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ON THE EVOLUTION EQUATION WITH A DYNAMIC HARDY-TYPE POTENTIAL

JANN-LONG CHERN, GYEONGHA HWANG, JIN TAKAHASHI AND EIJI YANAGIDA

Abstract. Motivated by the celebrated paper of Baras and Goldstein (1984), we study the heat equation with a dynamic Hardy-type singular potential. In particular, we are interested in the case where the singular point moves in time. Under appropriate conditions on the potential and initial value, we show the existence, non-existence and uniqueness of solutions, and obtain a sharp lower and upper bound near the singular point. Proofs are given by using solutions of the radial heat equation, some precise estimates for an equivalent integral equation and the comparison principle.

1. Introduction and main results

We consider positive solutions of an evolution equation of the form

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + V(x,t)u(x,t), & x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \setminus \{\xi(0)\}, \end{cases} \tag{1.1}$$

where $N \geq 3$ and the singular point $\xi : [0, \infty) \to \mathbb{R}^N$ is a given Hölder-continuous function. We assume that the potential $V(x,t)$ is continuous in $x \in \mathbb{R}^N \setminus \{\xi(t)\}$ and $t \in [0, \infty)$, and is unbounded as $x \to \xi(t)$. The initial value $u_0(x)$ is continuous, nonnegative and nontrivial for $x \neq \xi(0)$ and is bounded for $|x - \xi(0)| > 1$. By a solution of (1.1), we mean a function $u = u(x,t)$ that satisfies (1.1) in the classical sense and is bounded for $|x - \xi(t)| > 1$. The aim of this paper is to study the existence, non-existence and uniqueness of solutions, and a lower and upper bound near the singular point.

If $u_0 \in L^1_{loc}(\mathbb{R}^N)$, there exists a unique bounded solution $u = u_n$ of

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + V_n(x,t)u(x,t), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $V_n(x,t) := \min\{V(x,t), n\}$. Then by the comparison principle, $\{u_n(x,t)\}$ is a monotone increasing sequence for every (x,t). Hence if the sequence is bounded, then

$$u(x,t) := \lim_{n\to\infty} u_n(x,t)$$

exists and is bounded for $x \in \mathbb{R}^N \setminus \{\xi(t)\}$. In this case, the standard parabolic regularity implies that the limiting function $u(x,t)$ is a solution of (1.1). A solution defined in this way

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is called a minimal solution of (1.1). We note that this does not exclude the possibility that non-minimal solutions exist.

For the existence of a minimal solution of (1.1), Baras-Goldstein [2] (see also Cabré-Martel [4]) studied the case where the potential function is given by

$$V(x,t) = \frac{\lambda}{|x|^2}.$$

It was shown in [2] that the Hardy constant

$$\lambda_c := \left(\frac{N-2}{2}\right)^2 > 0.$$

is critical in the following sense:

(i) If $0 < \lambda \le \lambda_c$ and $|x|^{-\alpha_1} u_0 \in L^1_{loc}(\mathbb{R}^N)$, then (1.1) has a time-global minimal solution u. Moreover, for every $\varepsilon > 0$ and $T > 0$, there exists a constant $c > 0$ such that

$$u(x,t) \ge c|x|^{-\alpha_1}, \qquad 0 < |x| < 1,\ \varepsilon \le t \le T.$$

(ii) If $\lambda > \lambda_c$, then (1.1) is not well-posed, that is, there exist no positive solutions.

We note that if $\lambda < \lambda_c$, the quadratic equation

$$\alpha^2 - (N-2)\alpha + \lambda = 0$$

has two positive roots given by

$$\alpha_1 = \alpha_1(\lambda) := \sqrt{\lambda_c} - \sqrt{\lambda_c - \lambda}, \quad \alpha_2 = \alpha_2(\lambda) := \sqrt{\lambda_c} + \sqrt{\lambda_c - \lambda},$$

which satisfy

$$0 < \alpha_1(\lambda) < \frac{N-2}{2} < \alpha_2(\lambda) < N-2.$$

If $\lambda = \lambda_c$, then the equation has a double root $\alpha_1(\lambda_c) = \alpha_2(\lambda_c) = \sqrt{\lambda_c}$.

The paper [2] has attracted much attention, and many results have been obtained later for equations with singular potentials [4, 9, 10, 12, 15, 16, 23, 24]. However, no results have been obtained when the position of the singularity depends on time. In this paper, we consider the case where both the position and strength of the singularity depend on time. See [7, 14, 20, 22] for related works on such dynamic singularities.

For $r, R \in \mathbb{R}$ with $0 \le r < R \le \infty$, an interval $I \subset \mathbb{R}$ and $x_0 \in \mathbb{R}^N$, we introduce the following notation:

$$\begin{aligned} Q_{(r,R),I} &:= \left\{(x,t) \in \mathbb{R}^{N+1}; r < |x - \xi(t)| < R,\ t \in I\right\}, \\ A_R(x_0) &:= \{x \in \mathbb{R}^N; 0 < |x - x_0| < R\}, \\ B_R(x_0) &:= \{x \in \mathbb{R}^N; |x - x_0| < R\}. \end{aligned} \tag{1.3}$$

Throughout this paper, we assume that for $T > 0$ fixed, the following conditions hold for ξ, V and u_0:

(A1) There exist constants $\gamma \in (1/2, 1]$ and $C_\xi > 0$ such that $|\xi(t) - \xi(s)| \le C_\xi |t - s|^\gamma$ for all $t, s \in [0,T]$.

(A2) V is nonnegative and continuous on $Q_{(0,\infty),[0,T]}$, and is bounded on $Q_{(1,\infty),[0,T]}$.

(A3) u_0 is nonnegative, nontrivial and continuous on $\mathbb{R}^N \setminus \{\xi(0)\}$, and is bounded on $\mathbb{R}^N \setminus B_1(\xi(0))$.

Note that $V(x,t)$ may not be differentiable. In fact, we shall consider potentials such as $V(x,t) = \lambda|x-\xi(t)|^{-2}$, where $\xi(t)$ is not smooth in t.

First, we give a sufficient condition for the existence of a solution and its upper bound.

Theorem 1.1. *Assume that V satisfies*

$$0 \le V(x,t) \le \lambda|x-\xi(t)|^{-2}, \qquad (x,t) \in Q_{(0,R),[0,T]}$$

with some $\lambda \in (0,\lambda_c)$ and $R > 0$. If

$$0 \le u_0(x) \le C_1|x-\xi(0)|^{-k}, \qquad x \in A_R(\xi(0))$$

with some $k \in (0, \alpha_2(\lambda)+2)$ and $C_1 > 0$, then (1.1) *has a solution satisfying*

$$u(x,t) \le C_2|x-\xi(t)|^{-\alpha_1(\lambda)-\varepsilon}, \qquad (x,t) \in Q_{(0,R),[\tau,T]},$$

where $\varepsilon > 0$ and $\tau \in (0,T)$ are arbitrary and $C_2 = C_2(\varepsilon,\tau) > 0$ is a constant.

This result with $\varepsilon = 0$ was proved by Baras-Goldstein [2, Theorem 2.2 (i)] when the potential V does not depend on t.

Next, we give a lower bound of solutions.

Theorem 1.2. *Assume that V satisfies*

$$V(x,t) \ge \lambda|x-\xi(t)|^{-2}, \qquad (x,t) \in Q_{(0,R),[0,T]}$$

with some $\lambda \in (0,\lambda_c)$ and $R > 0$. Then any solution of (1.1) *satisfies*

$$u(x,t) \ge C|x-\xi(t)|^{-\alpha_1(\lambda)+\varepsilon}, \qquad (x,t) \in Q_{(0,R),[\tau,T]},$$

where $\varepsilon > 0$ and $\tau \in (0,T)$ are arbitrary and $C = C(\varepsilon,\tau) > 0$ is a constant.

We remark that when $\xi(t)$ does not move, the lower bound was obtained in [2, Theorem 2.1 (i)] by using the method of [17, 18] (see also [5] for a related result). In this paper, we shall give a much simpler proof.

Next, we show that any two solutions of (1.1) coincide with each other if their difference is small in some sense.

