Observer design for Interconnected Takagi-Sugeno systems with Immeasurable Premise Variables

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Observer design for Interconnected Takagi-Sugeno systems with Immeasurable Premise Variables

Lamia Ouhib¹, Redouane Kara¹

Abstract

This paper deals with observer design for a class of Interconnected Takagi-Sugeno systems with Immeasurable Premise Variables. Asymptotic stability and $\mathcal{D}$-stability of Luenberger-like Interconnected Multiple Observers are examined. Finsler’s Lemma is used to relax $\mathcal{D}$-stability conditions through the introduction of additional slack variables. The designed conditions are expressed as Linear Matrix Inequalities. The proposed approaches are applied in simulation to a Proton Exchange Membrane Fuel Cell (PEMFC) system and a four-tank system.

Keywords Interconnected Takagi-Sugeno Systems · Immeasurable Premise Variable · Proportional Observer · $\mathcal{D}$-Stability · $L_2$-Approach · Finsler’s Lemma · Proton Exchange Membrane Fuel Cell (PEMFC) · Four-tank system

1 Introduction

Over the past decades, Takagi-Sugeno (TS) fuzzy systems have been intensively investigated by the control community due to their ability to describe a broad class of nonlinear systems as a weighted convex sum of linear models in different regions of the state space. In this way, various linear control concepts and strategies can be extended to them [33], [10], [9].

For the stability analysis of TS systems, the Lyapunov theory has been widely used leading in most of the cases to Linear Matrix Inequality (LMI) based optimization problems [34],[17]. Thus, many automatic problems coping with stability, stabilization, observer design, diagnosis and fault tolerant control have been developed [22], [35].

Observer design for TS systems with Immeasurable Premise Variables (IPV) has been amply studied but still remains an open research problem. The result proposed in [2] is principally based on the Thau-Luenberger observer [36] and a Lipschitz hypothesis. In [4], the authors proposed a method to design an observer for systems subjected to unknown inputs and disturbances. Actuator fault estimation in presence of sensor disturbances using an adaptive observer has been treated in [23]. An observer-based controller for robust stabilization of uncertain TS systems is presented in [37].

Interconnected systems have been considered in the last twenty years to characterize the dynamics of complex systems presenting interactions between a large number of variables [29], [21]. In fact, they appear in a large variety of applications such as power systems, transportation networks and manufacturing processes. Recently, the TS formalism has been extended to the study of large scale interconnected systems [24], [19]. The authors of [1] have considered the stability and stabilization of this class of systems using multiple Lyapunov functions. Decentralized controllers design for systems with bounded disturbances and measurement noise is provided in [14].

In this paper, the observer design problem for nonlinear systems described by interconnected TS systems with Immeasurable Premise Variables is investigated. For this purpose, we examine the stability of Luenberger-like interconnected multiple observers. First, the asymptotic stability based on the $L_2$ approach is considered. Secondly, the concept of $\mathcal{D}$-stability is used to assign local observers eigenvalues of each multiple observer in a defined LMI region. The conservatism of the proposed $\mathcal{D}$-stability conditions is reduced using Finsler’s Lemma.

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via the introduction of additional decision variables. The remainder of the paper is structured as follows. In Section 2, the problem statement is introduced. Section 3 presents the main results concerning the design of the interconnected multiple observers. In Section 4, the performances of the proposed approaches are illustrated by numerical simulations for a Proton Exchange Membrane Fuel Cell (PEMFC) system and a four-tank system. Lastly, some conclusions are given in Section 5.

2 Preliminaries and problem statement

Let us consider a set of $n$ Interconnected Takagi-Sugeno subsystems $TS_i$ with Immeasurable Premise Variables $(i=1,\ldots,n)$

\[
TS_i : \begin{cases} 
\dot{x}_i(t) = \sum_{j=1}^{r_i} h_j^i(x_i(t))(A_{ji}^i x_i(t) + B_{ji}^i u_i(t) + \sum_{k=1}^{n} D_{ji}^k x_k(t)), \\
y_i(t) = C_i x_i(t), 
\end{cases} \tag{2.1}
\]

where $x_i(t) \in \mathbb{R}^{n_i}, u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^{p_i}$ are respectively the state, input and output vectors of the $i$-th subsystem. $A_{ji}^i, B_{ji}^i, D_{ji}^k, C_i$ are known constant matrices with appropriate dimensions. The matrices $D_{ji}^k$ denote state interactions between the $i$-th and the $k$-th subsystems with $k=1,\ldots,n$ and $k\neq i$. $r_i$ is the number of local models (vertices) of the subsystem $TS_i$. $h_j^i(x_i(t))$ are the state dependent activation functions of the $i$-th TS subsystem, which verify the following convexity properties

\[
\sum_{j=1}^{r_i} h_j^i(x_i(t)) = 1, \forall j_i \in [1,\ldots,r_i],
\]

\[
0 \leq h_j^i(x_i(t)) \leq 1. \tag{2.2}
\]

For the rest of the paper, the following notations are adopted. For any matrices $N$, $He(N)$ stands for $N+N^T$. The asterisk $(\ast)$ represents the transpose of the symmetric element in matrix expressions. The symbol $\otimes$ is the Kronecker product of matrices. For any matrices $N_j^i$, we set

\[
N_h = \sum_{j=1}^{r_i} h_j^i(x_i) N_j^i, \quad N_{\hat{h}} = \sum_{j=1}^{r_i} h_j^i(\hat{x}_i) N_j^i. \tag{2.3}
\]

In this present work, we are interested in the design of $n$ Luenberger-like Interconnected Multiple Observers $MO_i$ ($i=1,\ldots,n$)

\[
MO_i : \begin{cases} 
\dot{\hat{x}}_i(t) = \sum_{j=1}^{r_i} h_j^i(\hat{x}_i(t))(A_{ji}^i \hat{x}_i(t) + B_{ji}^i u_i(t) + \sum_{k=1}^{n} D_{ji}^k \hat{x}_k(t) + L_{ji}^i (y_i(t) - \hat{y}_i(t))), \\
\hat{y}_i(t) = C_i \hat{x}_i(t), 
\end{cases} \tag{2.4}
\]

where $\hat{x}_i(t)$ and $\hat{y}_i(t)$ are respectively the state and output vector of the $i$-th multiple observer. $L_{ji}^i$ are observer constant gains to be determined.

To have common activation functions between subsystems $TS_i$ and multiple observers $MO_i$, we rewrite the system (2.1) as follows

\[
\dot{x}_i(t) = \sum_{j=1}^{r_i} h_j^i(x_i(t))(A_{ji}^i x_i(t) + B_{ji}^i u_i(t) + \sum_{k=1}^{n} D_{ji}^k x_k(t) + \omega_j(t)), \tag{2.5}
\]

where

\[
\omega_j(t) = \sum_{j=1}^{r_i} \left[h_j^i(x_i(t)) - h_j^i(\hat{x}_i(t))\right] \left(A_{ji}^i x_i(t) + B_{ji}^i u_i(t) + \sum_{k=1}^{n} D_{ji}^k x_k(t)\right). \tag{2.6}
\]

Hence, the dynamics of the $i$-th state estimation error is given by
\[ \dot{e}_i(t) = (A_h - L_i C_i)e_i(t) + \sum_{k \neq i}^n D_i^k e_k(t) \]

or

\[ \dot{e}_i(t) = (A_h - L_i C_i)e_i(t) + \left[ \begin{array}{cccc} D_i^1 & \ldots & D_i^n \\ & I \end{array} \right] \begin{bmatrix} e_1(t) \\ \vdots \\ e_n(t) \end{bmatrix}, \quad k \neq i. \]  

(2.8)

To simplify the stability analysis, the vector \( \tilde{\omega}(t) = [e_i^T(t) \ldots e_i^T(t) \ \omega_j^T(t)]^T \) will be considered as a disturbance with respect to the state estimation error \( e_i(t) \).

**Problem statement.** The design objectives are presented in the following requirements:

1) **Asymptotic stability**: For \( i=1,\ldots, n \), the gain matrices \( L_{ij} \) are designed such that :
   - The dynamics of state estimation errors (8) are asymptotically stable when \( \tilde{\omega}_i(t) = 0 \);
   - Each transfer from the disturbance \( \tilde{\omega}_i(t) \) to the state estimation error \( e_i(t) \) is minimized (\( L_2 \)-gain analysis).

2) **\( \mathcal{D} \)-stability**: For \( i=1,\ldots, n \), the gain matrices \( L_{ij} \) are designed such that :
   - The dynamics of state estimation errors (8) are \( \mathcal{D} \)-stable when \( \tilde{\omega}_i(t) = 0 \);
   - Each transfer from the disturbance \( \tilde{\omega}_i(t) \) to the state estimation error \( e_i(t) \) is minimized.

3) In addition, \( \mathcal{D} \)-stability constraints are relaxed using the Finsler’s Lemma.

### 3 Main results

In this section, we propose LMI-based sufficient conditions to design the gain matrices \( L_{ij} \) so that the requirements expressed in the problem statement are pursued.

#### 3.1 Asymptotic stability

Asymptotic stability of the proposed Interconnected Multiple Observers (2.4) is stated in the following Theorem.

