship between two uncertain objects quantitatively and determines the relationship as fuzzy. Section 4 concludes the paper and proposes some future work.

2 Preliminary concepts

Topological relation between the objects is a key concept in GIS, used for query, analysis, and to check data consistency. Intersection matrix models such as 4 intersection [1] and 9 intersection [16] are basic and efficient methods to compute the relation between two objects in GIS.

2.1 Intersection matrix models

2.1.1 4 intersection matrix

Egenhofer et al. introduced the 4 intersection method [1] in 1990. The model is constructed using point set topological properties (interior and boundary) of the objects. For two objects $A$ and $B$ in a crisp topological space, the 4 intersection matrix is defined as follows:

$$
\begin{pmatrix}
IntA \cap IntB & IntA \cap BdB \\
BdA \cap IntB & BdA \cap BdB
\end{pmatrix}
$$

(1)

where $Int$, $Bd$ are the interior and boundary of a set in a crisp topological space. Mathematically, it is possible to generate $2^4$ different matrices, but some do not signify valid relations. For two area objects 8 different relations can be realizable.
2.1.2 9 intersection matrix

The 4 intersection model was later upgraded to 9 intersection model by Egenhofer and Franzosa [16] by including the exterior of the sets. The 9 intersection method between two crisp objects \( A \) and \( B \) is defined as follows:

\[
\begin{pmatrix}
\text{Int}A \cap \text{Int}B & \text{Int}A \cap \text{Bd}B & \text{Int}A \cap B^c \\
\text{Bd}A \cap \text{Int}B & \text{Bd}A \cap \text{Bd}B & \text{Bd}A \cap B^c \\
A^c \cap \text{Int}B & A^c \cap \text{Bd}B & A^c \cap B^c
\end{pmatrix}
\]  

where \( \text{Int}A, \text{Bd}A, A^c \) are respectively the interior, boundary, and exterior of a set \( A \) in a crisp topological space.

Mathematically, using this method is possible to get \( 2^9 \) different types relation. Still, due to several conditions, the total number of realizable relations between two area objects are restricted to eight only. The below Table (1) describes all those eight relations and their corresponding 9 intersection matrices.

<table>
<thead>
<tr>
<th>Relation</th>
<th>9 Intersection Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disjoint</td>
<td>( I_0 = \begin{pmatrix} 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 1 \ 1 &amp; 1 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>Contains</td>
<td>( I_1 = \begin{pmatrix} 1 &amp; 1 &amp; 1 \ 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>Inside</td>
<td>( I_2 = \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \ 1 &amp; 1 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>Equal</td>
<td>( I_3 = \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>Meet</td>
<td>( I_4 = \begin{pmatrix} 0 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 1 \ 1 &amp; 1 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>Covers</td>
<td>( I_5 = \begin{pmatrix} 1 &amp; 1 &amp; 1 \ 0 &amp; 1 &amp; 1 \ 0 &amp; 0 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>Covered-by</td>
<td>( I_6 = \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 1 &amp; 1 &amp; 0 \ 1 &amp; 1 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>Overlap</td>
<td>( I_7 = \begin{pmatrix} 1 &amp; 1 &amp; 1 \ 1 &amp; 1 &amp; 1 \ 1 &amp; 1 &amp; 1 \end{pmatrix} )</td>
</tr>
</tbody>
</table>

Table 1: 9 intersection matrices for area-area relations

2.2 Uncertain object modeling

Both the crisp 4 intersection and 9 intersection matrices effectively determine the relationship between two objects with sharp boundaries. However, contrary to that idea, most geographical objects have an imprecise boundary. It is tough to distinguish the boundary
between two natural objects in many cases. Uncertainty may also occur due to the positional inaccuracy of a particular object. As those uncertainties are widespread across the GIS database, uncertain object modeling is required for accuracy. Hence, the method to determine the relationships between objects must alter. The broad boundary model by Clementini and Di Felice [9] was one of the basic models to determine the relation between uncertain objects.

Figure 1: An area with broad boundary

Figure 1 describes a geographical area object $A$ with broad boundary, where $IntA$ denotes the interior and $\Delta A$ denotes the broad boundary of an uncertain area $A$. In the broad boundary method, one object is considered to have a minimal extend and a maximal extend. The minimal extend is constructed with a certain part of the object, while the area between the maximal and minimal extend contains the uncertain region. That uncertain region is defined as the broad boundary of the object. Clementini and Di Felice [9] upgraded the 9 intersection method replacing the boundary with broad boundary and discovered 44 different relations using this method. In 1995, Cohn et al. [10] introduced the famous Egg yolk method, based on the minimum and minimal extent of an uncertain object. The Egg yolk method was developed using the region connection calculus (RCC) theory [17].