Theorem 1.3. *Assume that V satisfies*

$$0 \le V(x,t) \le \lambda|x-\xi(t)|^{-2}, \qquad (x,t) \in Q_{(0,R),[0,T]}$$

with some $\lambda \in (0,\lambda_c)$ and $R > 0$. If u_1 and u_2 are solutions of (1.1) *with the same initial value such that*

$$|u_1(x,t) - u_2(x,t)| \le C|x-\xi(t)|^{-\alpha_2(\lambda)+\varepsilon}, \qquad (x,t) \in Q_{(0,R),[0,T]}$$

with some $C > 0$ and $\varepsilon \in (0, \alpha_2(\lambda))$, then $u_1 \equiv u_2$ in $Q_{(0,\infty),[0,T]}$.

In particular, this theorem implies that the solution given in Theorem 1.1 is unique, and hence it must be a minimal solution. We remark that we consider solutions which are bounded outside a neighborhood of the singular point. In the context of uniqueness, we can relax this assumption for an exponential growth (see Theorem 5.1).

The following two theorems show that the conditions in Theorem 1.1 are sharp for the existence of a solution.

Theorem 1.4. *Assume that V satisfies*

$$V(x,t) \ge \lambda|x-\xi(t)|^{-2}, \qquad (x,t) \in Q_{(0,R),[0,T]}$$

with some $\lambda \in (0,\lambda_c)$ and $R > 0$. If

$$u_0(x) \ge C|x-\xi(0)|^{-k}, \qquad x \in A_R(\xi(0))$$

with some $k > \alpha_2(\lambda) + 2$ *and* $C > 0$*, then* (1.1) *has no solution.*

Theorem 1.5. *Assume that* V *satisfies*

$$V(x,t) \geq \lambda |x - \xi(t)|^{-2}, \qquad (x,t) \in Q_{(0,R),[0,\tau]}$$

with some $\lambda > \lambda_c$*,* $R > 0$ *and* $\tau \in (0,T)$*. Then* (1.1) *has no solution for any* u_0*.*

We remark that Theorems 1.4 and 1.5 were proved in [2] when $\xi(t)$ does not move. In this paper, we prove these results in a totally different manner.

Proofs of the above theorems are almost self-contained, and are based on new ideas using some particular solutions of the radial heat equation, precise estimate of the integral representation formula of solutions and comparison principle. Finally we remark that the method used in this paper is applicable only when $\gamma > 1/2$ in (A1). In fact, the above theorems no longer hold if $\gamma < 1/2$ (see [19]).

This paper is organized as follows. In Section 2, we give some preliminary lemmas concerning the radial heat equation. In Section 3, we study the existence of a minimal solution. In Section 4, we give a lower bound of solutions. In Section 5, we prove the uniqueness of a solution, namely, if the difference of two solutions are small, then they coincide with each other. In Sections 6, we show the non-existence for large initial value. In Section 7, we discuss the supercritical case.

2. Radial heat equation

We start from a simple equation

$$u_t = \Delta u + \frac{\lambda}{|x|^2} u, \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

If the initial value is radially symmetric, then this equation is reduced to

$$v_t = v_{rr} + \frac{N-1}{r} v_r + \frac{\lambda}{r^2} v, \qquad r > 0. \tag{2.1}$$

When $0 < \lambda < \lambda_c$, we set $v(r,t) = r^{-\alpha_1(\lambda)} w(r,t)$ to obtain the radial heat equation

$$w_t = w_{rr} + \frac{d-1}{r} w_r, \qquad r > 0, \tag{2.2}$$

where $d = N - 2\alpha_1(\lambda) > 2$ corresponds to the spatial dimension. See [1, 3, 11, 13] for the analysis of the radial heat equation.

We first consider (2.2) with a nonnegative and nontrivial initial value $w_0(r)$,

Lemma 2.1. *If* $d > 2$*, then any nonnegative and nontrivial solution* w *of* (2.2) *satisfies* $w(0,t) > 0$ *for* $t > 0$*.*

Proof. Let $\tilde{w}_0 \in C_0^\infty((0,\infty))$ satisfy $0 \leq \tilde{w}_0 \leq w(\cdot,0)$ and $\tilde{w}_0 \not\equiv 0$. Define

$$\tilde{w}(r,t) = \int_0^\infty G(r,s,t)\tilde{w}_0(s)ds,$$

where G is the fundamental solution of (2.2) with $w_r(+0,t) = 0$ (see [1, 3, 11]). Then $\tilde{w}(0,t) > 0$ for $t > 0$. For $\varepsilon > 0$, we define a subsolution of (2.2) by

$$w_\varepsilon^-(r,t) = \max\{\tilde{w}(r,t) - \varepsilon r^{-d+2}, 0\}.$$

Then by comparison, any nonnegative and nontrivial solution w of (2.2) satisfies $w(r,t) \geq w_\varepsilon^-(r,t)$ for $r > 0$ and $t > 0$. Taking the limit as $\varepsilon \downarrow 0$, we obtain $w(r,t) \geq \tilde{w}(r,t)$. Since $\tilde{w}(0,t) > 0$, the proof is complete. ∎

Next, let us consider forward self-similar solutions of (2.2) of the form

$$w(r,t) = \frac{1}{\sigma(l)t^{l/2}}\varphi(\rho), \qquad \rho = \frac{r}{t^{1/2}}, \tag{2.3}$$

where $l > 0$ is a constant and $\sigma(l)$ is given in Lemma 2.2 (iii) below. Substituting this in (2.2), we see that φ must satisfy

$$\varphi_{\rho\rho} + \frac{d-1}{\rho}\varphi_\rho + \frac{\rho}{2}\varphi_\rho + \frac{l}{2}\varphi = 0, \qquad \rho > 0. \tag{2.4}$$

Lemma 2.2. *For $d > 2$ and $l > 0$, there exists a unique solution of* (2.4) *subject to the initial condition $\varphi(0) = 1$ and $\varphi_\rho(0) = 0$, and the solution has the following properties:*

(i) *If $0 < l < d$, then $\varphi(\rho) > 0$ and $\varphi_\rho(\rho) < 0$ for all $\rho > 0$.*
(ii) *If $l = d$, then $\varphi(\rho) = e^{-\rho^2/4}$.*
(iii) *If $0 < l < d$, there exists a constant $\sigma(l) > 0$ such that $\varphi(\rho) = \sigma(l)\rho^{-l} + o(\rho^{-l})$ as $\rho \to \infty$.*
(iv) *$\sigma(l) \downarrow 0$ as $l \uparrow d$.*

Proof. The existence and uniqueness of a solution of (2.4) with $\varphi(0) = 1$ and $\varphi_\rho(0) = 0$ can be proved in the same manner as [6, Lemma 3.1].

(i) Suppose that

$$z := \inf\{\rho > 0; \varphi(\rho) \le 0\} < \infty.$$

Then $\varphi(z) = 0$ and $\varphi_\rho(z) \le 0$. We rewrite (2.4) as

$$(\rho^{d-1}\varphi_\rho)_\rho + \frac{\rho^{d-l}}{2}(\rho^l\varphi)_\rho = 0, \tag{2.5}$$

and integrate this over $[0, z]$ to obtain

$$0 \ge z^{d-1}\varphi_\rho(z) = -\frac{1}{2}\int_0^z \rho^{d-l}(\rho^l\varphi)_\rho d\rho = \frac{d-l}{2}\int_0^z \rho^{d-1}\varphi\, d\rho > 0.$$

This contradiction implies that $\varphi(\rho) > 0$ for all $\rho > 0$. Moreover, we see from (2.4) that φ does not attain a positive local minimum. Hence $\varphi_\rho(\rho) < 0$ for all $\rho > 0$.

(ii) This part can be verified by direct computation.