**Theorem 1** For \( i=1,\ldots, n \), the Interconnected systems (2.8) are asymptotically stable and the \( L_2 \)-gains from \( \tilde{\omega}_i(t) \) to \( e_i(t) \) are bounded if there exist symmetric positive definite matrices \( P_i \), matrices \( X_j \) and positive scalars \( \gamma_i \) such that the following LMIs are satisfied

\[ \begin{bmatrix} \text{He}(PA_h - X_j C_i) + I & PD_i^1 & \ldots & PD_i^n & P \\ * & -\gamma_i I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & \ldots & -\gamma_i I & 0 \\ * & 0 & \ldots & 0 & -\gamma_i I \end{bmatrix} < 0. \]  

(3.1)

Multiple observers gain matrices are given by

\[ L_{ij}^* = P_i^{-1} X_j. \]  

(3.2)

**Attenuation levels** are

\[ \gamma_i = \sqrt{\gamma_i}. \]  

(3.3)

**Proof** Consider the Quadratic Lyapunov Function

\[ V(e_1, e_n) = \sum_{i=1}^n V_i(e_i) = \sum_{i=1}^n e_i^T P_i e_i, \quad P_i = P_i^T > 0. \]  

(3.4)
Its time derivative is

\[
\dot{V}(e_i,...,e_n) = \sum_{i=1}^{n} V_i(e_i) = \sum_{i=1}^{n} H(e_i P_i \dot{e}_i) = \sum_{i=1}^{n} H e_i (P_i A_{hi} - P_i L_{hi} C_i) e_i + e_i ^T P_i D_i^T e_i + e_i ^T P_i \omega_i \]

(3.5)

The dynamics of state estimation errors (2.8) are asymptotically stable (when \( \bar{o}_t = 0 \)) and the \( L_2 \)-gains from \( \bar{o}_t(t) \) to \( \bar{e}_t(t) \) are bounded if [3].

\[
\dot{V} + \sum_{i=1}^{n} (e_i ^T e_i - \gamma_i^2 \sum_{k=1}^{n} e_k ^T e_k - \gamma_i^2 \delta_i \omega_i ^T \omega_i) = \sum_{i=1}^{n} (\dot{V}_i(e_i) + e_i ^T e_i - \gamma_i^2 \sum_{k=1}^{n} e_k ^T e_k - \gamma_i^2 \delta_i \omega_i ^T \omega_i) < 0 .
\]

(3.6)

Replacing \( \dot{V} \) by its expression (3.5) in the inequality (3.6) gives

\[
\sum_{i=1}^{n} (e_i ^T (He_i A_{hi} - P_i L_{hi} C_i) + I) e_i + He_i \sum_{k=1}^{n} P_i D_i^T e_k + e_i ^T P_i \omega_i - \gamma_i^2 \sum_{k=1}^{n} e_k ^T e_k - \gamma_i^2 \delta_i \omega_i ^T \omega_i) < 0 .
\]

(3.7)

The previous inequality can be expressed in matrix form as

\[
\sum_{i=1}^{n} \left[ \begin{array}{cccc}
He_i A_{hi} - P_i L_{hi} C_i & P_i & D_i^T & P_i \\
* & -\gamma_i^2 I & \vdots & 0 \\
* & 0 & \ddots & 0 \\
* & 0 & \ddots & -\gamma_i^2 I \\
\end{array} \right] < 0, k \neq i ,
\]

(3.8)

where \( X^T = \left[ e_i ^T, e_i ^T, ..., e_i ^T, \omega_i ^T \right] \)

The inequality (3.8) holds, if

\[
\forall i = 1, \cdots, n : \left[ \begin{array}{cccc}
He_i A_{hi} - P_i L_{hi} C_i & P_i & D_i^T & P_i \\
* & -\gamma_i^2 I & \vdots & 0 \\
* & 0 & \ddots & 0 \\
* & 0 & \ddots & -\gamma_i^2 I \\
\end{array} \right] < 0, k \neq i .
\]

(3.9)

Finally, we consider the variable changes \( X_i = P_i L_{hi} \) and \( \gamma_i = \gamma_i^2 \) to get the sufficient conditions (3.1). This ends the proof. ■

3.2 \( \mathcal{D} \)-stability

The \( \mathcal{D} \)-stability consists in assigning the eigenvalues of a system in a defined region of the complex plane. This concept has been developed for linear systems [5] then extended to polytopic systems [26], [20]. In this section, we propose sufficient LMI conditions to design Interconnected Multiple Observers \( MO_i \) (2.4) with \( \mathcal{D} \)-stability constraints. In this context, let us first give some definitions.

**Definition 1 (LMI region) [6]:** A LMI region is a subset \( \mathcal{D} \) of the complex plane that is defined as

\[
\{ \bar{z} \in C : L + z \bar{M} + z^* \bar{M}^T < 0 \}
\]

(3.10)

where \( L = [L_{kl}] \in \mathbb{R}^{m \times m} \) is a symmetric matrix and \( M = [M_{kl}] \in \mathbb{R}^{m \times m} \). The matrix function

\[
f_\bar{z}(z) = L + z \bar{M} + z^* \bar{M}^T = [L_{kl} + M_{kl} z + M_{lk} z^*]_{l \in \mathbb{C}, k \in \mathbb{C}}
\]

is called the characteristic function of \( \mathcal{D} , z^* \) denotes the conjugate of the complex \( z \).

**Definition 2 (\( \mathcal{D} \)-Stability) [6]:** Given a LMI region defined by (3.10), a nonlinear system \( \dot{x} = f(x)x \) is \( \mathcal{D} \)-stable if there exists a Lyapunov function \( V(x(t)) \) satisfying

\[
\lim_{t \to \infty} \frac{1}{2} V(x(t)) \in \mathcal{D}, i.e.
\]
\( L \otimes V(x(t)) + M \otimes \frac{1}{2} \dot{V}(x(t)) + M^T \otimes \frac{1}{2} \tilde{V}(x(t)) < 0 \)  

(3.12)

Let us recall some useful properties of the Kronecker product.

**Properties of the Kronecker product** [11], [18]

For any scalar \( \mu \) and matrices \( A, B \) and \( C \) of appropriate dimensions, the following properties hold

\[
\begin{align*}
A \otimes (B + \mu C) &= (A \otimes B) + \mu (A \otimes C) \\
(A + \mu B) \otimes C &= (A \otimes C) + \mu (B \otimes C) \\
(A \otimes C)(B \otimes D) &= (AB) \otimes (CD) \\
(A \otimes B)^T &= A^T \otimes B^T \\
(A \otimes B)^{-1} &= A^{-1} \otimes B^{-1} \\
\|A \otimes B\|_2 &= \|A\|_2 \cdot \|B\|_2
\end{align*}
\]

(3.13) - (3.18)

**Combining \( \mathcal{D} \)-stability and \( \mathcal{L}_2 \) approach**

The inequality (3.12) is modified as follows

\[
L \otimes V(x(t)) + M \otimes \frac{1}{2} \dot{V}(x(t)) + M^T \otimes \frac{1}{2} \tilde{V}(x(t)) + (I \otimes \gamma^2)(I \otimes \omega^2)(I \otimes \omega) < 0
\]

where \( \omega(t) \) is considered as a disturbance and \( \gamma \) is a positive scalar.

Inequality (3.19) can also be expressed as

\[
L \otimes V(x(t)) + M \otimes \frac{1}{2} \dot{V}(x(t)) + M^T \otimes \frac{1}{2} \tilde{V}(x(t)) < -(I \otimes \gamma^2)(I \otimes \omega^2)(I \otimes \omega)
\]

(3.20)

Depending on the LMI region, the following results are obtained from the inequality (3.20).

**Case 1: Left half plane with a decay \( \beta \)**

In this case, \( L = 2\beta \) and \( M = I \). The inequality (3.20) becomes

\[
2\beta V(x(t)) + \dot{V}(x(t)) < -(I \otimes \gamma^2)(I \otimes \omega^2). 
\]

(3.21)

We integrate the previous inequality from 0 to \( T_f \)

\[
2\beta \int_0^{T_f} \dot{V}(x(t)) dt + \int_0^{T_f} \ddot{V}(x(t)) dt < -\int_0^{T_f} (I \otimes \gamma^2)(I \otimes \omega^2) dt + \int_0^{T_f} (I \otimes \omega^2) dt. 
\]

(3.22)

Since \( \int_0^{T_f} 2\beta V(x(t)) dt + \int_0^{T_f} \dot{V}(x(t)) dt \) is positive, we have

\[
\int_0^{T_f} (I \otimes \gamma^2)(I \otimes \omega^2) dt < \gamma^2 \int_0^{T_f} (I \otimes \omega^2) dt.
\]

(3.23)

**Case 2: Circle centered at \( (q,0) \) with radius \( s \)**

The matrices \( L \) and \( M \) are defined as

\[
L = \begin{bmatrix} -s & q \\
q & -s \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\
0 & 0 \end{bmatrix}.
\]

(3.24)

The matrices \( L \) and \( M \) are replaced in the inequality (3.20)
\[
\begin{bmatrix}
-sV(x(t)) + \frac{1}{2} V(x(t)) \\
q + \frac{1}{2} V(x(t))
\end{bmatrix} < -(I \otimes y^T)(I \otimes y) + \gamma^2(I \otimes \omega^T)(I \otimes \omega). \tag{3.25}
\]

**Lemma 1 (Schur complement) [3]** Given constant matrices \(M, L, P\) of appropriate dimensions where \(M\) and \(P\) are symmetric, then \(P>0\) and \(M + L^T P^{-1} L < 0\) if and only if
\[
\begin{bmatrix}
M & L^T \\
L & -P
\end{bmatrix} < 0. \tag{3.26}
\]