Both the broad boundary and Egg yolk methods can produce relationships between two uncertain area objects. Still, both methods have the following limitations: Uncertain line objects and uncertain point objects are not defined and relation between such objects has not been described. As these methods are not based on point-set topology, it is not guaranteed that the relationships would remain unchanged under topological transformations. Also, gradual changes of the thematic attributes of the geographical objects are ignored in these types of models.

### 2.3 Fuzzy set for modeling uncertain objects in GIS

All the models we have discussed till now are based on the crisp set. Thus those models cannot handle the uncertainty properly. Many of the researchers [18, 21] advocated fuzzy modeling for a superior description of the geographical objects. The fuzzy set, defined by Zadeh, [22] is an excellent tool for handling uncertainty in a data model.
2.3.1 Fuzzy set [22]

Definition 2.1 Let $X \neq \phi$ be an ordinary set and $I = [0, 1]$. Let $\mu_A : X \rightarrow I$ be a function for $A \subset X$, where $\mu_A$ (or simply $A(x)$) is said to be the membership function of $A$ for all $x \in X$. The set $A$ along with the membership function is said to be the $I$ fuzzy set on $X$.

For two fuzzy set $A$ and $B$, few standard fuzzy set operations are as follows;

- Union $(A \lor B)(x) = \max[A(x), B(x)]$.
- Intersection $(A \land B)(x) = \min[A(x), B(x)]$.
- Complement $A^c(x) = 1 - A(x)$.

As we described earlier, the $9$ intersection method is based on topological theory, allowing the relations between objects to be invariant under topological transformations. To achieve the same feature in fuzzy object modeling, the relation computing tools must be constructed using the topological theory. In fuzzy object modeling, the fuzzy topological theory would be the perfect candidate for the same.

2.3.2 Fuzzy topology

Chang introduced the fuzzy topology [23] in 1968.

Definition 2.2 For a set $X \neq \phi$ and $I$ be the unit interval and $\tau \subset I^X$ be a collection of fuzzy sets, $\tau$ is said to be a fuzzy topology (FT) for $X$, if

1. $0_X, 1_X \in \tau$,
2. $A, B \in \tau \Rightarrow A \land B \in \tau$,
3. $(A_j)_{j \in J} \in \tau \Rightarrow \bigvee_{j \in J} A_j \in \tau$, where $J$ is an index set.

The members of $\tau$ are called open fuzzy sets, and complement of elements of $\tau$ are said to be closed fuzzy sets.

For a fuzzy set $A$ in a FT space, the interior of $A$, denoted by $\text{Int}A$, is defined as the supremum of all fuzzy open set contained in $A$, the closure of $A$, denoted by $\text{Cl}A$ is defined as the infimum of the all fuzzy closed set that contains $A$, and the exterior (Ext) of $A$, denoted as $A^\circ$, is defined as $A^\circ = (\text{Cl}A)^c$.

Each of the definitions in the above paragraph is due to [23].

2.3.3 Fuzzy boundary

The fuzzy boundary of a fuzzy set in a FT space is an important aspect that requires special attention in this study. Contrary to the crisp case, the boundary of a fuzzy set can have a non-empty intersection with the interior and the exterior of the set. Actually, it almost mimics the attributes of geographical objects found in nature. Those objects hardly have any distinguishable boundary.

In the present literature, we have different definitions [24–28] of the fuzzy boundary. Among all these definitions, the one introduced by Pu and Liu [25] is mostly used in the literature.
**Definition 2.3** The boundary of $A$ denoted by $BdA$, defined as $BdA = ClA \land ClA^c$.

The definition of fuzzy boundary is an important aspect as the uncertainty is mostly narrated using the definition of boundary. Also, the boundary of an object is used in relationship obtaining models, viz. 4 intersection matrix, 9 intersection matrix. A 4 intersection matrix between two objects is constructed using the interior and boundary of the objects. In contrast, a 9 intersection matrix between two objects is constructed with the objects’ interior, boundary, and exterior.

### 2.3.4 Fuzzy topological models

Contrary to the crisp model, the interior, boundary, and exterior of a fuzzy set in a FTS may not be mutually disjoint. That inspired Tang and Kainz [11] to find different mutually disjoint fuzzy topological properties other than the interior, boundary, and exterior. They defined and established the core, outer, and fringe [11] of a fuzzy set as mutually disjoint topological properties. Using these mutually disjoint parts of a fuzzy set, they introduced the fuzzy 9 intersection matrix. The same study also introduced i-closure, c-closure, b-closure, i-boundary, and c-boundary. Using core, b-closure, c-boundary, and outer, they also proposed a $4 \times 4$ intersection matrix for two fuzzy sets and generated 152 topological relations.

On a different note, Liu and Shi [12] introduced a new technique to construct a 9 intersection method for the fuzzy set using $\alpha$ induced fuzzy topology, where the value of $\alpha$ can be any real number between 0 to 1. Here, the interior, boundary, and exterior of an objects is dependent on the user-provided $\alpha$ value. They proposed a 9 intersection matrix to determine quantitative relations between the fuzzy objects in FTS [13, 29].