(iii) We take $\gamma \in (l, l+2)$ and $\theta > 1$ such that

$$\theta^{-2}|\gamma(\gamma+2-d)| < \frac{\gamma - l}{4}, \qquad \theta^{\gamma-l}\frac{\gamma-l}{4} - |l(l+2-d)| > 0, \tag{2.6}$$

and define a function

$$g(\rho) := \rho^{-l} - \theta^{\gamma-l}\rho^{-\gamma}.$$

Note that $g(\theta) = 0$ and $g(\rho) > 0$ for $\rho > \theta$. By direct substitution, we have

$$\begin{aligned}
g_{\rho\rho} + \frac{d-1}{\rho}g_\rho + \frac{\rho}{2}g_\rho + \frac{l}{2}g &= l(l+2-d)\rho^{-l-2} - \theta^{\gamma-l}\gamma(\gamma+2-d)\rho^{-\gamma-2} + \theta^{\gamma-l}\frac{\gamma-l}{2}\rho^{-\gamma} \\
&\ge -|l(l+2-d)|\rho^{-l-2} - \theta^{\gamma-l-2}|\gamma(\gamma+2-d)|\rho^{-\gamma} + \theta^{\gamma-l}\frac{\gamma-l}{2}\rho^{-\gamma} \\
&\ge -|l(l+2-d)|\rho^{-l-2} + \theta^{\gamma-l}\frac{\gamma-l}{4}\rho^{-\gamma} \\
&\ge \left\{\theta^{\gamma-l}\frac{\gamma-l}{4} - |l(l+2-d)|\right\}\rho^{-l-2}.
\end{aligned}$$

Hence we obtain

$$g_{\rho\rho} + \frac{d-1}{\rho} g_\rho + \frac{\rho}{2} g_\rho + \frac{l}{2} g > 0, \qquad \rho \geq \theta. \tag{2.7}$$

We rewrite (2.4) as

$$(\rho^{d-1} e^{\rho^2/4} \varphi_\rho)_\rho + \frac{l}{2} \rho^{d-1} e^{\rho^2/4} \varphi = 0.$$

Integrating this on $[0, \rho]$, we easily see that $\varphi_\rho(\rho) = O(\rho)$ as $\rho \to 0$. Multiplying this by $g(\rho)$, integrating it on (θ, ρ) by parts and using (2.7), we obtain

$$\rho^{d-1} e^{\rho^2/4} (\varphi_\rho(\rho) g(\rho) - \varphi(\rho) g_\rho(\rho)) + \theta^{d-1} e^{\theta^2/4} \varphi(\theta) g_\rho(\theta) < 0.$$

Since $\varphi(\theta) > 0$ and $g_\rho(\theta) > 0$, we obtain $\varphi_\rho g - \varphi g_\rho < 0$ for $\rho > \theta$. This implies that $\varphi(\rho)/g(\rho)$ is decreasing in $\rho \in (\theta, \infty)$.

On the other hand, using $g(\rho) \leq \rho^{-l}$, multiplying (2.5) by ρ^{l-d}, integrating it on $(0, \tilde{\rho})$ for $\tilde{\rho} > \theta$ and applying (i), we obtain

$$\begin{aligned} \frac{\varphi(\tilde{\rho})}{2g(\tilde{\rho})} \geq \frac{1}{2} \tilde{\rho}^l \varphi(\tilde{\rho}) &= -\int_0^{\tilde{\rho}} \rho^{l-d} (\rho^{d-1} \varphi_\rho)_\rho d\rho \\ &= -(d-l) \int_0^{\tilde{\rho}} \rho^{l-2} \varphi_\rho d\rho - \left[\rho^{l-1} \varphi_\rho\right]_0^{\tilde{\rho}} \\ &\geq -(d-l) \int_0^{\theta} \rho^{l-2} \varphi_\rho d\rho > 0. \end{aligned}$$

(Here $\rho^{l-1}\varphi_\rho \to 0$ as $\rho \to 0$ and $\rho^{l-2}\varphi_\rho$ is integrable on $[0, \tilde{\rho}]$, because $l > 0$ and $\varphi_\rho(\rho) = O(\rho)$ as $\rho \to 0$.) Hence $\varphi(\rho)/g(\rho)$ is bounded below on (θ, ∞), so that $\varphi(\rho)/g(\rho)$ converges to a positive constant as $\rho \to \infty$.

(iv) Assume that $d - \delta < l < d$ for some small $\delta > 0$. Then we can take θ and γ in (2.6) to be independent of l. For any $\varepsilon > 0$, we take $\rho_0 > \theta$ such that

$$e^{-\rho_0^2/4} \leq \varepsilon g(\rho_0).$$

Then by (ii) and the continuity of φ with respect to l, there exists $\delta > 0$ such that $\varphi(\rho_0) \leq 2\varepsilon g(\rho_0)$ for $l \in (d - \delta, d)$. Since $\varphi(\rho)/g(\rho)$ is decreasing, we obtain $\sigma(l) \leq 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the proof is complete. ☐

Lemma 2.3. *If $0 < \lambda < \lambda_c$ and $0 \leq k < \alpha_2(\lambda) + 2$, then*

$$v(r,t) = v(r,t;k) = \frac{r^{-k}}{\sigma(l)} \left(\frac{r}{t^{1/2}}\right)^l \varphi\left(\frac{r}{t^{1/2}}\right),$$

satisfies (2.1), where $l = k - \alpha_1(\lambda)$ and σ and φ are given in Lemma 2.2 with $d = N - 2\alpha_1(\lambda)$. Moreover, the function v has the following properties:

(i) *$v(r,t;k) \to r^{-k}$ as $t \downarrow 0$ uniformly on (δ, ∞), where $\delta > 0$ is arbitrary.*
(ii) *$v(r,t;k) \to \infty$ as $k \uparrow \alpha_2(\lambda) + 2$ uniformly on any compact subset of $(0,\infty) \times (0,\infty)$.*

Proof. (i) If $w(r,t)$ satisfies (2.2), then $v(r,t) = r^{-\alpha_1(\lambda)} w(r,t)$ satisfies (2.1). By $\lambda < \lambda_c$, we have $d > 2$ and $l < d$. Hence (2.2) has a positive solution given by (2.3). On the other hand, by Lemma 2.2 (iii), we have

$$v(r,t;k) = \frac{r^{-k}}{\sigma(l)} \left(\frac{r}{t^{1/2}}\right)^l \varphi\left(\frac{r}{t^{1/2}}\right) \to r^{-k} \text{ as } t \downarrow 0.$$

(ii) We remark that $l \uparrow d$ is equivalent to $k \uparrow N-\alpha_1(\lambda)=\alpha_2(\lambda)+2$. Then by Lemma 2.2 (iv), we have $\sigma(l)\to 0$ and $\varphi(\rho)\to e^{-\rho^2/4}$ as $k\uparrow \alpha_2(\lambda)+2$. □

3. Existence in the subcritical case

In this section, we prove Theorem 1.1 by using a uniform upper estimate of the solution u_n for the cut-off problem. We use solutions of a simple equation $v_t=\Delta v+\lambda'|x|^{-2}v$ $(0<\lambda'<\lambda_c)$.

Proof of Theorem 1.1. For $n\geq 1$, set $V_n:=\min\{V,n\}$. Let u_n be a unique solution of the approximate problem (1.2). Note that u_n satisfies

$$u_n(x,t)=\int_{\mathbb{R}^N}G(x-y,t)u_0(y)dy+\int_0^t\int_{\mathbb{R}^N}G(x-y,t-s)V_n(y,s)u_n(y,s)dyds. \tag{3.1}$$

For a while, we fix n, and set $\tilde{u}(x,t):=u_n(x+\xi(t),t)$ and $\tilde{V}_n(x,t):=V_n(x+\xi(t),t)$. Then $\tilde{u}$ satisfies $\tilde{u}(x,t)=\Phi[\tilde{u}](x,t)$, where

$$\begin{aligned}\Phi[\tilde{u}](x,t):=&\int_{\mathbb{R}^N}G(x-y+\xi(t)-\xi(0),t)u_0(y+\xi(0))dy\\&+\int_0^t\int_{\mathbb{R}^N}G(x-y+\xi(t)-\xi(s),t-s)\tilde{V}_n(y,s)\tilde{u}(y,s)dyds.\end{aligned} \tag{3.2}$$

Let $\varepsilon>0$ be arbitrarily given. For $k<\alpha_2(\lambda)+2$, we take constants $\delta=\delta(\varepsilon)$ and $0<T_\delta\leq T$ independent of n such that

$$\begin{aligned}&c_0\lambda<\lambda_c,\qquad k<\alpha_2(c_0\lambda)+2,\qquad \alpha_1(c_0\lambda)<\alpha_1(\lambda)+\varepsilon,\\&c_0:=(1-\delta)^{-\frac{N}{2}}\exp\left(\frac{(1-\delta)C_\xi^2}{4\delta}T_\delta^{2\gamma-1}\right)>1,\end{aligned}$$

where C_ξ is a constant given by (A1).