The Schur complement allows to write
\[
-sV(x(t)) + s^2 V^{-1}(x(t)) \left( q + \frac{1}{2} V(x(t)) \right) < -(I \otimes y^T)(I \otimes y) + \gamma^2(I \otimes \omega^T)(I \otimes \omega). \tag{3.27}
\]

\[
\left( \frac{q^2}{s} - s \right) V(x(t)) + \frac{q}{s} V(x(t)) \left( \frac{1}{4s} \frac{V^2(x(t))}{V(x(t))} \right) < -(I \otimes y^T)(I \otimes y) + \gamma^2(I \otimes \omega^T)(I \otimes \omega). \tag{3.28}
\]

or
\[
\left( \frac{q^2}{s} - s \right) V(x(t)) + \frac{q}{s} V(x(t)) \left( \frac{1}{4s} \frac{V^2(x(t))}{V(x(t))} \right) < -(I \otimes y^T)(I \otimes y) + \gamma^2(I \otimes \omega^T)(I \otimes \omega). \tag{3.29}
\]

The inequality (3.29) is integrated from 0 to \(T_f\)
\[
\left( \frac{q^2}{s} - s \right) \int_0^{T_f} V(x(t)) \, dt + \frac{q}{s} \int_0^{T_f} V(x(t)) \, dt < -\int_0^{T_f} (I \otimes y^T)(I \otimes y) \, dt + \gamma^2 \int_0^{T_f} (I \otimes \omega^T)(I \otimes \omega) \, dt. \tag{3.30}
\]

By choosing \(q > s\), the term \(\left( \frac{q^2}{s} - s \right) \int_0^{T_f} V(x(t)) \, dt + \frac{q}{s} \int_0^{T_f} V(x(t)) \, dt\) is positive and
\[
\int_0^{T_f} (I \otimes y^T)(I \otimes y) \, dt < \gamma^2 \int_0^{T_f} (I \otimes \omega^T)(I \otimes \omega) \, dt. \tag{3.31}
\]

**Case 3: Conic sector with an apex \((a,0)\) and an angle \(\theta\) relative to the imaginary axe**

\[
L = \begin{bmatrix}
-2\alpha \cos \theta & 0 \\
0 & -2\alpha \cos \theta
\end{bmatrix},
\quad
M = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}. \tag{3.32}
\]

The matrices \(L\) and \(M\) are replaced in the inequality (3.20)
\[
\begin{bmatrix}
-2\alpha \cos \theta V(x(t)) + V(x(t)) \cos \theta \\
0
\end{bmatrix} < -(I \otimes y^T)(I \otimes y) + \gamma^2(I \otimes \omega^T)(I \otimes \omega). \tag{3.33}
\]

Thanks to the Schur complement, we have
\[
-2\alpha \cos \theta V(x(t)) + V(x(t)) \cos \theta < -(I \otimes y^T)(I \otimes y) + \gamma^2(I \otimes \omega^T)(I \otimes \omega). \tag{3.34}
\]

Once again, the inequality (3.35) is integrated from 0 to \(T_f\)
\[
-2\alpha \cos \theta \int_0^{T_f} V(x(t)) \, dt + \cos \theta \int_0^{T_f} V(x(t)) \, dt < -\int_0^{T_f} (I \otimes y^T)(I \otimes y) \, dt + \gamma^2 \int_0^{T_f} (I \otimes \omega^T)(I \otimes \omega) \, dt. \tag{3.36}
\]

By choosing \(\alpha = 0\) and \(\theta\) so that \(\cos \theta > 0\), the left term \(-2\alpha \cos \theta \int_0^{T_f} V(x(t)) \, dt + \cos \theta \int_0^{T_f} V(x(t)) \, dt\) of the above inequality, and
\[
\int_0^T (I \otimes y^T)(I \otimes y) dt < \gamma^2 \int_0^T (I \otimes \omega^T)(I \otimes \omega) dt.
\] (3.37)

Using the fact that \( \|z(t)\|_2^2 = \int_0^T z^T(t)z(t) dt \), the inequality (3.37) becomes
\[
\|I \otimes y\|_2^2 < \gamma^2 \|I \otimes \omega\|_2^2.
\] (3.38)

Using the property (3.18) of the Kronecker product yields
\[
\|y\|_2^2 < \gamma^2 \|\omega\|_2^2.
\] (3.39)

Since the positive scalar \( \gamma \) represents the attenuation level between the disturbance \( \omega(t) \) and the output \( y(t) \), the constraint (3.19) guarantees the \( \mathcal{D} \)-stability of the system and the boundedness of the \( L_2 \)-gain from \( \omega(t) \) to \( y(t) \) and will be used in the following theorem.

**Theorem 2** For given matrices \( L \) and \( M \) defining a convenient LMI region. For \( i=1, \ldots, n \), the interconnected systems (2.8) are \( \mathcal{D} \)-stable and the \( L_2 \)-gains from \( \tilde{\omega}_i(t) \) to \( e_i(t) \) are bounded if there exist symmetric positive definite matrices \( P_i \), matrices \( X_{ik} \) and positive scalars \( \gamma_i \) such that the following constraints are satisfied
\[
\min_{\gamma_i, X_{ik}} \gamma_i \leq \gamma_i \leq \gamma_i^{\text{max}}.
\]

Multiple observers gain matrices are given by
\[
L_{ih} = P_i^{-1} X_{ih}.
\] (3.41)

**Attenuation levels are**
\[
\gamma_i = \sqrt{\gamma_i}.
\] (3.42)

**Proof** Considering the same Lyapunov function as above, the dynamics of state estimation errors (2.8) are \( \mathcal{D} \)-stable (when \( \tilde{\omega}_i = 0 \)) and the \( L_2 \)-gains from \( \tilde{\omega}_i(t) \) to \( e_i(t) \) are bounded if
\[
\sum_{i=1}^n (L \otimes V_i + M \otimes \frac{1}{2} \tilde{V}_i + M^T \otimes \frac{1}{2} \tilde{V}_i + (I \otimes e_i^T)(I \otimes e_i) - \gamma_i^2 \sum_{k \neq i} (I \otimes e_k^T)(I \otimes e_k) - \gamma_i^2 (I \otimes \omega^T)(I \otimes \omega)) < 0.
\] (3.43)

Then, we replace \( \tilde{V}_i \) by their expressions in the inequality (3.43) and use the property of the Kronecker product (3.15)
\[
\sum_{i=1}^n ((I \otimes e_i^T)(L \otimes P_i + He(M \otimes (P_i A_{ih} - P_i L_{ih} C_i)) + I)(I \otimes e_i) + He((I \otimes e_i^T)(M \otimes \sum_{k \neq i} P D_k^T)(I \otimes e_i^T)) +
He((I \otimes e_i^T)(M \otimes P_i)(I \otimes \omega_i)) - \gamma_i^2 \sum_{k \neq i} (I \otimes e_k^T)(I \otimes e_k) - \gamma_i^2 (I \otimes \omega^T)(I \otimes \omega_i)) < 0.
\] (3.44)

The previous inequality in matrix form gives
with
\[
Y^T = \begin{bmatrix}
I \otimes e_i^T & I \otimes e_i^T & \cdots & I \otimes e_i^T & I \otimes e_i^T
\end{bmatrix}
\]

The inequality (3.45) is satisfied if
\[
\begin{bmatrix}
L \otimes P_i + He(M \otimes (P_iA_h - P_iL_h C_i)) + I & M \otimes P_i D_{h}^1 & \cdots & M \otimes P_i D_{h}^k & M \otimes P_i D_{h}^k
\end{bmatrix}
\]
\[
\begin{bmatrix} \ast & -\gamma_i^2 I & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\ast & 0 & \cdots & -\gamma_i^2 I & 0 \\
\ast & 0 & \cdots & 0 & -\gamma_i^2 I
\end{bmatrix}
\]
\[
< 0, k \neq i.
\]

The sufficient conditions (3.40) are obtained using the variable changes $X_i = P_i L_h$ and $\bar{\gamma}_i = \gamma_i^2$. ■

3.3 Finsler lemma relaxation of $\mathcal{D}$-stability constraints

In this section, we propose to use the so-called Finsler’s Lemma to reduce the conservatism of $\mathcal{D}$-stability constraints. This lemma provides more degrees of freedom through additional slack variables to obtain more relaxed stability conditions. Before stating Theorem 3, let’s recall this important lemma and some others.

Lemma 2 (Finsler’s Lemma) \([13]\) Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times n}$ such that $\text{rank}(R) < n$, the following expressions are equivalent

\(a)\) $x^T Q x < 0, \forall x \in \{x \in \mathbb{R}^n; x \neq 0, Rx = 0\}$,

\[ (3.47) \]

\(b)\) $\exists Y \in \mathbb{R}^{n \times m}; Q + YR + R^T Y^T < 0.$

\[ (3.48) \]

Lemma 3 (Congruence) \([3]\) Let two matrices $M$ and $N$, if $M$ is negative (resp. positive) definite and if $N$ is a full rank matrix, then the matrix $N^T M N$ is negative (resp. positive) definite.