Later, in 2010, Tang-Kainz [14] defined a special fuzzy topological space, namely crisp fuzzy topological space (CFTS). They established that in a CFTS, the interior, boundary, and exterior are mutually disjoint. Thus 9 intersection matrices can be formed using usual topological properties.

The above-mentioned fuzzy topological methods help us to find several relationships among uncertain objects. Each of the fuzzy 9 intersection matrices can distinguish 44 different relations between two area objects, similar to the relations acquired using the broad boundary [9] method. The main problem with those topological models is that despite any intersection value between the topological properties, the matrix entries are considered 1 if the intersection is non-empty and 0 otherwise. The model by Liu and Shi [13] though focused on determining the quantitative relations, but practically the relations are based on the emptiness and non-emptiness of the intersection values. Another problem with those models is that the fuzzy topological properties are very tough to compute, and the relations are not realizable in any linguistic terms.

As the geographical objects are fuzzy, it is reasonable to presume that the relations between the objects are also fuzzy rather than crisp always. Several studies [15, 30, 31] in the literature have suggested that the topological relations between two objects should be expressed using fuzzy set. Rather than expressing the relations qualitatively, these methods express the using degree of membership (degree of disjoint, degree of contains) that a
relation hold. Among these studies, Bjørke \[15\] attempted to construct a model to represent fuzzy topological relationships using linguistic variables. The method computes a 4 intersection matrix, where the matrix entries are fuzzy valued. Thus, the entries vary from 0 to 1 rather than 0 or 1 only. The fuzzy membership is calculated by taking the highest membership of the intersection. This procedure is similar to that of Dilo \[32\] (Details study has been done in that article). After producing the fuzzy 4-intersection matrix, the membership of a particular relation is obtained by calculating the similarity between the known crisp intersection matrix corresponding to the relation and the fuzzy 4-intersection matrix. Linguistic terms such as ‘lightly’, ‘somewhat’, ‘mostly’, ‘clearly’ are used for different membership ranges to express the relationship verbally.

Upon looking into several studies, one can conclude that the exterior is one of the major parts that help distinguish between uncertain objects with another. As a result, 9 intersection matrix provides us more relations as that of 4 intersection matrix \[9, 11, 29\]. But the model by Bjørke does not consider the exterior of the objects, so the model’s effectiveness gets reduced due to ignorance of the exterior part. In 2005, Du et al. \[33\] introduced a model to study the positional uncertainty of a crisp object. They defined a unified fuzzy 9 intersection method that determines the relation between fuzzy objects, crisp objects, and crisp objects. In the follow-up paper, \[34\] they introduced the fuzzy 9 intersection method computed by the raster method and also with the vector method. Next, they constructed 9 intersection matrices between crisp objects and fuzzy objects and between two fuzzy objects. Though this method computes a 9 intersection matrix whose entries are fuzzy valued, the method does not reveal anything regarding the relationship perspective.

### 3 Fuzzy relationships between uncertain geographical objects

In light of the above discussion, we suggest a fuzzy topology-based method for generating topological relationship between two fuzzy objects in terms of membership values. For that we first construct the fuzzy valued 9 intersection matrix for two fuzzy objects. Next, the fuzzy memberships corresponding to relationships are deduced by calculating the similarity between fuzzy valued matrix and the 9 intersection matrices corresponding to known relationships listed in Table \[1\].

#### 3.1 Fuzzy valued 9 intersection matrix

The determination of fuzzy relationships was studied by Bjørke \[15\] using the fuzzy 4 intersection matrix. This paper considers the fuzzy 9 intersection matrix instead of the fuzzy 4 intersection matrix. This fuzzy valued 9 intersection matrix for two fuzzy sets \(A\) and \(B\) is up-gradation of classical 9 intersection matrix, given by the following matrix.

\[
\begin{pmatrix}
\ell(\text{Int}A \land \text{Int}B) & \ell(\text{Int}A \land B \text{D}B) & \ell(\text{Int}A \land B^-) \\
\ell(B \text{D}A \land \text{Int}B) & \ell(B \text{D}A \land B \text{D}B) & \ell(B \text{D}A \land B^-) \\
\ell(A^- \land \text{Int}B) & \ell(A^- \land B \text{D}B) & \ell(A^- \land B^-)
\end{pmatrix}
\] (3)
where \( h \) denotes the height of a fuzzy set. The following points are to be noted while constructing the fuzzy valued 9 intersection matrix:

- The truth-degree \( \text{[32]} \) of the intersections are considered as the matrix element, rather than 0 and 1 only.

- Truth-degree is the height of the intersection denoted by \( h \). For example, \( h(\text{Int}A \land \text{Int}B) = \max\{\min(\text{Int}A(x), \text{Int}B(x))\} \).

- The boundary definition given by Piu and Liu \( \text{[25]} \) is considered. The membership of boundary is multiplied by 2, to raise the domain of membership of boundary to \([0, 1]\).