We claim that there exist w_n and w^+ on $t\in[0,T_\delta]$ such that $w_n=\Phi[w_n]$ and $w_n\leq w^+$ for $t\in[0,T_\delta]$. Once we prove this claim, by the uniqueness of solutions of (1.2), we obtain

$$u_n(x,t)=\tilde{u}(x-\xi(t),t)=w_n(x-\xi(t),t)\leq w^+(x-\xi(t),t),\qquad (x,t)\in Q_{(0,\infty),[0,T_\delta]}, \tag{3.3}$$

where $Q_{(0,\infty),[0,T_\delta]}$ is given in (1.3). We prove the claim by using solutions of a simple problem

$$\begin{cases}v_t=\Delta v+\dfrac{c_0\lambda}{|x|^2}v, & (x,t)\in A_\infty(0)\times(0,\infty),\\ v(x,0)=|x|^{-k}, & x\in A_\infty(0),\end{cases}$$

for which we denote a solution by $v=v(x,t;k)$. Note that the existence of a solution v follows from $k<\alpha_2(c_0\lambda)+2$ and Lemma 2.3.

By (A2), (A3) and the assumption on V, there exists a constant $M>0$ independent of n such that

$$\begin{aligned}&\tilde{V}_n(x,t)\leq\lambda|x|^{-2}+M, && (x,t)\in A_\infty(0)\times[0,T],\\&u_0(x+\xi(0))\leq M(|x|^{-k}+1), && x\in A_\infty(0).\end{aligned} \tag{3.4}$$

Define

$$w^+(x,t):=c_0Me^{c_0Mt}\{v((1-\delta)^{1/2}x,t;k)+v((1-\delta)^{1/2}x,t;0)\}.$$

We will see that

$$\Phi[w^+](x,t)\leq w^+(x,t),\qquad (x,t)\in A_\infty(0)\times[0,T_\delta]. \tag{3.5}$$

Direct computations show that w^+ is a solution of the following problem:

$$\begin{cases} w_t^+ = \dfrac{1}{1-\delta}\Delta w^+ + \dfrac{c_0\lambda}{(1-\delta)|x|^2}w^+ + c_0 M w^+, & (x,t)\in A_\infty(0)\times(0,\infty),\\ w^+(x,0) = c_0M((1-\delta)^{-\frac{k}{2}}|x|^{-k}+1), & x\in A_\infty(0), \end{cases}$$

which is equivalent to

$$\begin{aligned} w^+(x,t) = &\int_{\mathbb{R}^N} G\left(x-y, \frac{t}{1-\delta}\right) c_0M((1-\delta)^{-\frac{k}{2}}|y|^{-k}+1)dy \\ &+ \int_0^t\int_{\mathbb{R}^N} G\left(x-y,\frac{t-s}{1-\delta}\right)\left\{\frac{c_0\lambda}{(1-\delta)|y|^2}+c_0M\right\} w^+(y,s)dyds. \end{aligned}$$

By (A1), for $0\le s<t\le T_\delta$, we have

$$\begin{aligned} |x-y+\xi(t)-\xi(s)|^2 &\ge (1-\delta)|x-y|^2 - \frac{1-\delta}{\delta}|\xi(t)-\xi(s)|^2 \\ &\ge (1-\delta)|x-y|^2 - \frac{(1-\delta)C_\xi^2}{\delta}(t-s)^{2\gamma}, \\ G(x-y+\xi(t)-\xi(s),t-s) &\le c_0 G\left(x-y,\frac{t-s}{1-\delta}\right). \end{aligned} \tag{3.6}$$

These computations together with (3.2) and (3.4) yield

$$\begin{aligned} \Phi[w^+](x,t) &\le \int_{\mathbb{R}^N} c_0 G\left(x-y,\frac{t}{1-\delta}\right) u_0(y+\xi(0))dy \\ &\quad + \int_0^t\int_{\mathbb{R}^N} c_0G\left(x-y,\frac{t-s}{1-\delta}\right)\tilde{V}(y,s)w^+(y,s)dyds \\ &\le \int_{\mathbb{R}^N} G\left(x-y,\frac{t}{1-\delta}\right) c_0M(|y|^{-k}+1)dy \\ &\quad + \int_0^t\int_{\mathbb{R}^N} G\left(x-y,\frac{t-s}{1-\delta}\right)\left(\frac{c_0\lambda}{|y|^2}+c_0M\right) w^+(y,s)dyds \\ &\le w^+(x,t) \end{aligned}$$

for $t\in(0,T_\delta]$. This proves (3.5).

We inductively define a sequence $\{w_i\}$ by

$$w_0(x,t) := \int_{\mathbb{R}^N} G(x-y+\xi(t)-\xi(0),t)u_0(y+\xi(0))dy$$

and $w_i := \Phi[w_{i-1}]$ $(i\ge 1)$. By (3.2), (3.4) and (3.6), we have $0\le w_0 = \Phi[0]\le w^+$, and so $\Phi[0]\le\Phi[w_0]\le\Phi[w^+]$. Then $\Phi[w^+]\le w^+$ implies $0\le w_0\le w_1\le w^+$. Similarly, we have

$$0\le\Phi[0]\le\Phi[w_0]=w_1\le\Phi[w_1]=w_2\le\Phi[w^+]\le w^+.$$

Repeating this procedure yields

$$0\le w_1(x,t)\le w_2(x,t)\le\cdots\le w_i(x,t)\le\cdots\le w^+(x,t)<\infty$$

for $(x,t)\in Q_{(0,\infty),[0,T_\delta]}$, and the limit $w_n(x,t):=\lim_{i\to\infty}w_i(x,t)$ exists for each $(x,t)\in Q_{(0,\infty),[0,T_\delta]}$. Then w_n is the desired function stated in the claim, and hence (3.3) follows.

Since w^+ is independent of n, we see from (3.3) that the limit $u(x,t):=\lim_{n\to\infty}u_n(x,t)$ exists and satisfies $u(x,t)\le w^+(x-\xi(t),t)$ for each $(x,t)\in Q_{(0,\infty),[0,T_\delta]}$. In addition, the standard regularity theory for linear parabolic equations implies that the limiting function u satisfies (1.1) on $[0, T_\delta]$.

Let $\tau \in (0, T_\delta)$ and let $\sigma(\cdot; k)$ and $\varphi(\cdot; k)$ be as in Lemma 2.2 with $l = k - \alpha_1(c_0\lambda)$ and $d = N - 2\alpha_1(c_0\lambda)$. Then by Lemma 2.3, there exists a constant $C = C(\delta, \tau) > 0$ such that

$$\begin{aligned} w^+(x,t) &= c_0 M e^{c_0 M t} \left[\frac{\{(1-\delta)^{\frac{1}{2}}|x|\}^{-\alpha_1(c_0\lambda)}}{\sigma(k-\alpha_1(c_0\lambda); k)} t^{-\frac{k-\alpha_1(c_0\lambda)}{2}} \varphi(t^{-\frac{1}{2}}(1-\delta)^{\frac{1}{2}}|x|; k) \right. \\ &\quad \left. + \frac{\{(1-\delta)^{\frac{1}{2}}|x|\}^{-\alpha_1(c_0\lambda)}}{\sigma(-\alpha_1(c_0\lambda); 0)} t^{\frac{\alpha_1(c_0\lambda)}{2}} \varphi(t^{-\frac{1}{2}}(1-\delta)^{\frac{1}{2}}|x|; 0) \right] \\ &\le C|x|^{-\alpha_1(c_0\lambda)} \end{aligned}$$

for $(x,t) \in A_R(0) \times [\tau, T_\delta]$. Recall that $\alpha_1(c_0\lambda) < \alpha_1(\lambda) + \varepsilon$ and that δ is determined by ε. Then there exists a constant $C = C(\varepsilon, \tau) > 0$ such that

$$u(x,t) \le w^+(x - \xi(t), t) \le C|x - \xi(t)|^{-\alpha_1(\lambda)-\varepsilon}$$

for $(x,t) \in Q_{(0,R),[\tau,T_\delta]}$. Since the equation is linear and T_δ is determined by ε, we see that u has a continuation as a solution of (1.1) for $t \in [0, T]$ satisfying

$$u(x,t) \le C|x - \xi(t)|^{-\alpha_1(\lambda)-\varepsilon}, \qquad (x,t) \in Q_{(0,R),[\tau,T]},$$

where $\varepsilon > 0$ and $\tau \in (0, T)$ are arbitrary and $C = C(\varepsilon, \tau) > 0$ is a constant. □