Lemma 4 (Young’s inequality) \([38]\) For any matrices $X$ and $Y$, the following property is verified for any positive matrix $A$

\[
H e(X^T Y) \leq X^T A^{-1} X + Y^T A Y, \quad \Lambda > 0.
\]

\[ (3.49) \]

Remark 1 Let’s consider the following Bilinear Matrix Inequality (BMI)

\[
P - X^T S X < 0, S > 0.
\]

\[ (3.50) \]

Young’s relation can be used to obtain an LMI in $X$ (when the new variable $Y$ is fixed)

\[
P - X^T Y - Y^T X + Y^T S^{-1} Y < 0.
\]

\[ (3.51) \]

Theorem 3 For given matrices $L$ and $M$ defining a convenient LMI region. For $i=1,\ldots,n$, the interconnected systems (2.8) are $\mathcal{D}$-stable and the $L_2$-gains from $\tilde{w}_h(t)$ to $e_i(t)$ are bounded if $\exists \epsilon_i > 0$, symmetric positive definite matrices $P_i$, matrices $X_i$, $Z_i$, $V_i$, $W_i$ and positive scalars $\bar{\gamma}_i$ such that the following constraints are satisfied $\forall k \in [1, m] / k \neq i$

\[
\min_{P_i, X_i, Z_i, V_i, W_i, \bar{\gamma}_i} \bar{\gamma}_i
\]
where

\[ \Gamma_{i}^j > 0 \quad \text{and} \quad \Lambda_i > 0. \]

Multiple observers gain matrices are given by

\[ L_{h_j} = P_{i}^{-1} X_{h_j}^{i}. \quad (3.53) \]

Attenuation levels are

\[ \gamma_i = \sqrt{\nu_i}. \quad (3.54) \]

**Proof.** To prove the Theorem 3, we use the same \( \mathcal{D} \)-stability constraint (3.43) as above

\[ \sum_{i=1}^{n} (L \otimes V_i + M \otimes \frac{1}{2} \tilde{V}_i + M^T \otimes \frac{1}{2} \tilde{V}_i) + (I \otimes e_i) (I \otimes e_i) - \gamma_i^2 \sum_{k=1}^{n} (I \otimes e_i) (I \otimes e_k) - \gamma_i^2 (I \otimes \omega_i) (I \otimes \omega_i) < 0. \quad (3.55) \]

We replace \( V_i \) and \( \tilde{V}_i \) by their respective expressions \( e_i \otimes P_i e_i \) and \( He(e_i \otimes \tilde{P} \tilde{e}_i) \) in the previous inequality and use the property (3.15) of the Kronecker product

\[ \sum_{i=1}^{n} ((I \otimes e_i) \otimes L \otimes P_i + I(\otimes e_i) + He((I \otimes e_i) \otimes M \otimes P_i)(\otimes e_i)) - \gamma_i^2 \sum_{k=1}^{n} (I \otimes e_i) (I \otimes e_k) - \gamma_i^2 (I \otimes \omega_i) (I \otimes \omega_i) < 0. \quad (3.56) \]
Inequality (3.56) in matrix form gives

$$\sum_{i=1}^{n} Z^T \begin{bmatrix} L \otimes P_i + I & 0 & \cdots & 0 & 0 & M \otimes P_i \\ 0 & -\gamma_i^2 I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\gamma_i^2 I & 0 & 0 \\ M^T \otimes P_i & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} Z < 0, k \neq i, \quad (3.57)$$

where $Z^T = [I \otimes e_i^T \ I \otimes e_i^T \ I \otimes e_i^T \ I \otimes e_i^T]$

The inequality (3.57) holds, if

$$\forall i = 1, \ldots, n : Z^T \begin{bmatrix} L \otimes P_i + I & 0 & \cdots & 0 & 0 & M \otimes P_i \\ 0 & -\gamma_i^2 I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\gamma_i^2 I & 0 & 0 \\ M \otimes P_i & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} Z < 0, k \neq i. \quad (3.58)$$

In the other hand, from expression (2.7), we can write

$$\forall i = 1, \ldots, n : M \otimes \dot{e}_i = (M \otimes (A_{h_i} - L_{h_i} C_i))e_i + (M \otimes \sum_{k=4}^{n} D_{h_i}^k e_k) + (M \otimes \omega_i) \quad (3.59)$$

Using the property (3.15) of the Kronecker product, we have

$$\forall i = 1, \ldots, n : (M \otimes I)(I \otimes \dot{e}_i) = M \otimes (A_{h_i} - L_{h_i} C_i)(I \otimes e_i) + \sum_{k=4}^{n} (M \otimes D_{h_i}^k)(I \otimes e_k) + (M \otimes I)(I \otimes \omega_i) \quad (3.60)$$

The previous expression in matrix form

$$\forall i = 1, \ldots, n : \begin{bmatrix} M \otimes (A_{h_i} - L_{h_i} C_i) & M \otimes D_{h_i}^1 & \cdots & M \otimes D_{h_i}^k & (M \otimes I) - (M \otimes I) \end{bmatrix} \begin{bmatrix} (I \otimes e_i) \\ (I \otimes e_i) \\ \vdots \\ (I \otimes e_i) \\ (I \otimes e_i) \end{bmatrix} = 0, k \neq i. \quad (3.61)$$

Then, using Finsler's Lemma, the inequality (3.58) under the constraint (3.61) leads to

$$\forall i = 1, \ldots, n : \begin{bmatrix} L \otimes P_i + I & 0 & \cdots & 0 & 0 & M \otimes P_i \\ 0 & -\gamma_i^2 I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\gamma_i^2 I & 0 & 0 \\ M^T \otimes P_i & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_i^1 \\ U_i^2 \\ \vdots \\ U_i^k \\ U_i^l \end{bmatrix} + \begin{bmatrix} M \otimes (A_{h_i} - L_{h_i} C_i) & M \otimes D_{h_i}^1 & \cdots & M \otimes D_{h_i}^k & (M \otimes I) - (M \otimes I) \end{bmatrix} \begin{bmatrix} (I \otimes e_i) \\ (I \otimes e_i) \\ \vdots \\ (I \otimes e_i) \\ (I \otimes e_i) \end{bmatrix} = 0, k \neq i. \quad (3.62)$$

We choose: $U_i^1 = (I \otimes P_i), U_i^2 = (I \otimes e_i(W_i^k)^-1 P_i), U_i^3 = (I \otimes e_i(V_i^k)^-1 P_i), U_i^4 = (I \otimes e_i(Y_i^k)^-1 P_i)$ with $e_i > 0$, so the expression (3.62) becomes
We divide the inequality (3.63) as shown above

\[ \forall i = 1, \ldots, n: M_i = \begin{bmatrix} M_{i1}^T & M_{i2}^T \end{bmatrix} < 0. \] (3.64)

The following matrix is considered

\[ N_i = \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix} \] (3.65)

with

\[ Q = \begin{bmatrix} I \otimes V_i^T & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & I \otimes V_i^T & 0 & 0 \\ 0 & \cdots & 0 & I \otimes W_i & 0 \\ 0 & \cdots & 0 & 0 & I \otimes Y_i \end{bmatrix}. \] (3.66)

Applying the congruence transformation (Lemma 3) yields

\[ \forall i = 1, \ldots, n: N_i^T M_i N_i < 0. \] (3.67)

\[ \forall i = 1, \ldots, n: \begin{bmatrix} M_{i1}^T Q^T & M_{i2}^T Q^T \end{bmatrix} < 0. \] (3.68)

After the substitution of \( M_{i1}, M_{i2}, M_{i3} \) and \( Q \), we have

\[ \forall i = 1, \ldots, n: \begin{bmatrix} L \otimes P + He(M \otimes (P A_k - P L_k C_i)) + I \\ M \otimes (e_i(V_i^T)^T P A_k - e_i(V_i^T)^T P L_k C_i) + (M^T \otimes D_i^T P) \end{bmatrix} < 0, k \neq i. \] (3.69)
Based on remark 1, the diagonal elements (2,2), (4,4) and (5,5) of the inequality (3.69) can be bounded as

\[-\gamma^2 \left( I \otimes V_{h_i}^T \right) + He(M \otimes \varepsilon_i P D_{hi}^T V_{h_i}^T) \leq He(M \otimes \varepsilon_i P D_{hi}^T V_{h_i}^T) - He(M \otimes \varepsilon_i V_{h_i}^T) + \gamma^{-2} (M \otimes \varepsilon_i R_{h_i}^T) (M^T \otimes \varepsilon_i R_{h_i}^T). \]  

(3.70)

\[-\gamma^2 \left( I \otimes V_{h_i}^T \right) + He(M \otimes \varepsilon_i P D_{hi}^T V_{h_i}^T) \leq He(M \otimes \varepsilon_i P D_{hi}^T V_{h_i}^T) - He(M \otimes \varepsilon_i V_{h_i}^T) + \gamma^{-2} (M \otimes \varepsilon_i R_{h_i}^T) (M^T \otimes \varepsilon_i R_{h_i}^T). \]  

(3.71)

\[-\gamma^2 \left( I \otimes W_{h_i}^T \right) + He(M \otimes \varepsilon_i P W_{h_i}^T) \leq He(M \otimes \varepsilon_i P W_{h_i}^T) - He(M \otimes \varepsilon_i W_{h_i} S_{h_i}^T) + \gamma^{-2} (M \otimes \varepsilon_i S_{h_i}^T) (M^T \otimes \varepsilon_i S_{h_i}^T). \]  

(3.72)