The following Table 2 describes several classifications of obtained matrix \( (3) \) based on different conditions.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Matrix</th>
<th>Observations</th>
</tr>
</thead>
</table>
| 1 \( \text{Int}A \land \text{Int}B = \phi, \text{bd}A \land \text{bd}B = \phi \) | \[
\begin{pmatrix}
0 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & 1
\end{pmatrix}
\] | \( h(\text{Int}A) \quad h(\text{Int}B) \quad h(\text{Bd}A) \quad h(\text{Bd}B) \) 1 | The condition signifies the disjoint relation between \( A \) and \( B \). \( x_5 \) will be always non zero and the value of the remaining other \( x_i \in [0, 1] \) .

| 2 \( \text{Int}A \land \text{Int}B = \phi, \text{bd}A \land \text{bd}B \neq \phi \) | \[
\begin{pmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & 1
\end{pmatrix}
\] | \( h(\text{Int}A) \quad h(\text{Int}B) \quad h(\text{Bd}A) \quad h(\text{Bd}B) \) 1 | \( x_5 \) will be equal to zero. All the remaining \( x_i \) except \( x_1 \) could be ranging from 0 to 1 and \( x_1 \) will be always positive.

| 3 \( \text{Int}A \land \text{Int}B \neq \phi, \text{bd}A \land \text{bd}B = \phi \) | \[
\begin{pmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & 1
\end{pmatrix}
\] | \( h(\text{Int}A) \quad h(\text{Int}B) \quad h(\text{Bd}A) \quad h(\text{Bd}B) \) 1 | In this case \( x_1, x_5 \) will be always non-zero and values of other \( x_i \)'s will be anything between 0 and 1.

| 4 \( \text{Int}A \land \text{Int}B \neq \phi, \text{bd}A \land \text{bd}B \neq \phi \) | \[
\begin{pmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & 1
\end{pmatrix}
\] | \( h(\text{Int}A) \quad h(\text{Int}B) \quad h(\text{Bd}A) \quad h(\text{Bd}B) \) 1 | In this case \( x_5 \) is always non zero and the value of the remaining other \( x_i \) could be ranging from 0 to 1.

Table 2: Classifications of fuzzy valued 9 intersection matrix based on several conditions.

Table 2 describes the possible fuzzy valued 9 intersection matrix based on those conditions. In reality, it is quite challenging to determine the relationship between any two fuzzy objects by observing the matrix only. We use similarity computation that provides the fuzzy membership of the relations by computing the similarity between \( (3) \) and the known relation matrices.
3.2 Measurement of similarity

We use 2 different techniques to compute the similarity between the fuzzy valued matrix and the known crisp intersection matrix.

3.2.1 Similarity _1_

This particular similarity calculation is denoted by \( S_1 \). For a fuzzy valued matrix \( A_f \), and a crisp intersection matrix \( B_c \), the similarity is defined as,

\[
\mu_{S_1}(A_f, B_c) = 1 - \frac{1}{n} \sum_{i=1}^{n} |A_f(x_i) - B_c(x_i)|
\]  

(4)

In particular, for 9 intersection matrix, \( n = 9 \), we consider the matrix as a fuzzy set with 9 elements and use (4) to find the similarity.

For the crisp intersection matrices \( B_c \)'s are chosen from the Table 1.

3.2.2 Similarity _2_

We denote the similarity as \( S_2 \), for a fuzzy valued matrix \( A_f \), and a crisp intersection matrix \( B_c \), similarity \( S_2 \) is defined as,

\[
\mu_{S_2}(A_f, B_c) = \left( \prod_{i=1}^{n} |1 - B_c(x_i) - A_f(x_i)| \right)^{\frac{1}{n}}
\]  

(5)

**Example 3.0.1** Let for two fuzzy regions \( C \) and \( D \) the fuzzy valued 9 intersection matrix calculated using (3) is given by

\[
A_f = \begin{pmatrix}
0.7 & 0.9 & 0.8 \\
0.2 & 0.2 & 0.7 \\
0.2 & 0.1 & 1
\end{pmatrix}
\]

To calculate similarity between \( A_f \) and \( I_2 \), we consider \( A_f \) as a fuzzy set defined by

\[
A_f = \{(x_1, 0.7), (x_2, 0.9), (x_3, 0.8), (x_4, 0.2), (x_5, 0.2), (x_6, 0.7), (x_7, 0.2), (x_8, 0.1), (x_9, 1)\}
\]

and

\[
I_2 = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 1), (x_5, 0), (x_6, 0), (x_7, 1), (x_8, 1), (x_9, 1)\}
\]

Then using (4) the similarity between \( A_f \) and \( I_2 \) is calculated as

\[
\mu_{S_1}(A_f, I_2) = 1 - \frac{1}{9} \sum_{i=1}^{9} |A_f(x_i) - I_2(x_i)| = 0.40
\]

As \( I_2 \) represents the relation ‘inside’, thus from the obtained similarity we conclude that \( C \) is inside \( D \) with membership 0.40.