Remark 3.1. For Theorem 1.1, we can relax (A1) as follows. Let

$$h(t) := \limsup_{s \downarrow 0} \frac{|\xi(t+s) - \xi(t)|}{s^{1/2}},$$

which is well-defined if $\xi(t)$ is at least 1/2-Hölder continuous. If $h(t)$ is sufficiently small for $t > 0$, then the proof given above holds true. If $\xi(t)$ is γ-Hölder continuous in $t \in [0, \infty)$ with some $\gamma > 1/2$, then $h(t)$ is identically equal to 0.

4. Lower bound of solutions

In this section, assuming the existence of a minimal solution of (1.1), we shall derive a lower bound of solutions by using a solution of a simple equation $v_t = \Delta v + \lambda' |x|^{-2} v$ $(0 < \lambda' < \lambda_c)$.

Proof of Theorem 1.2. Fix $\tau \in (0, T)$, $t_1 \in [\tau, T]$ and $\varepsilon > 0$. We take constants $\delta = \delta(\varepsilon) > 0$ and $T_\delta \in (0, \tau)$ independent of t_1 such that

$$\alpha_1(c_1\lambda) > \alpha_1(\lambda) - \varepsilon, \qquad c_1 := (1+\delta)^{-\frac{N}{2}} \exp\left(-\frac{(1+\delta)C_\xi^2}{4\delta} T_\delta^{2\gamma-1} \right) < 1.$$

Let u be a minimal solution of (1.1). Recall that u is obtained by $u(x,t) = \lim_{n\to\infty} u_n(x,t)$, where u_n is a unique solution of the integral equation (3.1). We set $\tilde{u}_n(x,t) := u_n(x + \xi(t), t)$. Then $\tilde{u}_n$ satisfies $\tilde{u}_n = \Phi[\tilde{u}_n]$, where Φ is defined by (3.2). From the same computations as in (3.6), it follows that

$$\begin{aligned} |x - y + \xi(t) - \xi(s)|^2 &\le (1+\delta)|x-y|^2 + \frac{(1+\delta)C_\xi^2}{\delta}(t-s)^{2\gamma}, \\ G(x - y + \xi(t) - \xi(s), t-s) &\ge c_1 G\left(x - y, \frac{t-s}{1+\delta}\right) \end{aligned}$$

for $t_0 := t_1 - T_\delta \le s < t \le t_1$. Then for $t_0 < t \le t_1$, by writing $\tilde{V}_n(y,s) := V_n(y+\xi(s),s)$, we have

$$\tilde{u}_n(x,t) \ge \tilde{v}_n(x,t) := \int_{\mathbb{R}^N} c_1 G\left(x-y, \frac{t}{1+\delta}\right) u_0(y+\xi(0))dy + \int_{t_0}^{t}\int_{\mathbb{R}^N} c_1 G\left(x-y,\frac{t-s}{1+\delta}\right)\tilde{V}_n(y,s)\tilde{u}_n(y,s)dyds.$$

Direct computations show that

$$\begin{cases} \tilde{v}_{n,t} = \dfrac{1}{1+\delta}\Delta\tilde{v}_n + c_1\tilde{V}_n\tilde{u}_n \ge \dfrac{1}{1+\delta}\Delta\tilde{v}_n + c_1\tilde{V}_n\tilde{v}_n, & (x,t)\in A_\infty(0)\times(t_0,t_1], \\ \tilde{v}_n(x,t_0) = \displaystyle\int_{\mathbb{R}^N} c_1 G\left(x-y,\frac{t_0}{1+\delta}\right)u_0(y+\xi(0))dy > 0, & x\in A_\infty(0). \end{cases}$$

Note that similar argument is used in [8]. By (A2) and the assumption on V, there exists a constant $M>0$ independent of n such that

$$\tilde{V}(x,t) \ge \lambda|x|^{-2} - M, \qquad (x,t)\in A_\infty(0)\times[0,T], \tag{4.1}$$

Hence we obtain

$$c_1\tilde{V}_n\tilde{v}_n \ge \min\left\{\frac{c_1\lambda}{(1+\delta)|x|^2} - c_1 M, c_1 n\right\}\tilde{v}_n.$$

Let $\tilde{v}_n^-$ be a unique solution of

$$\begin{cases} \tilde{v}_{n,t}^- = \dfrac{1}{1+\delta}\Delta\tilde{v}_n^- + \min\left\{\dfrac{c_1\lambda}{(1+\delta)|x|^2} - c_1M, c_1 n\right\}\tilde{v}_n^-, & x\in A_\infty(0),\ t>t_0, \\ \tilde{v}_n^-(x,t_0) = \tilde{v}_0^-(|x|), & x\in\mathbb{R}^N. \end{cases}$$

Here the initial value $\tilde{v}_0^- = \tilde{v}_0^-(|x|)$ is a positive and radially symmetric function satisfying $\tilde{v}_0^-(|x|) \le \tilde{v}_n(x,t_0)$ and $(\tilde{v}_0^-)_r(0) = 0$. We see that $\tilde{v}_n$ is a supersolution of the problem for $\tilde{v}_n^-$, and hence the comparison principle for bounded functions implies $\tilde{v}_n \ge \tilde{v}_n^-$. This together with $\tilde{u}_n \ge \tilde{v}_n$ yields $\tilde{u}_n \ge \tilde{v}_n^-$. Letting $n\to\infty$, we obtain

$$\tilde{u} \ge \tilde{v}^-. \tag{4.2}$$

Here $\tilde{u}(x,t) := u(x+\xi(t),t)$ and $\tilde{v}^-$ is a minimal solution of

$$\begin{cases} \tilde{v}_t^- = \dfrac{1}{1+\delta}\Delta\tilde{v}^- + \left\{\dfrac{c_1\lambda}{(1+\delta)|x|^2} - c_1M\right\}\tilde{v}^-, & x\in A_\infty(0),\ t>t_0, \\ \tilde{v}^-(x,t_0) = \tilde{v}_0^-(|x|), & x\in\mathbb{R}^N. \end{cases}$$

Since the initial value $\tilde{v}_0^-$ is radially symmetric, the solution $\tilde{v}_n^-$ is also radially symmetric, and so the limit $\tilde{v}^-$ is radially symmetric. Thus we can rewrite $\tilde{v}^-(x,t) = \tilde{u}^-((1+\delta)^{1/2}r, t-t_0)$ $(r=|x|)$, where $\tilde{u}^-$ is a solution of the corresponding problem

$$\begin{cases} \tilde{u}_t^- = \tilde{u}_{rr}^- + \dfrac{N-1}{r}\tilde{u}_r^- + \left(\dfrac{c_1\lambda}{r^2} - c_1M\right)\tilde{u}^-, & r>0,\ t\in(0,T_\delta], \\ \tilde{u}^-(r,0) = \tilde{v}_0^-((1+\delta)^{-\frac{1}{2}}r), & r\ge 0. \end{cases}$$

Set $w(r,t) := e^{c_1Mt}r^{\alpha_1(c_1\lambda)}\tilde{u}^-(r,t)$. From

$$(\alpha_1(c_1\lambda))^2 - (N-2)\alpha_1(c_1\lambda) + c_1\lambda = 0,$$

it follows that w satisfies

$$w_t = w_{rr} + \frac{N - 2\alpha_1(c_1\lambda) - 1}{r} w_r, \qquad r > 0, t \in (0, T_\delta].$$