Hence, the inequality (3.69) becomes

\[
\begin{bmatrix}
L \otimes I + He(M \otimes (P A_{k,l} - P L_{k,l} C)) + I & * & \cdots & * \\
M \otimes (\varepsilon_i P A_{k,l} - \varepsilon_i P L_{k,l} C) + (M^T \otimes V_{k,l}^T) P & He(M \otimes \varepsilon_i P D_{k,l}^T V_{k,l}^T) - He(M \otimes \varepsilon_i V_{k,l}^T) & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
M \otimes (\varepsilon_i P A_{k,l} - \varepsilon_i P L_{k,l} C) + (M^T \otimes V_{k,l}^T) P & (M \otimes \varepsilon_i P D_{k,l}^T V_{k,l}^T) + (M^T \otimes \varepsilon_i V_{k,l}^T) & \cdots & * \\
M \otimes (\varepsilon_i P A_{k,l} - \varepsilon_i P L_{k,l} C) + (M^T \otimes W_{k,l} P) & (M \otimes \varepsilon_i P D_{k,l}^T V_{k,l}^T) + (M^T \otimes \varepsilon_i W_{k,l} P) & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
M \otimes (\varepsilon_i P A_{k,l} - \varepsilon_i P L_{k,l} C) & (M \otimes \varepsilon_i P D_{k,l}^T V_{k,l}^T) - (M^T \otimes \varepsilon_i Y_{k,l}) & \cdots & * \\
\end{bmatrix}
\]

\[
He(M \otimes \varepsilon_i P D_{k,l}^T V_{k,l}^T) - He(M \otimes \varepsilon_i V_{k,l}^T) = * + \Omega_k^i \quad \text{for } k \neq i.
\]  

(3.73)

The matrix \( \Omega^i_k \) is given by

\[
\Omega_k^i = \gamma^{-2} \begin{bmatrix}
0 & M \otimes \varepsilon_i R_{h_i}^T & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
+ \gamma^{-2} \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
M \otimes \varepsilon_i S_{h_i}^T & 0 \\
\end{bmatrix}
\]

(3.74)

The application of the Schur complement (Lemma 1) to the inequality (3.73) gives
The inequality (3.75) can also be expressed as

![Equation Image]

The inequality (3.75) can also be expressed as

![Equation Image]
Applying Young’s inequality to (3.77), the Schur complement to (3.76) and using the variable changes
\[ X_{ij} = P_{ij}L_{ij}, Z_{ij} = P_{ij}Z_{ij}^T \] and \( \bar{\gamma}_i = \gamma_i^2 \) yields to the sufficient condition (3.52).

**Remark 2** The Finsler’s based \( D \)-stability constraints are LMIs for given \( R_{ij}, S_{ij} \) and \( \epsilon_i \). Thus, in this paper, the matrices \( R_{ij} \) and \( S_{ij} \) and the scalars \( \epsilon_i \) are fixed and are not decision variables.

## 4 Simulation results

### 4.1 Application to a PEMFC system

Polymer Electrolyte Membrane Fuel Cells (PEMFC) are one of the most promising clean energy conversion devices because of their light weight, rapid start-up, low operating temperature, high energy density and non-polluting features [16], [31], [7]. PEMFCs are extensively used as power sources in numerous stationary and portable applications [8].

In PEMFCs hydrogen (\( \text{H}_2 \)) and oxygen (\( \text{O}_2 \)) are used to produce electricity, water and heat. The hydrogen enters at the anode. There, the catalyst layer separates hydrogen molecules into protons (\( \text{H}^+ \)) and electrons (\( \text{e}^- \)). This reaction releases heat. The electrolyte made of polymer membrane permits the transfer of the protons to the cathode enabling the electrons to flow through an external load circuit. At the same time the oxygen enters at the cathode and combines with electrons and protons to form water (\( \text{H}_2\text{O} \)).

![Fig.1. Working principle of a PEMFC](image_url)

A complete PEMFC system includes in general four auxiliary sub-systems [27]:

(i) hydrogen supply system, (ii) air supply system (iii) cooling system and (iv) humidification system. Before entering the cathode, the air generated by a compressor is cooled down and humidified. The hydrogen stored in
high pressure tank is transmitted to the anode after humidification. Finally, the current is produced by electrochemical reactions between hydrogen and oxygen (Fig. 2).

Fig. 2. Fuel Cell Stack with auxiliary systems [32]

### 4.1.1 Nonlinear model

The four states nonlinear model considered in this study is based on the work of [32]. The hydrogen pressure, stack temperature and humidity are assumed to be controlled accurately. The cooler and the humidifier are neglected. The model focuses principally on the variables related to air supply sub-system and the dynamic of the cathode pressure

\[
\begin{align*}
    \dot{p}_{ca}(t) &= (c_1 + c_8)(p_{sm}(t) - p_{ca}(t)) + c_4\alpha(p_{ca}) (c_2 - p_{ca}(t)) - c_7I_{st}(t) \\
    \dot{p}_{N2}(t) &= c_8(p_{sm}(t) - p_{ca}(t)) - c_4p_{ca}(t) + (c_5 - c_4)p_{N2}(t) + (c_6 - c_2c_4)P_{N2}(t) \\
    \dot{\omega}_{cp}(t) &= -c_9\omega_{cp}(t) - \frac{c_{10}}{\omega_{cp}(t)} \left( \frac{p_{sm}(t)}{c_{11}} \right)^{c_{12}} \right] \right) W_{cp}(\omega_{cp}, p_{sm}) + c_{13}v_{cm}(t) \\
    \dot{p}_{sm}(t) &= c_{14} \left[ 1 + c_{15} \left( \frac{p_{sm}(t)}{c_{11}} \right)^{c_{12}} \right] W_{cp}(\omega_{cp}, p_{sm}) - c_{16}(p_{sm}(t) - p_{ca}(t)) \\
\end{align*}
\]

(4.5)

with

\[
\alpha(p_{ca}) = \begin{cases} 
    c_{17}p_{ca}(t) \left( \frac{c_{11}}{p_{ca}(t)} \right)^{c_{18}} & \text{if } \frac{c_{11}}{p_{ca}(t)} > c_{19} \\
    c_{20}p_{ca}(t) & \text{if } \frac{c_{11}}{p_{ca}(t)} \leq c_{19} 
\end{cases}
\]

(4.6)

where \( p_{ca}(t) \) is the cathode pressure, \( p_{N2}(t) \) is the nitrogen partial pressure, \( \omega_{cp}(t) \) is the angular velocity of the compressor and \( p_{sm}(t) \) is the air pressure in the supply manifold.

The inputs are the compressor motor voltage \( v_{cm}(t) \) and the stack current \( I_{st} \).

The system outputs are the cathode pressure \( y_1(t) = p_{ca} \) and the air pressure in the supply manifold \( y_2(t) = p_{sm} \).

The compressor air mass flow \( W_{cp}(t) \) has been approximated in [12] as
\[ W_{cp}(t) = -\frac{W_{cP}^{\text{max}}}{\omega_{CP}^{\text{max}}} \left[ \frac{\omega_{CP}(t)}{q} \right] \left[ 1 - e^{\lambda(t)} \right] \]  

with

\[ \lambda(t) = -\frac{\omega_{CP}^2(t)}{s + \frac{\omega_{CP}(t)}{q}} - P_{sm}(t) \]

where

\[ r = 15 \text{, } q = 462.25 \text{ rad/s}^2 \text{Pa}, \omega_{cp}^{\text{max}} = 11500 \text{ rad/s}, \quad P_{sm}^{\text{min}} = 50000 \text{ Pa}, \quad s = 100000 \text{ Pa}, \quad W_{cp}^{\text{max}} = 0.0975 \text{ kg/s}. \]

The constants \( c_i, \ i \in \{1, \ldots, 20\} \) and the PEMFC stack system parameters are defined respectively in Tables 2 and 3 in the Appendix 1.

### 4.1.2 Interconnected T-S models

First, we divide the nonlinear model of the PEMFC system (4.1-4.5) into two dynamics as follows

\[
S_1 : \begin{cases} 
\dot{p}_{aw}(t) &= (c_1 + c_8 X_p(t) - p_{aw}(t)) + \frac{c_2 \alpha(p_{aw})}{c_4 p_{aw}(t) + (c_5 - c_4) p_{N2}(t) + (c_6 - c_2 c_4)} (c_2 - p_{aw}(t)) - c_7 I(t) \\
\dot{y}_1(t) &= p_{aw}(t) 
\end{cases}
\]

\[
S_2 : \begin{cases} 
\dot{p}_{sm}(t) &= c_{14} \left( 1 + c_{15} \left( \frac{p_{sm}(t)}{c_{11}} \right)^{1/2} - 1 \right) \left[ W_{cp}(\omega_{cp}, p_{sm}) - c_{16} (-p_{aw}(t) + p_{sm}(t)) \right] \\
y_2(t) &= p_{sm}(t) 
\end{cases}
\]