Using (4) and (5) the similarities calculated by \( S_1 \) and \( S_2 \) between \( A_f \) and for all 8 crisp 9 intersection matrices are obtained in following Table 3.
<table>
<thead>
<tr>
<th>Fuzzy matrix</th>
<th>Crisp matrix</th>
<th>Relationship</th>
<th>$\mu_{S_1}$</th>
<th>$\mu_{S_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_f$</td>
<td>$I_0$</td>
<td>Disjoint</td>
<td>0.53</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>$I_1$</td>
<td>Contain</td>
<td>0.82</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>$I_2$</td>
<td>Inside</td>
<td>0.40</td>
<td>0.29</td>
</tr>
<tr>
<td></td>
<td>$I_3$</td>
<td>Equal</td>
<td>0.56</td>
<td>0.43</td>
</tr>
<tr>
<td></td>
<td>$I_4$</td>
<td>Meet</td>
<td>0.47</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>$I_5$</td>
<td>Cover</td>
<td>0.76</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>$I_6$</td>
<td>Covered-by</td>
<td>0.33</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>$I_7$</td>
<td>Overlap</td>
<td>0.53</td>
<td>0.40</td>
</tr>
</tbody>
</table>

Table 3: Similarity between fuzzy and crisp 9 intersection matrices

where $\mu_{S_1}$ and $\mu_{S_2}$ denotes the membership of a relation respectively calculated using the similarity $S_1$ and $S_2$ and $I_n$, $n = 0, 2, \ldots, 7$ crisp matrix described in Table 1. For example, one can conclude from the Table 3, that $C$ overlaps $D$ with membership 0.53 according to the similarity computation $S_1$, while the relation $C$ overlaps $D$ has membership 0.40 according to $S_2$. If there is any component in the fuzzy value matrix which has empty intersection with the corresponding component of the crisp intersection matrix, then it can be proved that the relation, which crisp intersection matrix indicates, would have zero membership for similarity $2$.

**Theorem 3.1** If $A_f(x_i) = (B_c)^c(x_i)$ for any $i$, $0 \leq i \leq n$, then $\mu_{S_2}(A_f, B_c) = 0$.

**Proof:** Let $A_f(x_i) = (B_c)^c(x_i)$ for any $i = r$.

Then $|1 - B_c(x_r) - A_f(x_r)| = |1 - (B_c(x_r) + (B_c)^c(x_r))| = 0$.

Eventually, $(\prod_{i=1}^{n} |1 - B_c(x_i) - A_f(x_i)|)^{\frac{1}{n}} = 0$.

$\Rightarrow \mu_{S_2}(A_f, B_c) = 0$

Theorem 3.1 may not be true for the similarity $1$, i.e., we may found that $\mu_{S_1}(A_f, B_c) \neq 0$ even if $A_f(x_i) = (B_c)^c(x_i)$ for some $i$, $0 \leq i \leq n$. The following example verifies the fact.

**Example 3.1.1** Consider the fuzzy valued 9 intersection matrix $A_f = \begin{pmatrix} 0.4 & 0.6 & 0.7 \\ 0.5 & 1 & 0.8 \\ 0.4 & 0.3 & 1 \end{pmatrix}$

Then $\mu_{S_2}(A_f, I_2) = 0$, but $\mu_{S_1}(A_f, I_2) = 0.39 \neq 0$.

In case of similarity computation $S_1$, $\mu_{S_1}(A_f, B_c) = 0$ iff $A_f(x_i) = (B_c)^c(x_i)$, $\forall x_i$, $i = 1, 2, \ldots 8$.

With the resulting fuzzy valued 9 intersection matrix, we now have two options for computing similarity to determine the memberships associated with each relationship. So the question is, which particular similarity computation between $S_1$ and $S_2$ should be preferred?
Argument for choosing $S_1$

We have several studies [9,10] that have examined the relationships in terms of their conceptual neighbourhood. For example, the relation ‘meet’ is the conceptual neighbourhood of the relations ‘disjoint’ and ‘overlap’. This conceptual neighbourhood property is well expressed by the similarity computation $S_1$. As a result, we find that $\mu_{S_1}(I_4, I_0)$ and $\mu_{S_1}(I_4, I_7)$ are higher than $\mu_{S_1}(I_0, I_i), i = 1, 2, 3, 5, 6$. $S_1$ is preferable, if the conceptual neighbourhood between the relationships are favoured.

Argument for choosing $S_2$

During the computation of the fuzzy valued 9 intersection matrix, if we find that the height of one intersection value is crisp, that indicates we have a definite knowledge about the intersection value. For example, assume that $h(\text{Int}A \land \text{Int}B) = 0$ we have exact knowledge that the interiors of both objects are not intersecting each other, at all. The membership of the relations contains, inside, equal, cover, covered-by, and overlap may then be argued to be zero. Because all these mentioned relationships require the intersection between interiors to be non-zero. $S_2$ can be used, when this argument is preferable.