By Lemma 2.1, we have $w(0,t) > 0$ for $t \in (0, T_\delta]$. Then there exists a constant $C = C(\delta, T_\delta) > 0$ independent of t_1 such that $w(r,t) \ge C$ for $(r,t) \in [0,R] \times [T_\delta/2, T_\delta]$, and hence

$$\tilde{u}^-(r,t) \ge Ce^{-c_1 M t} r^{-\alpha_1(c_1\lambda)}, \qquad (r,t) \in (0,R] \times [T_\delta/2, T_\delta].$$

This together with (4.2) and $\alpha_1(c_1\lambda) > \alpha_1(\lambda) - \varepsilon$ shows that

$$\begin{aligned} u(x,t) &= \tilde{u}(x - \xi(t), t) \ge \tilde{v}^-(x - \xi(t), t) = \tilde{u}^-((1+\delta)^{\frac{1}{2}}|x - \xi(t)|, t - t_0) \\ &\ge C \frac{e^{-c_1 M (t - t_0)}}{(1+\delta)^{\alpha_1(c_1\lambda)/2}} |x - \xi(t)|^{-\alpha_1(c_1\lambda)} \ge Ce^{-c_1 M T} |x - \xi(t)|^{-\alpha_1(c_1\lambda)} \ge C|x - \xi(t)|^{-\alpha_1(\lambda) + \varepsilon} \end{aligned}$$

for $(x,t) \in Q_{(0,R],[t_0 + T_\delta/2, t_0 + T_\delta]}$, where the above constants C may have different values within the same line. In particular, since $t_1 = t_0 + T_\delta$ and T_δ depends on τ, there exists a constant $C = C(\delta, \tau) > 0$ such that $u(x, t_1) \ge C|x - \xi(t_1)|^{-\alpha_1(\lambda)+\varepsilon}$ for $x \in A_R(\xi(t_1))$. Since $t_1 \in [\tau, T]$ is arbitrary, C is independent of t_1 and δ is determined by ε, we obtain

$$u(x,t) \ge C|x - \xi(t)|^{-\alpha_1(\lambda)+\varepsilon}, \qquad (x,t) \in Q_{(0,R),[\tau,T]},$$

where $\varepsilon > 0$ and $\tau \in (0,T)$ are arbitrary and $C = C(\varepsilon, \tau) > 0$ is a constant. $\square$

5. Uniqueness of solutions

We prove the uniqueness of solutions. The proof is based on nontrivial modifications of the method of Marchi [16, pp. 1075–1079]. In that paper, the position of the singularity does not move in time, while in our paper, the singularity moves in time and its motion may not be smooth.

Note that, in the definition of solutions of (1.1), we assume the boundedness of solutions for $|x - \xi(t)| \ge 1$. However, for uniqueness, this can be relaxed as follows, and so we prove the following result which is stronger than Theorem 1.3.

Theorem 5.1. *Assume that V satisfies*

$$0 \le V(x,t) \le \lambda |x - \xi(t)|^{-2}, \qquad (x,t) \in Q_{(0,R),[0,T]}$$

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. If u_1 and u_2 are solutions of (1.1) *with the same initial data such that*

$$|u_1(x,t) - u_2(x,t)| \le C|x - \xi(t)|^{-\alpha_2(\lambda)+\varepsilon} e^{C|x-\xi(t)|^2}, \qquad (x,t) \in Q_{(0,\infty),[0,T]}$$

with some $C > 0$ and $\varepsilon \in (0, \alpha_2(\lambda))$, then $u_1 \equiv u_2$ on $Q_{(0,\infty),[0,T]}$.

Proof. Let $\varepsilon \in (0, \alpha_2(\lambda))$. We prepare constants $\delta = \delta(\varepsilon)$ and $0 < T_\delta \le T$ similarly to the proof of Theorem 1.1,

$$c_0\lambda < \lambda_c, \qquad \alpha_1(c_0\lambda) < \alpha_1(\lambda) + \frac{\varepsilon}{2}, \qquad k := \alpha_1(\lambda) + \frac{\varepsilon}{2} < \alpha_2(c_0\lambda) + 2,$$

$$c_0 := (1-\delta)^{-\frac{N}{2}} \exp\left(\frac{(1-\delta)C_\xi^2}{4\delta} T_\delta^{2\gamma - 1} \right) > 1.$$

In addition, we take T_δ so small that

$$T_\delta^{-\frac{N}{2}} \le e^{\frac{1}{16 T_\delta}}. \tag{5.1}$$

Fix $t_0 \in (0, T_\delta]$. Since u_1 and u_2 have the same initial data, we have

$$\int_0^{t_0} \int_{\mathbb{R}^N} (u_1 - u_2)(\phi_t + \Delta\phi + V\phi) dx dt = \int_{\mathbb{R}^N} (u_1(x, t_0) - u_2(x, t_0))\phi(x, t_0) dx \tag{5.2}$$

for any $\phi \in C_0^{2,1}(\mathbb{R}^N \times [0, t_0])$. We will construct a suitable test function ϕ.

Let $d > 0$ and let $\theta \in C_0^\infty(B_d(\xi(t_0)))$ be nonnegative. We write $d_0 := \sup_{t\in[0,T]} |\xi(t)|$ and fix $R_0 := d_0 + d + 1$. For $R > 2R_0$, we consider the problem

$$\begin{cases} w_{n,t} + \Delta w_n + V_n(x, t)w_n = 0, & (x, t) \in B_R(0) \times (0, t_0), \\ w_n(x, t) = 0, & (x, t) \in \partial B_R(0) \times (0, t_0), \\ w_n(x, t_0) = \theta(x), & x \in B_R, \end{cases}$$

where $V_n := \min\{V, n\}$. By taking $\tilde{w}_n(x, \tau) = w_n(x, t)$ $(\tau = t_0 - t)$, we see that this problem is equivalent to

$$\begin{cases} \tilde{w}_{n,\tau} = \Delta \tilde{w}_n + V_n(x, t_0 - \tau)\tilde{w}_n, & (x, \tau) \in B_R(0) \times (0, t_0), \\ \tilde{w}_n(x, \tau) = 0, & (x, \tau) \in \partial B_R(0) \times (0, t_0), \\ \tilde{w}_n(x, 0) = \theta(x), & x \in B_R. \end{cases} \tag{5.3}$$

First, we estimate w_n as follows. In order to construct a supersolution of (5.3), we consider a unique solution u_n of

$$\begin{cases} u_{n,\tau} = \Delta u_n + V_n(x, t_0 - \tau)u_n, & x \in A_\infty(\xi(t_0 - \tau)),\ t \in (0, t_0], \\ u_n(x, 0) = \theta(x), & x \in \mathbb{R}^N. \end{cases}$$

Setting $\tilde{u}_n(x, \tau) := u_n(x + \xi(t_0 - \tau), \tau)$ and $\tilde{V}_n(x, \tau) := V_n(x + \xi(t_0 - \tau), t_0 - \tau)$, we have

$$\begin{aligned} \tilde{u}_n(x, \tau) = & \int_{\mathbb{R}^N} G(x - y + \xi(t_0 - \tau) - \xi(t_0), \tau)\theta(y + \xi(t_0)) dy \\ & + \int_0^\tau \int_{\mathbb{R}^N} G(x - y + \xi(t_0 - \tau) - \xi(t_0 - s), \tau - s)\tilde{V}_n(y, s)\tilde{u}_n(y, s) dy ds. \end{aligned}$$

Then by the same argument as in the proof of Theorem 1.1, we can prove that

$$\begin{aligned} & u_n(x, \tau) = \tilde{u}_n(x - \xi(t_0 - \tau), \tau) \\ & \quad \le c_0 M e^{c_0 M\tau} \frac{\{(1 - \delta)^{\frac{1}{2}} |x - \xi(t_0 - \tau)|\}^{-\alpha_1(c_0\lambda)}}{\sigma(k - \alpha_1(c_0\lambda))} \tau^{-\frac{k - \alpha_1(c_0\lambda)}{2}} \varphi\left(\tau^{-\frac{1}{2}} (1 - \delta)^{\frac{1}{2}} |x - \xi(t_0 - \tau)|\right) \end{aligned}$$

for $x \in A_\infty(\xi(t_0 - \tau))$ and $\tau \in [0, t_0]$, where σ and φ are given in Lemma 2.2 with $l = k - \alpha_1(c_0\lambda)$ and $d = N - 2\alpha_1(c_0\lambda)$, and M is a constant independent of n such that

$$\begin{aligned} \tilde{V}_n(x, \tau) \le \lambda|x|^{-2} + M, \qquad & (x, \tau) \in A_\infty(0) \times [0, T], \\ \theta(x + \xi(t_0)) \le M|x|^{-k}, \qquad & x \in A_\infty(0). \end{aligned}$$