Then, the two sub-systems (4.8) and (4.9) are expressed in quasi-LPV forms

\[
\begin{bmatrix} \dot{p}_{aw}(t) \\ \dot{y}_1(t) \end{bmatrix} = \begin{bmatrix} c_7 - c_8 \\ -c_8 \end{bmatrix} \begin{bmatrix} p_{aw}(t) \\ p_{N2}(t) \end{bmatrix} + \begin{bmatrix} c_6 \\ -c_6 \end{bmatrix} \begin{bmatrix} I(t) \\ p_{aw}(t) \end{bmatrix}
\]

\[
S_1 : \begin{bmatrix} \dot{p}_{aw}(t) \\ \dot{y}_1(t) \end{bmatrix} = \begin{bmatrix} -c_7 \\ 0 \end{bmatrix} \begin{bmatrix} p_{aw}(t) \\ p_{N2}(t) \end{bmatrix} + \begin{bmatrix} c_6 \\ -c_6 \end{bmatrix} \begin{bmatrix} I(t) \\ p_{aw}(t) \end{bmatrix}
\]
\[
\begin{bmatrix}
\dot{\omega}_{cp}(t) \\
\dot{p}_{sm}(t)
\end{bmatrix} = 
\begin{bmatrix}
-c_9 - c_{10} z_2(p_{sw}) z_4(\omega_{cp}, p_{sm}) & z_5(\omega_{cp}) \\
-c_4 (1 + c_{15} z_5(p_{sm})) z_6(\omega_{cp}, p_{sm}) & -c_4 c_{16} (1 + c_{15} z_5(p_{sm}))
\end{bmatrix} \begin{bmatrix}
\omega_{cp}(t) \\
p_{sm}(t)
\end{bmatrix} + \begin{bmatrix}
c_{13} \\
0
\end{bmatrix} \gamma_n(t)
\]
\]

(4.11)

\[\begin{align*}
S_2 : & \sum_{j=1}^{4} \begin{bmatrix}
0 \\
0
\end{bmatrix}^T \begin{bmatrix}
c_4 (1 + c_{15} z_5(p_{sm})) \\
c_4 (1 + c_{15} z_5(p_{sm}))
\end{bmatrix} \begin{bmatrix}
p_{sm}(t) \\
p_{sm}(t)
\end{bmatrix} \\
y_2(t) = & \begin{bmatrix}
0 \\
1
\end{bmatrix} \begin{bmatrix}
\omega_{cp}(t) \\
p_{sm}(t)
\end{bmatrix}
\end{align*}\]

where the premise variables used in the T-S representation of each sub-system are

\[z_1(p_{ca}, P_{N2}) = \frac{c_3 \alpha(p_{ca})}{c_4 p_{ca}(t) + (c_5 - c_4) P_{N2}(t) + (c_6 - c_2 c_4)} ; z_2(P_{N2}) = \frac{1}{P_{N2}(t)} ; z_3(p_{sw}) = \left(\frac{p_{sw}(t)}{c_{11}}\right)^{c_{12}} - 1;\]

\[z_4(\omega_{cp}, p_{sm}) = \frac{W_{max cp}}{\omega_{max cp} \left(1 - e^{x(t)}\right)} ; z_5(\omega_{cp}) = \frac{1}{\omega_{cp}(t)}\]

Using the Lemma 5 (Appendix 2), each premise variable is expressed as follows

\[z_1 = F_{i1}(z_1) z_{i1} + F_{i2}(z_1) z_{i2},\]
\[z_2 = F_{i3}(z_2) z_{i2} + F_{i4}(z_2) z_{i2},\]
\[z_3 = F_{i5}(z_3) z_{i3} + F_{i6}(z_3) z_{i3},\]
\[z_4 = F_{i7}(z_4) z_{i4} + F_{i8}(z_4) z_{i4},\]
\[z_5 = F_{i9}(z_5) z_{i5} + F_{i10}(z_5) z_{i5} ,\]

\[F_{i3} = \frac{z_{j1} - z_{j2}}{z_{j1} - z_{j2}} ; F_{i4} = \frac{z_{j1} - z_{j2}}{z_{j1} - z_{j2}} = 1 - F_{i3}(z_j) ; z_{j1} = \max(z_j) ; z_{j2} = \min(z_j) .\]

Hence, the PEMFC stack system can be described by two interconnected TS models

\[\begin{align*}
TS_1 : & \dot{x}_1(t) = \sum_{i=1}^{4} h_{i1}(x_i) (A_{i1} x_i(t) + B_{i1} I_n(t) + D_{i1} x_{2}(t)) \\
y_1(t) = C_1 x_1(t)
\end{align*}\]

(4.12)

\[\begin{align*}
TS_2 : & \dot{x}_2(t) = \sum_{j=1}^{4} h_{j2}(x_2) (A_{j2} x_2(t) + B_{j2} v_m(t) + D_{j2} x_1(t)) \\
y_2(t) = C_2 x_2(t)
\end{align*}\]

(4.13)

Tables 4 and 5 give the matrices \(A_{i1}, D_{i1}, B_{i1}\) and the activation functions \(h_{i1}\) of sub-systems TS1 and TS2 (see Appendix 3).

4.1.3 Observer design

- Test of Theorem 1 (asymptotic stability)

The Theorem 1 is applied to the interconnected T-S models (4.12-4.13) via the LMI constraints (3.1). Simulation results are depicted in Figs.3-6.
Fig. 3. Estimate of cathode pressure and its error (Theorem 1)

Fig. 4. Estimate of nitrogen partial pressure and its error (Theorem 1)

Fig. 5. Estimate of compressor angular velocity and its error (Theorem 1)
As could be seen from simulation results, the observation of PEMFC stack system variables is performed correctly and the state estimation errors converge properly to zero.

- **Test of Theorem 2 (D-stability)**

The LMI region considered in this section is presented in Fig. 7. It is the combination of three elementary regions: the left half plane with a decay rate $\beta$, a conic sector defined by its apex at $(\alpha,0)$ with an angle $\theta$ with the imaginary axe and a circle at $(q,0)$ with a radius $s$. According to definition 1, this LMI region is defined by the two matrices $L$ and $M$:

$$
L = \begin{bmatrix}
2\beta & 0 & 0 & 0 & 0 \\
0 & -2\alpha \cos \theta & 0 & 0 & 0 \\
0 & 0 & -2\alpha \cos \theta & 0 & 0 \\
0 & 0 & 0 & -s & q \\
0 & 0 & 0 & q & -s
\end{bmatrix},
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 & 0 \\
0 & -\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Simulation results of Theorem 2 are presented in figs. 8-13 for $\beta=3.5$, $\alpha=0$, $\theta=\pi/2.25$, $s=20$ and $q=21$. 

**Fig.6.** Estimate of air pressure in supply manifold and its error (Theorem 1)

**Fig.7.** Usual LMI region
Fig. 8. Estimate of cathode pressure and its error (Theorem 2)

Fig. 9. Estimate of nitrogen partial pressure and its error (Theorem 2)

Fig. 10. Estimate of compressor angular velocity and its error (Theorem 2)
State estimations shown in these figures are satisfactory; state estimation errors converge to zero and the eigenvalues of local observers of each multiple observer are contained in the desired LMI region. The optimal attenuation levels are $\gamma_1 = 232, 1674$ and $\gamma_2 = 5,0028$.

- **Test of Theorem 3 (Finsler’s lemma based D-stability)**

Simulation results of Theorem 3 are illustrated by figs.13-17. Finsler’s Lemma based observation achieves better values of attenuation levels for $\varepsilon = 10^{-6}$ i.e. $\gamma_1 = 0,0044$. 

---

![Fig.11. Estimate of air pressure in supply manifold and its error (Theorem 2)](image1.png)

![Fig.12. Distribution of local observers eigenvalues (Theorem 2)](image2.png)
Fig. 13. Estimate of cathode pressure and its error (Theorem 3)

Fig. 14. Estimate of nitrogen partial pressure and its error (Theorem 3)

Fig. 15. Estimate of compressor angular velocity and its error (Theorem 3)
4.2 Application to a four-tank system

4.2.1 Description and nonlinear model

The process shown in Fig.18 consists of four interconnected water tanks and two pumps. The inputs are the voltages $v_1$ and $v_2$ to the two pumps and the outputs are water levels in the two lower tanks. Mass balances and Bernoulli’s law yield to the nonlinear model (4.14) [15]. $A_i$ is the cross section of tank $i$, $a_i$ is the cross section of the drain in tank $i$, $h_i$ is the water level in tank $i$. $\gamma_1$ is the ratio of flow in tank 1 to flow in tank 4. $\gamma_2$ is the ratio of flow in tank 2 to flow in tank 3. Parameters values of the plant are presented in Table 1.
\begin{align}
\dot{h}_1 &= -\frac{a_1}{A_1} \sqrt{2gh_1} + \frac{a_1}{A_1} \sqrt{2gh_0} + \frac{\gamma_1 k_1}{A_1} v_i \\
\dot{h}_2 &= -\frac{a_2}{A_2} \sqrt{2gh_2} + \frac{a_2}{A_2} \sqrt{2gh_1} + \frac{\gamma_2 k_2}{A_2} v_2 \\
\dot{h}_4 &= -\frac{a_4}{A_4} \sqrt{2gh_4} + \frac{a_4}{A_4} \sqrt{2gh_3} + \frac{(1-\gamma_2) k_2}{A_2} v_2 \\
\dot{h}_4 &= -\frac{a_4}{A_4} \sqrt{2gh_4} + \frac{(1-\gamma_1) k_1}{A_1} v_i
\end{align}
(4.14)

**Table 1. Parameters values of the quadruple tank system**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^0$</td>
<td>Nominal levels</td>
<td>14,1;11,2;7,2;4,7 cm</td>
</tr>
<tr>
<td>$v^0$</td>
<td>Nominal pump settings</td>
<td>50%; 50%</td>
</tr>
<tr>
<td>$a_i$</td>
<td>Area of the drain in Tank $i$</td>
<td>2,3 cm²</td>
</tr>
<tr>
<td>$A_i$</td>
<td>Area of the Tank $i$</td>
<td>730 cm²</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>Ratio of flow in tank 1 to flow in tank 4</td>
<td>0,333</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>Ratio of flow tank 2 to flow in tank 3</td>
<td>0,307</td>
</tr>
<tr>
<td>$k_i$</td>
<td>Pump proportionality constants</td>
<td>5,51;6,58 cm³/(s%)</td>
</tr>
<tr>
<td>$g$</td>
<td>The acceleration of gravity</td>
<td>981 cm/s²</td>
</tr>
</tbody>
</table>