Here we have not judged the superiority of the arguments. It would be better to let GIS experts choose between $S_1$ and $S_2$, according to their studies. The similarity computation formula (4) for $S_1$, can be also expressed as,

$$\mu_{S_1}(A_f, B_c) = \beta \left[ (A_f \land B_c) \lor (A_c \land B_c) \right],$$

and the similarity computation $S_2$ can be also expressed as,

$$\mu_{S_2}(A_f, B_c) = \omega \left[ (A_f \land B_c) \lor (A_c \land B_c) \right],$$

where $\land$ and $\lor$ are respectively the fuzzy intersection and union defined in 2.3.1 and $\beta$ and $\omega$ respectively calculates the arithmetic mean and the geometrical mean of the all the entries of the matrix

$$\left[ (A_f \land B_c) \lor (A_c \land B_c) \right]$$

(8)

Whereas, the formula used in Bjørke method is given by following

$$\mu(A_f, B_c) = \gamma \left[ (A_f \land B_c) \lor (A_c \land B_c) \right],$$

(9)

where $\gamma$ chooses the minimum of all the matrix elements of the matrix (8).

Thus each of the discussed similarities compute the matrix (8) and then calculate the membership by using $\beta$, $\omega$ or $\gamma$. In particular, $\gamma$ is dependent on the minimum matrix elements only. Thus if we have several matrices with a common minimum value, then the $\gamma$ would give the same set of membership values for each matrix. The other matrix elements would have no impact to the result. Thus, we may come up with several sets of relations that may be different in reality but non-distinguishable by Bjørke’s calculation of similarity or $\gamma$. On the other hand, the similarity $S_1$ and $S_2$ both method are formulated in a way that value of the membership is dependent on each of the elements of the matrix.
Though the fuzzy valued 9 intersection matrix is preferable to perform the similarity com-
putation for $S_1$ and $S_2$, they can also be applicable for the fuzzy valued 4 intersection
matrix. For experimental purposes, we use $S_1$ and $S_2$ for 4 intersection matrix in the fol-
lowing example 3.1.2. Next, we compare the methods through example by calculating all
the corresponding relationship memberships.

Example 3.1.2

Let us suppose that the 4 intersection fuzzy val-
ued matrix for $A$ and $B$ are \[
\begin{pmatrix}
0.9 & 0.9 \\
0.3 & 0.1
\end{pmatrix}.
\]

Figure 2 illustrates two fuzzy regions $A$ and $B$. The higher saturation of the colors in the fig-
ure denotes higher membership of the interior, and as we move towards the boundary, the sat-
uration gets decreased and membership of the boundary increases. After a certain stage (where
$\mu_{CIA} = \mu_{CIA'}$) the membership of boundary de-
creases as the saturation decreases.

We apply $S_1$, $S_2$, and Bjørke’s method of finding similarity. The following Table 2 describes the Figure 2: Two uncertain geographical fuzzy relations between $A$ and $B$ using the stated objects similarity computations:

It is quite evident from the Example 3.1.2 that the relation $A$ covers $B$ has a larger mem-
bership than $A$ is covered by $B$. Though both of our proposed similarity computations
support the fact, the Bjørke’s method produces the same membership for both relations.
The following graph in Figure 3 represents the membership corresponding to the similarity computations we discussed in the Table 4. The graph reveals that both the proposed similarity computations produce different membership corresponding to the different relationships. On the contrary, Bjørke’s method is unable to distinguish all the relations but can only reveal the relations with the highest membership. A fuzzy set is expected to have different membership related to its attributes, which expresses the true nature of the fuzzy set. It can be verified from the graph in Figure 3 that, all the relations are well expressed using similarities $S_1$ and $S_2$.

Both crisp 9 intersection and 4 intersection matrices produce the same set of relations be-
tween two area objects. Also, the previous Example 3.1.2 supports the superiority of the newly proposed similarity computation over the former method for 4 intersection matrices. So question will arise about our choice of 9 intersection matrix than 4 intersections, as 9 intersection is computationally larger than the 4 intersection matrix. Previous stud-
ies [9,11,13] in the literature already proved the supremacy of the 9 intersection matrix over
the 4 intersection matrix. The studies have shown that the 9 intersection matrix produces
44 different relations between two uncertain area objects, while we only have 11 different
relations if 4 intersection matrix is used. So the inclusion of exterior in the 4 intersection model brings further refinement to the relationships. As a result, 33 additional relations are available for 9 intersection matrices. Our concern of interest is the membership of

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Table 4: Relations corresponding to different similarity calculations