By $l < d$ and Lemma 2.2 (iii), there exists a constant $C = C(\delta, T_\delta) > 0$ such that

$$u_n(x, \tau) \le C|x - \xi(t_0 - \tau)|^{-l - \alpha_1(c_0\lambda)}, \qquad (x, \tau) \in x \in A_\infty(\xi(t_0 - \tau)),\ \tau \in [0, t_0].$$

Since $l = \alpha_1(\lambda) + (\varepsilon/2) - \alpha_1(c_0\lambda)$ and u_n is a supersolution of (5.3), we obtain

$$w_n(x, t) = \tilde{w}_n(x, t_0 - t) \le u_n(x, t_0 - t) \le C|x - \xi(t)|^{-\alpha_1(\lambda) - \frac{\varepsilon}{2}}, \qquad (x, t) \in B_R(0) \times [0, t_0]. \tag{5.4}$$

Let us next show that there exists a constant $C > 0$ which depends on R_0 but is independent of n, δ and R such that

$$|\nabla w_n| \le C e^{-\frac{(R-1)^2}{16T_\delta}} \quad \text{on } \partial B_R(0) \times (0, t_0). \tag{5.5}$$

Since $\theta = 0$ in $\mathbb{R}^N \setminus B_{R_0}(0)$, we see that $\tilde{w}_n$ is a solution of

$$\begin{cases} g_\tau = \Delta g + V_n(x, t_0 - \tau) g, & (x,\tau) \in (B_R(0) \setminus \overline{B_{R_0}(0)}) \times (0, t_0), \\ g(x,\tau) = 0, & (x,\tau) \in \partial B_R(0) \times (0, t_0), \\ g(x,\tau) = \tilde{w}_n(x,\tau), & (x,\tau) \in \partial B_{R_0}(0) \times (0, t_0), \\ g(x,0) = 0, & x \in B_R(0) \setminus \overline{B_{R_0}(0)}. \end{cases} \tag{5.6}$$

By (A2) and (5.4), we can choose positive constants C_1, C_2 and C_3 independent of n, δ and R such that

$$\sup_{(\mathbb{R}^N \setminus \overline{B_{R_0}}) \times [0, T_\delta)} V \le C_1, \qquad \sup_{\partial B_{R_0} \times (0,t_0)} \tilde{w}_n \le C_2, \qquad C_3 e^{C_1 T_\delta} (8\pi T_\delta)^{-\frac{N}{2}} e^{-\frac{R_0^2}{4T_\delta}} \ge C_2.$$

We define

$$W_1^+(x,\tau) := C_3 e^{C_1(T_\delta + \tau)} G(x, T_\delta + \tau).$$

Then we can check that W_1^+ is a supersolution of (5.6), and hence $\tilde{w}_n \le W_1^+$ in $(B_R(0) \setminus \overline{B_{R_0}(0)}) \times (0, t_0)$. In particular,

$$\tilde{w}_n \le W_1^+ \qquad \text{on } \partial B_{R-1}(0) \times (0, t_0). \tag{5.7}$$

On the other hand, $\tilde{w}_n$ is also a solution of

$$\begin{cases} h_\tau = \Delta h + V_n(x, t_0 - \tau) h, & (x,\tau) \in (B_R(0) \setminus \overline{B_{R-1}(0)}) \times (0, t_0), \\ h(x,\tau) = 0, & (x,\tau) \in \partial B_R(0) \times (0, t_0), \\ h(x,\tau) = \tilde{w}_n(x,\tau), & (x,\tau) \in \partial B_{R-1}(0) \times (0, t_0), \\ h(x,0) = 0, & x \in B_R(0) \setminus \overline{B_{R-1}(0)}. \end{cases} \tag{5.8}$$

We define

$$W_2^+(x,\tau) := e^{C_1(T_\delta + \tau)} \left(C_4 |x|^{2-N} - C_5 \right),$$

where $C_4 > 0$ and $C_5 > 0$ are positive constants such that

$$R^{2-N} C_4 - C_5 = 0, \qquad (R-1)^{2-N} C_4 - C_5 = \sup_{\partial B_{R-1}(0) \times (0,t_0)} W_1^+.$$

Note that

$$C_4 = \frac{1}{(R-1)^{2-N} - R^{2-N}} \sup_{\partial B_{R-1}(0) \times (0,t_0)} W_1^+.$$

By (5.7), we see that W_2^+ is a supersolution of (5.8) so that $\tilde{w}_n \le W_2^+$. This together with $\tilde{w}_n = W_2^+ = 0$ on $\partial B_R(0) \times (0, t_0)$ shows that $|\nabla \tilde{w}_n| \le \left| \nabla W_2^+ \right|$ on $\partial B_R(0) \times (0, t_0)$. By the definition of W_2^+ and W_1^+ and using (5.1), we have

$$\begin{aligned} |\nabla W_2^+| &= \frac{(N-2) R^{-1} (R-1)^{N-2}}{R^{N-2} - (R-1)^{N-2}} C_3 e^{2C_1(T_\delta + \tau)} \{4\pi (T_\delta + \tau)\}^{-\frac{N}{2}} e^{-\frac{(R-1)^2}{4(T_\delta + \tau)}} \\ &\le (N-2) C_3 e^{4C_1 T_\delta} (4\pi T_\delta)^{-\frac{N}{2}} e^{-\frac{(R-1)^2}{8T_\delta}} \\ &\le (N-2) C_3 e^{4C_1 T} (4\pi)^{-\frac{N}{2}} e^{-\frac{(R-1)^2}{16T_\delta}} \end{aligned}$$

for $(x,\tau) \in \partial B_R(0) \times (0,t_0)$. Recall $\tilde{w}_n(x,\tau) = w_n(x,t)$ $(\tau = t_0 - t)$. Then there exists a constant $C > 0$ independent of n, δ and R such that (5.5) holds.

Next, for $r > 0$ small, we take $\psi_r \in C_0^\infty(B_R(0))$ such that

$$\begin{cases} \psi_r = 1 & \text{for } |x| \le R - 2r, \\ \psi_r = 0 & \text{for } |x| \ge R - r, \\ 0 \le \psi_r \le 1, \quad |\nabla \psi_r| \le Cr^{-1}, & |\Delta \psi_r| \le Cr^{-2}, \end{cases}$$

where $C > 0$ is a constant independent of R and r. Then $\phi_{r,n} := w_n \psi_r$ satisfies

$$(\phi_{r,n})_t + \Delta \phi_{r,n} + V\phi_{r,n} = w_n \Delta \psi_r + 2\nabla w_n \cdot \nabla \psi_r + w_n \psi_r (V - V_n),$$

so that (5.2) with $\phi = \phi_{r,n}$ implies

$$\begin{aligned} \int_{\mathbb{R}^N} (u_1(x,t_0) - u_2(x,t_0))\theta(x)dx &= I_{r,n} + J_{r,n} \\ := \int_0^{t_0} \int_{R-2r \le |x| \le R-r} &(u_1 - u_2)\left(w_n \Delta \psi_r + 2\nabla w_n \cdot \nabla \psi_r\right) dxdt \\ + \int_0^{t_0} \int_{\mathbb{R}^N} &(u_1 - u_2) w_n \psi_r (V - V_n) dxdt. \end{aligned} \tag{5.9}$$