### 4.2.2 Interconnected T-S models

The nonlinear model of the quadruple tank system is divided into two dynamics

\begin{align}
S_1: \dot{h}_1 &= -\frac{a_1}{A_1} \sqrt{2gh_1} + \frac{a_1}{A_1} \sqrt{2gh_0} + \frac{\gamma_1 k_1}{A_1} v_i \\
\dot{h}_2 &= -\frac{a_2}{A_2} \sqrt{2gh_2} + \frac{a_2}{A_2} \sqrt{2gh_1} + \frac{\gamma_2 k_2}{A_2} v_2 \\
\dot{h}_4 &= -\frac{a_4}{A_4} \sqrt{2gh_4} + \frac{a_4}{A_4} \sqrt{2gh_3} + \frac{(1-\gamma_2) k_2}{A_2} v_2 \\
\dot{h}_4 &= -\frac{a_4}{A_4} \sqrt{2gh_4} + \frac{(1-\gamma_1) k_1}{A_1} v_i
\end{align}
(4.15)

A quasi-LPV formulation is proposed for the sub-systems (4.15) and (4.16)

\begin{align}
S_1:\begin{bmatrix}
\dot{h}_1 \\
\dot{h}_2 \\
y_1
\end{bmatrix} = \begin{bmatrix}
\frac{a_1}{A_1} \sqrt{2gh_1} & \frac{a_1}{A_1} \sqrt{2gh_0} & \frac{\gamma_1 k_1}{A_1} \\
0 & \frac{a_2}{A_2} \sqrt{2gh_2} & \frac{\gamma_2 k_2}{A_2} \\
0 & 0 & \frac{(1-\gamma_1) k_1}{A_1}
\end{bmatrix} \begin{bmatrix}
h_1 \\
h_2 \\
v_i
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\gamma_1 k_1
\end{bmatrix} \begin{bmatrix}
h_1 \\
h_2 \\
v_i
\end{bmatrix}
\end{align}
(4.17)
The LMIs constraints (3.1) are applied to the interconnected TS systems (4.19)-(4.20). State estimation errors are

The premise variables used in the TS representation of each sub-system are given by

\[
\begin{bmatrix}
    h_2 \\
    h_3
\end{bmatrix} = \begin{bmatrix}
    \frac{a_2}{A_2} \sqrt{2gh_1} \\
    \frac{a_2}{A_2} \sqrt{2gh_1} \\
    0
\end{bmatrix} \\begin{bmatrix}
    h_2 \\
    h_3
\end{bmatrix} + \begin{bmatrix}
    0 \\
    0
\end{bmatrix} \begin{bmatrix}
    \frac{a_4}{A_4} \sqrt{2gh_1} \\
    \frac{a_4}{A_4} \sqrt{2gh_1} \\
    0
\end{bmatrix} \begin{bmatrix}
    h_2 \\
    h_3
\end{bmatrix} + \begin{bmatrix}
    \frac{\gamma_2 k_2}{A_2} \frac{1}{(1-\gamma_2)k_2} v_2
\end{bmatrix}
\]

(4.18)

\[
y_2 = \begin{bmatrix}
    1 \\
    0
\end{bmatrix} \begin{bmatrix}
    h_2 \\
    h_3
\end{bmatrix}
\]

The premise variables used in the TS representation of each sub-system are given by

\[
z_1 = \frac{\sqrt{2gh_1}}{h_1 + h_2}, \quad z_2 = \frac{\sqrt{2gh_1}}{h_1}, \quad z_3 = \frac{\sqrt{2gh_1}}{h_3}, \quad z_4 = \frac{\sqrt{2gh_2}}{h_2 + h_3}
\]

Thus, the four-tank system can be represented by two interconnected TS models

\[
\begin{align*}
\text{TS}_1: & \quad \dot{x}_1(t) = \sum_{j=1}^{8} h_{j1}^2 (x_j) \left( A_{j1}^1 x_1(t) + B_{j1}^1 v_1(t) + D_{j1}^1 x_j(t) \right) \\
& \quad y_1(t) = C_1 x_1(t) \\
\text{TS}_2: & \quad \dot{x}_2(t) = \sum_{j=1}^{8} h_{j2}^2 (x_j) \left( A_{j2}^1 x_2(t) + B_{j2}^1 v_2(t) + D_{j2}^1 x_j(t) \right) \\
& \quad y_2(t) = C_2 x_2(t)
\end{align*}
\]

(4.19)

(4.20)

where

\[
A_1^1 = A_2^1 = \begin{bmatrix}
    \frac{a_1}{A_1} z_{1,1} & -\frac{a_1}{A_1} z_{1,1} \\
    0 & -\frac{a_4}{A_4} z_{2,1}
\end{bmatrix}, \quad A_1^2 = \begin{bmatrix}
    \frac{a_1}{A_1} z_{1,1} & -\frac{a_1}{A_1} z_{1,1} \\
    0 & -\frac{a_4}{A_4} z_{2,1}
\end{bmatrix}, \quad A_1^3 = \begin{bmatrix}
    \frac{a_1}{A_1} z_{1,1} & -\frac{a_1}{A_1} z_{1,1} \\
    0 & -\frac{a_4}{A_4} z_{2,1}
\end{bmatrix}, \quad A_1^4 = \begin{bmatrix}
    \frac{a_1}{A_1} z_{1,1} & -\frac{a_1}{A_1} z_{1,1} \\
    0 & -\frac{a_4}{A_4} z_{2,1}
\end{bmatrix},
\]

\[
A_1^5 = A_1^6 = \begin{bmatrix}
    \frac{a_1}{A_1} z_{1,2} & -\frac{a_1}{A_1} z_{1,2} \\
    0 & -\frac{a_4}{A_4} z_{2,2}
\end{bmatrix}, \quad B_1^1 = \begin{bmatrix}
    \frac{\gamma_1 k_1}{A_1} \\
    \frac{1}{1-\gamma_1} k_1
\end{bmatrix}, \quad D_1^1 = D_1^2 = D_1^3 = D_1^4 = \begin{bmatrix}
    0 & 0 \\
    0 & 0
\end{bmatrix},
\]

\[
A_1^7 = A_1^8 = \begin{bmatrix}
    \frac{a_2}{A_2} z_{4,1} & -\frac{a_2}{A_2} z_{4,1} \\
    0 & -\frac{a_3}{A_3} z_{3,1}
\end{bmatrix}, \quad A_1^9 = \begin{bmatrix}
    \frac{a_2}{A_2} z_{4,1} & -\frac{a_2}{A_2} z_{4,1} \\
    0 & -\frac{a_3}{A_3} z_{3,1}
\end{bmatrix}, \quad A_1^{10} = \begin{bmatrix}
    \frac{a_2}{A_2} z_{4,1} & -\frac{a_2}{A_2} z_{4,1} \\
    0 & -\frac{a_3}{A_3} z_{3,1}
\end{bmatrix}, \quad A_1^{11} = \begin{bmatrix}
    \frac{a_2}{A_2} z_{4,1} & -\frac{a_2}{A_2} z_{4,1} \\
    0 & -\frac{a_3}{A_3} z_{3,1}
\end{bmatrix},
\]

\[
B_1^2 = \begin{bmatrix}
    \frac{\gamma_2 k_2}{A_2} \frac{1}{(1-\gamma_2)k_2} \\
    \frac{1}{1-\gamma_2} k_2
\end{bmatrix}, \quad D_1^5 = D_1^6 = D_1^7 = D_1^8 = \begin{bmatrix}
    0 & 0 \\
    0 & 0
\end{bmatrix},
\]

4.2.3 Observer design

- Test of Theorem 1 (asymptotic stability)

The LMIs constraints (3.1) are applied to the interconnected TS systems (4.19)-(4.20). State estimation errors are shown in fig. 19.
Test of Theorem 2 (D-stability)

State estimation errors and eigenvalues distribution of local observers are presented in fig.20 and fig.21 for $\beta=0.05$, $\alpha=0$, $\theta=3\pi/11$, $s=0.095$ and $q=0.14$. The values of attenuation levels are $\gamma_1=1.0192\times10^3$ and $\gamma_2=202.5059$. 
• **Test of Theorem 3 (Finsler’s lemma based $\mathcal{D}$-stability)**

We apply Theorem 3 to four-tank process via the constraints (3.52). Simulation results are depicted in fig.22 and fig.23. As for the PEMFC example, Finsler’s based algorithm presents better values of attenuation levels for $\varepsilon_i=10^{-6}$ ($\gamma_1=0.0067$ and $\gamma_2=0.0059$).