<table>
<thead>
<tr>
<th>$A_f$</th>
<th>Relation($A, B$)</th>
<th>Corresponding crisp matrix</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$Bjørke's$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Disjoint</td>
<td>$(\begin{smallmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{smallmatrix})$</td>
<td>0.47</td>
<td>0.33</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>Contain</td>
<td>$(\begin{smallmatrix} 1 &amp; 1 \ 0 &amp; 0 \end{smallmatrix})$</td>
<td>0.82</td>
<td>0.82</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>Inside</td>
<td>$(\begin{smallmatrix} 1 &amp; 0 \ 1 &amp; 0 \end{smallmatrix})$</td>
<td>0.57</td>
<td>0.47</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>Equal</td>
<td>$(\begin{smallmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{smallmatrix})$</td>
<td>0.47</td>
<td>0.27</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>Meet</td>
<td>$(\begin{smallmatrix} 0 &amp; 0 \ 0 &amp; 1 \end{smallmatrix})$</td>
<td>0.27</td>
<td>0.19</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>Cover</td>
<td>$(\begin{smallmatrix} 1 &amp; 1 \ 0 &amp; 1 \end{smallmatrix})$</td>
<td>0.62</td>
<td>0.47</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>Covered-by</td>
<td>$(\begin{smallmatrix} 1 &amp; 0 \ 1 &amp; 1 \end{smallmatrix})$</td>
<td>0.37</td>
<td>0.27</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>Overlap</td>
<td>$(\begin{smallmatrix} 1 &amp; 1 \ 1 &amp; 1 \end{smallmatrix})$</td>
<td>0.52</td>
<td>0.38</td>
<td>0.1</td>
</tr>
</tbody>
</table>

the relation. We would like to enquire whether the inclusion of exterior makes any major difference in the membership values. For that, we have constructed the following Example 3.1.3

Example 3.1.3

![Figure 4: Two different overlap relations](image)
Let the 4 intersection matrices for both the relations (4a) and (4b) in the Figure 4 are given by 
(0.9 0.9 0.9 0.9) 
As both of the relation matrices are equal, thus both of them will produce the same set of membership for each of the relations. In both cases of the example, we have supposed that in each of the intersection matrices, Int-Int, Int-Bd, Bd-Int, and Bd-Bd have membership 0.9. In Relation 1, the interiors of both objects are almost inside the closure of the other object. Hence Int-Ext and Ext-Int have less membership. Whereas in Relation 2, the interior of C is not totally inside the closure of D. Thus both Int-Ext and Ext-Int have larger membership than Relation 1.

So we can suppose the corresponding 9 intersection matrices with respect to Relation 1 and Relation 2, are of the following form:

\[ R1 = \begin{pmatrix} 0.9 & 0.9 & 0.2 \\ 0.9 & 0.9 & 0.8 \\ 0.1 & 0.9 & 1 \end{pmatrix}, \quad \text{and} \quad R2 = \begin{pmatrix} 0.9 & 0.9 & 0.8 \\ 0.9 & 0.9 & 0.8 \\ 0.7 & 0.9 & 1 \end{pmatrix} \]

Applying the similarity\(_1\) and similarity\(_2\), memberships of fuzzy relation between A and B and between C and D are calculated and results are displayed in the following Table.
The Table 5 shows that both the proposed similarity computation Similarity1 and Similarity2 distinguish relations between A, B and C, D, when we consider the fuzzy valued 9 intersection matrices. But memberships of relations between A, B and that between C, D are identical, calculated via any of the similarity computation techniques if 4 intersection matrix is considered. This example justifies our choice of 9 intersection matrix over 4 intersection matrix.

The following Figure 5 further illustrates the change in the relationship due to the consideration of the exterior in the intersection matrix. We have considered that the fuzzy valued relation matrix between two area objects as following:

\[
A_f = \begin{pmatrix}
0.9 & 0.6 & y \\
0.9 & 0.5 & y \\
y & y & 1
\end{pmatrix}
\]

Now if the value of y varies from 0.1 to 0.5, with 0.1 interval and keeping other four entries (which are the entries for 4 intersection matrix) of the matrix unchanged.

![Figure 5: Visual representation of different relations due to different exteriors](image)

The variation in the membership due to the changes of y is described in the following bar chart in Figure 6.
Each of the relations $R_i$ in Figure 6 has identical 4 intersection matrix, though the relations are not identical. The consideration of the exterior using 9 intersection matrix provides a better realizable distinction between the relations, which is expressed using the Figure 5 and corresponding bar diagram in Figure 6.