![Fig.22. Four-tank system state estimation errors (Theorem 3)](image)

![Fig.23. Distribution of local observers eigenvalues (Theorem 3)](image)

5 Conclusion

The problem of observer design for interconnected Takagi-Sugeno systems with Immeasurable Premise Variables has been investigated in this paper. Asymptotic and $\mathcal{D}$-stability of Luenberger-like Interconnected Multiple Observers are examined. In addition, Finsler’s lemma is introduced to reduce the conservatism of $\mathcal{D}$-stability conditions through the introduction of extra decision variables providing more flexibility to the design. Finally, two simulation examples are presented to illustrate the effectiveness of the proposed approaches.
Appendix 1. Constants and parameters of the PEMFC model

Table 2. Constants of the PEMFC model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$RT_i k_{in,out} \left( \frac{\lambda_{O_2,atm}}{1 + \lambda_{O_2,atm}} \right)$</td>
<td></td>
</tr>
<tr>
<td>$c_2$</td>
<td>$P_{sat}$</td>
<td></td>
</tr>
<tr>
<td>$c_3$</td>
<td>$\frac{RT_i}{V_{in}}$</td>
<td></td>
</tr>
<tr>
<td>$c_4$</td>
<td>$M_{O_2}$</td>
<td></td>
</tr>
<tr>
<td>$c_5$</td>
<td>$M_{N_2}$</td>
<td></td>
</tr>
<tr>
<td>$c_6$</td>
<td>$M_e P_{out}$</td>
<td></td>
</tr>
<tr>
<td>$c_7$</td>
<td>$\frac{RT_i k_{in}}{4V_{in}}$</td>
<td></td>
</tr>
<tr>
<td>$c_8$</td>
<td>$\frac{RT_i k_{in}}{M_{N_2} V_{in}} \left( 1 - \frac{\lambda_{O_2,atm}}{1 + \lambda_{O_2,atm}} \right)$</td>
<td></td>
</tr>
<tr>
<td>$c_9$</td>
<td>$\frac{\eta_p k_{in}}{J_{sp} R_{in}}$</td>
<td></td>
</tr>
<tr>
<td>$c_{10}$</td>
<td>$\frac{C_p T_{atm}}{J_{sp} R_{in}}$</td>
<td></td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>$P_{atm}$</td>
<td></td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>$\frac{y-1}{y}$</td>
<td></td>
</tr>
<tr>
<td>$c_{13}$</td>
<td>$\frac{\eta_p k_{in}}{J_{sp} R_{in}}$</td>
<td></td>
</tr>
<tr>
<td>$c_{14}$</td>
<td>$\frac{RT_i}{M_{O_2} V_{in}}$</td>
<td></td>
</tr>
<tr>
<td>$c_{15}$</td>
<td>$\frac{1}{\eta_p}$</td>
<td></td>
</tr>
<tr>
<td>$c_{16}$</td>
<td>$k_{in,out}$</td>
<td></td>
</tr>
<tr>
<td>$c_{17}$</td>
<td>$\frac{C_p A_p}{\sqrt{RT_i}} \left( \frac{2y}{y-1} \right)$</td>
<td></td>
</tr>
<tr>
<td>$c_{18}$</td>
<td>$\frac{1}{y}$</td>
<td></td>
</tr>
<tr>
<td>$c_{19}$</td>
<td>$\frac{2}{y+1} \left( \frac{y+1}{y+2} \right)$</td>
<td></td>
</tr>
<tr>
<td>$c_{20}$</td>
<td>$\frac{C_p A_p}{\sqrt{RT_i}} \left( \frac{2}{y+2} \right)$</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{atm}$</td>
<td>$\frac{M_e}{y_{O_2,atm} M_{O_2} + (1 - y_{O_2,atm}) M_{N_2}}$</td>
<td></td>
</tr>
<tr>
<td>$P_{in}$</td>
<td>$\frac{1}{\lambda_{atm}} P_{sat}$</td>
<td></td>
</tr>
<tr>
<td>$x_{O_2,atm}$</td>
<td>$\frac{y_{O_2,atm} M_{O_2} + (1 - y_{O_2,atm}) M_{N_2}}{y_{O_2,atm} M_{O_2} + (1 - y_{O_2,atm}) M_{N_2}}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Parameters of the PEMFC model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_p$</td>
<td>Compressor efficiency</td>
<td>0,8</td>
</tr>
<tr>
<td>$\eta_{in}$</td>
<td>Motor mechanical efficiency</td>
<td>0,98</td>
</tr>
<tr>
<td>$J_{sp}$</td>
<td>Compressor inertia</td>
<td>$5 \times 10^{-4}$ kg.m²²</td>
</tr>
<tr>
<td>$R_{in}$</td>
<td>Motor winding resistance</td>
<td>0,82 Òi</td>
</tr>
<tr>
<td>$k_i$</td>
<td>Motor torque constant</td>
<td>0,0153 N.m/A</td>
</tr>
<tr>
<td>$k_c$</td>
<td>Motor back-emf constant</td>
<td>0,0153 V/rad/s</td>
</tr>
<tr>
<td>$M_{air,atm}$</td>
<td>Air molar mass</td>
<td>$29 \times 10^{-3}$ kg/mol</td>
</tr>
<tr>
<td>$M_{O_2}$</td>
<td>Oxygen molar mass</td>
<td>32. $10^{-3}$ kg/mol</td>
</tr>
<tr>
<td>$M_{N_2}$</td>
<td>Nitrogen molar mass</td>
<td>$28 \times 10^{-3}$ kg/mol</td>
</tr>
<tr>
<td>$M_v$</td>
<td>Vapor molar mass</td>
<td>$18 \times 10^{-3}$ kg/mol</td>
</tr>
<tr>
<td>$y_{O_2,atm}$</td>
<td>Oxygen mole fraction</td>
<td>0,21</td>
</tr>
<tr>
<td>$V_{in}$</td>
<td>Cathode volume</td>
<td>0,01 m³</td>
</tr>
<tr>
<td>$k_{in,out}$</td>
<td>Supply manifold orifice constant</td>
<td>$0,3629 \times 10^{-3}$ kg/(Pa.s)</td>
</tr>
</tbody>
</table>

Appendix 2. Convex Polytopic Transformation (CPT)

Lemma 5 [25] Consider a continuous and bounded function $h(x(t),u(t))$ from $[x_0, x_1] \times [u_0, u_1]$ to $\mathbb{R}$, with $x_0, x_1 \in \mathbb{R}^n$, $u_0, u_1 \in \mathbb{R}^m$. There exist two functions

$$F_i : [x_0, x_1] \times [u_0, u_1] \rightarrow [0,1] (i = 1, 2),$$

with $F_i$ being the function that maps the input to its corresponding interval in the output.
\((x(t), u(t)) \rightarrow F_1(x(t), u(t))\).

with \(F_1(x(t), u(t)) + F_2(x(t), u(t)) = 1\) such as

\[
\hat{h}(x(t), u(t)) = F_1(x(t), u(t))h_1 + F_2(x(t), u(t))h_2.
\]

The functions \(F_1\) and \(F_2\) are given by

\[
F_1(x(t), u(t)) = \frac{\hat{h}(x(t), u(t)) - h_2}{h_1 - h_2}, \quad F_2(x(t), u(t)) = \frac{\hat{h}(x(t), u(t))}{h_1 - h_2}.
\]

for all \(h_1 \geq \max_{x,u}(h(x, u))\) and \(h_2 \leq \min_{x,u}(h(x, u))\). In particular, one can choose \(h_2 = \max_{x,u}(h(x, u))\), \(h_2 = \min_{x,u}(h(x, u))\).

**Appendix 3. Matrices and activation functions of sub-systems TS_1 and TS_2 of the PEMFC system**

**Table 4. Matrices and activation functions of subsystem TS_2**

<table>
<thead>
<tr>
<th>Local model</th>
<th>(A^1_{11})</th>
<th>(B^1_{11})</th>
<th>(D^1_{11})</th>
<th>(h^1_{11}(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-c_1 - c_4 - z_{1,1} - c_2 z_{1,2} z_{1,1})</td>
<td>(-c_7)</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{11, F_{21}})</td>
</tr>
<tr>
<td>2</td>
<td>(-c_1 - c_4 - z_{1,1} - c_2 z_{1,2} z_{1,1})</td>
<td>(-c_7)</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{11, F_{21}})</td>
</tr>
<tr>
<td>3</td>
<td>(-c_1 - c_4 - z_{1,1} - c_2 z_{1,2} z_{1,1})</td>
<td>(-c_7)</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{11, F_{21}})</td>
</tr>
<tr>
<td>4</td>
<td>(-c_1 - c_4 - z_{1,1} - c_2 z_{1,2} z_{1,1})</td>
<td>(-c_7)</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{11, F_{21}})</td>
</tr>
</tbody>
</table>

**Table 5. Matrices and activation functions of subsystem TS_2**

<table>
<thead>
<tr>
<th>Local model</th>
<th>(A^2_{12})</th>
<th>(B^2_{12})</th>
<th>(D^2_{12})</th>
<th>(h^2_{12}(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
<tr>
<td>2</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
<tr>
<td>3</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
<tr>
<td>4</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
<tr>
<td>5</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
<tr>
<td>6</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
<tr>
<td>7</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
<tr>
<td>8</td>
<td>(-c_9 - c_{10} z_{2,3} z_{1,2} z_{1,3})</td>
<td>(c_{13})</td>
<td>(0, c_1 + c_4)</td>
<td>(F_{31, F_{43}, F_{3,5}})</td>
</tr>
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Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Availability of data and material
Not available.

Competing interests
The authors declare that they have no competing interests.

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Authors’ contributions
Mrs Lamia OUHIB developed the theoretical formalism, performed the analytic calculations and performed the numerical simulations. Both Mrs Lamia OUHIB and Pr Redouane KARA contributed to the final version of the manuscript. Pr Redouane KARA supervised the project.

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References