### 3.3 Applications of the proposed model

#### 3.3.1 Linguistic variable

Non-exactness features can be expressed logically using fuzzy set as this generalized set provides a basis for approximate reasoning. Some of the non-exact features such as fairly disjoint, completely inside, very near are often subject of query in GIS models. These queries can be acknowledged by introducing a quantifier associated with a linguistic variable. The concept of fuzzy logic associated with linguistic variables was introduced by Zadeh [35] in 1975. In the following, we demonstrate an example of the application of our proposed method in solving GIS problems that requires the concept of linguistic variable.
Example 3.1.4 Let $A$ be an urban area and $B$, $C$, $D$, $E$, and $F$ are rural areas around $A$, illustrated by the Figure 7. The high contrast of color demonstrates the higher membership of road connectivity in each area. Let a project start to build a road between the least connected rural area and the urban area to increase connectivity. The fuzzy valued 9 intersection matrix between $A$ and each of these areas should be constructed to address this issue through our proposed model. The required matrix entries can be generated using the data available on field in this respect. Then we will find the membership of disjoint relation by computing the similarity with $I_0$. For instance, let the below table indicate the membership of disjoint relation.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Areas</th>
<th>Membership</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disjoint</td>
<td>$(A, B)$</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>$(A, C)$</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>$(A, D)$</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>$(A, E)$</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>$(A, F)$</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Table 6: Membership of disjoint relations

The rural area having highest disjoint membership is considered to be the least connected. In this case the area $E$ will be selected.

3.3.2 Example associated with a linguistic variable

A linguistic variable can be easily associated with the fuzzy relationship we achieved from our proposed method. For example, let $\mu_S(A_f, B_c)$ denotes the similarity between a fuzzy
valued 9 intersection matrix $A_f$ and a crisp relation $B_c$, let $L(x)$ is a quantifier, defined as

$$L(x) = \begin{cases} 
\text{weekly} & \text{if } \mu_S(A_f, B_c) \leq 0.10 \\
\text{slightly} & \text{if } 0.10 < \mu_S(A_f, B_c) \leq 0.30 \\
\text{fairly} & \text{if } 0.30 < \mu_S(A_f, B_c) \leq 0.60 \\
\text{strongly} & \text{if } 0.60 < \mu_S(A_f, B_c) \leq 0.90 \\
\text{clearly} & \text{if } \mu_S(A_f, B_c) > 0.90 
\end{cases}$$

(10)

By associating the linguistic variable defined by Equation (10) with Example 3.1.4, if we run a query to find all areas ‘strongly’ disjoint to the area $A$, we will get $E$ as the response to the query.

Though, it seems that the relationship with highest membership grade is the only concern, the other relationships also have the significance in explaining the overall relationship between two objects. Rather than concentrating only on the relation with highest membership, the relation between two uncertain objects in GIS can be well expressible if we consider three of the relations (the more relations we use, the more clarity we get) with highest membership degrees.

For example, assume that the relation matrices obtained for $A, B$ is $R1 = \begin{pmatrix} 0.7 & 0.8 & 0.9 \\
0.9 & 0.1 & 0.5 \\
0.4 & 0.7 & 1 \end{pmatrix}$, and that of $C, D$ is $R2 = \begin{pmatrix} 0.7 & 0.6 & 0.5 \\
0.8 & 0.6 & 0.6 \\
0.5 & 0.7 & 1 \end{pmatrix}$.

Then the relation between $A$ and $B$ is $\{(\text{overlap}, 0.67), (\text{contains}, 0.64), (\text{inside}, 0.60)\}$ and the relation between $C$ and $D$ is $\{(\text{overlap}, 0.67), (\text{covered-by}, 0.62), (\text{inside}, 0.60)\}$.

Consideration of only the highest membership will produces both the relations as overlap with 0.67 membership, hence it will not provide as much as clarity about the differences of the relations that we are getting by considering three highest memberships. If we use the quantifier (10), the relation between $A$ and $B$ is $\{\text{strongly overlap, strongly contains and fairly inside}\}$ and while the relation between $C$ and $D$ is $\{\text{strongly overlap, strongly covered-by and fairly inside}\}$.

4 Conclusions and future works

The paper proposes a method to calculate fuzzy relations between uncertain geographical areas. The proposed method is a quantitative method that calculates a fuzzy valued 9 intersection matrix for two uncertain objects. The paper classified the fuzzy valued 9 intersection matrix based on the intersection between interiors and boundaries of two objects. The classification shows the resemblance of the fuzzy valued 9 intersection matrix despite of different conditions. Further, two similarity computations have been introduced to determine the membership of the relations between two objects. Similarities are computed between our obtained fuzzy valued 9 intersection matrix and 8 known relations obtained from crisp 9 intersection matrix model. Next, examples has been provided to prove the
superiority of both the similarity computation techniques over the former methods. We further illustrated the superiority of 9 intersection matrix model over 4 intersection matrix model by some examples. The paper next demonstrates a possible example where the proposed method can be applicable. Next a possible association of linguistic variable with fuzzy membership is illustrated, which can respond to a linguistic query.

A further study is needed to compare the two similarity computation introduced in this paper. The present paper only considers the relations between two area objects; further study is needed to investigate the area-line and line-line objects relations.

Conflict of interest

The authors declare that they have no conflict of interest.

References