We first estimate $I_{r,n}$. Note that

$$\sup_{\substack{R-2r \le |x| \le R-r \\ t \in (0,t_0)}} w_n \le Cr \sup_{\substack{R-2r \le |x| \le R \\ t \in (0,t_0)}} |\nabla w_n(x,t)|$$

with some $C > 0$ independent of r, n and R. Since $|u_1 - u_2| \le Ce^{2Cd_0^2} e^{2C|x|^2}$ for $x \in \mathbb{R}^N \setminus B_1(\xi(t))$ and $t \in [0,T]$, we have

$$\begin{aligned} \limsup_{r\to 0} |I_{r,n}| &\le Ce^{2CR^2} \limsup_{r \to 0} \int_0^{t_0} \int_{R-2r \le |x| \le R-r} \left(r^{-2} w_n + r^{-1} |\nabla w_n| \right) dxdt \\ &\le Ce^{2CR^2} \limsup_{r \to 0} \left(r^{-1} \sup_{\substack{R-2r \le |x| \le R \\ t \in (0,t_0)}} |\nabla w_n(x,t)| \int_0^{t_0} \int_{R-2r \le |x| \le R-r} dxdt \right). \end{aligned}$$

From (5.5), it follows that

$$\begin{aligned} \limsup_{r \to 0} |I_{r,n}| &\le Ce^{2CR^2} \sup_{|x|=R, t \in (0,t_0)} |\nabla w_n(x,t)| \limsup_{r\to 0} (r^{-1} CrR^{N-1}) \\ &\le CR^{N-1} e^{-\frac{(R-1)^2}{16T_\delta} + 2CR^2} \le \tilde{C} R^{N-1} e^{-\frac{R^2}{64T_\delta} + 2\tilde{C}R^2} \end{aligned}$$

with some constants $C, \tilde{C} > 0$ independent of n and R. Thus, by replacing $T_\delta > 0$ with a small constant depending on $\tilde{C}$, we have

$$\limsup_{r \to 0} |I_{r,n}| \le CR^{N-1} e^{-\frac{R^2}{128 T_\delta}}$$

with some constant $C > 0$ depending on T_δ but not on n and R, and hence

$$\lim_{R \to 0} \left(\limsup_{n \to \infty} \left(\limsup_{r \to 0} I_{r,n} \right) \right) = 0.$$

Finally, we estimate $J_{r,n}$. From (5.4) and the assumption on $|u_1 - u_2|$, it follows that

$$|J_{r,n}| \le C \int_0^{t_0} \int_{B_R(0)} |x-\xi(t)|^{-\alpha_2(\lambda)+\varepsilon} |x-\xi(t)|^{-\alpha_1(\lambda)-\frac{\varepsilon}{2}} |V - V_n| dx dt,$$

where $C > 0$ is a constant independent of r and n. Thus

$$\limsup_{r\to 0} |J_{r,n}| \le C \int_0^{t_0} \int_{\mathbb{R}^N} |x-\xi(t)|^{-\alpha_1(\lambda)-\alpha_2(\lambda)+\frac{\varepsilon}{2}} |V - V_n| \chi_{B_R(0)}(x) dx dt.$$

Here $\chi_{B_R(0)}$ is the characteristic function on $B_R(0)$. By $\alpha_1(\lambda) + \alpha_2(\lambda) = N - 2$, $0 \le V_n \le V$, (A2) and (A3), we have

$$|x-\xi(t)|^{-\alpha_1(\lambda)-\alpha_2(\lambda)+\frac{\varepsilon}{2}} |V - V_n| \chi_{B_R(0)}(x) \le C |x-\xi(t)|^{-N+\frac{\varepsilon}{2}} \chi_{B_R(0)}(x)$$

for a constant $C > 0$ independent of r and n. The right-hand side of this inequality is independent of n and belongs to $L^1(\mathbb{R}^N \times (0, t_0))$. Hence by $V_n(x,t) \to V(x,t)$ as $n \to \infty$ and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \left(\limsup_{r\to 0} |J_{r,n}| \right) = 0.$$

The left-hand side of (5.9) is independent of r, n and R, and so by taking $\limsup_{r\to 0}$, $\limsup_{n\to\infty}$ and $\lim_{R\to 0}$ in (5.9), we obtain

$$\int_{\mathbb{R}^N} (u_1(x,t_0) - u_2(x,t_0)) \theta(x) dx = 0.$$

Since $t_0 \in (0, T_\delta]$ and θ are arbitrarily chosen, $u_1 \equiv u_2$ on $Q_{(0,\infty),[0,T_\delta]}$. We can repeat the same argument as above to obtain $u_1 \equiv u_2$ on $Q_{(0,\infty),[0,T]}$. □

6. Nonexistence for large initial data

In this section,we show the non-existence of a solution when the initial value is large. Our idea is to compare the solution of (1.1) with a self-similar solution of the radial heat equation (2.1). In the proof, Lemma 2.2 (iv) plays an essential role.

Proof of Theorem 1.4. We take constants δ and $0 < T_\delta \le T$ such that

$$\alpha_2(\lambda) + 2 < \alpha_2(c_1\lambda) + 2 < k,$$
$$c_1 := (1+\delta)^{-\frac{N}{2}} \exp\left(-\frac{(1+\delta) C_\xi^2}{4\delta} T_\delta^{2\gamma-1} \right) < 1.$$

Without loss of generality, it suffices to show the non-existence of solutions of (1.1) with u_0 satisfying that $u_0(x) \ge C|x - \xi(0)|^{-\alpha_2(c_1\lambda)-2}$ for $x \in A_\infty(\xi(0))$.

Contrary to the conclusion, suppose that there exists a solution of (1.1) for $t \in [0, T_\delta]$ with an initial data satisfying $u_0(x) \ge C|x - \xi(0)|^{-\alpha_2(c_1\lambda)-2}$ for $x \in A_\infty(\xi(0))$ with some constant $C > 0$. Then there exists $C_0 > 0$ such that, for each $0 < \tilde{k} < \alpha_2(c_1\lambda) + 2$, (1.1) also has a solution $u_{(\tilde{k})}$ if

$$u_0(x) \ge C_0 |x - \xi(0)|^{-\tilde{k}}, \qquad x \in A_\infty(\xi(0)). \tag{6.1}$$

Let u_n be a unique solution of the integral equation (3.1) with u_0 satisfying (6.1). We set $\tilde{u}_n(x,t) := u_n(x + \xi(t), t)$. Then $\tilde{u}_n$ satisfies $\tilde{u}_n = \Phi[\tilde{u}_n]$, where Φ is defined by (3.2). By the same argument as in Section 4, we have

$$\tilde{u}_n(x,t) \geq \tilde{v}_n(x,t) := \int_{\mathbb{R}^N} c_1 G\left(x-y, \frac{t}{1+\delta}\right) u_0(y+\xi(0))dy + \int_0^t \int_{\mathbb{R}^N} c_1 G\left(x-y, \frac{t-s}{1+\delta}\right) \tilde{V}_n(y,s)\tilde{u}_n(y,s)dyds$$

for $t \in (0, T_\delta)$. Then $\tilde{v}_n$ satisfies

$$\begin{cases} \tilde{v}_{n,t} \geq \dfrac{1}{1+\delta}\Delta \tilde{v}_n + \min\left\{\dfrac{c_1\lambda}{(1+\delta)|x|^2} - c_1 M, c_1 n\right\}\tilde{v}_n, & (x,t) \in A_\infty(0) \times (0, T_\delta), \\ \tilde{v}_n(x,0) \geq c_1 C_0 |x|^{-\tilde{k}}, & x \in A_\infty(0), \end{cases}$$

where $M > 0$ is a constant given by (4.1). Let $\tilde{v}_n^-$ be a unique solution of

$$\begin{cases} \tilde{v}_{n,t}^- = \dfrac{1}{1+\delta}\Delta \tilde{v}_n^- + \min\left\{\dfrac{c_1\lambda}{(1+\delta)|x|^2} - c_1 M, c_1 n\right\}\tilde{v}_n^-, & (x,t) \in A_\infty(0) \times (0, T_\delta), \\ \tilde{v}_n^-(x,0) = c_1 C_0 |x|^{-\tilde{k}}, & x \in A_\infty(0). \end{cases}$$

By the comparison principle for bounded functions, we have $\tilde{v}_n^- \leq \tilde{v}_n$. This together with $\tilde{v}_n \leq \tilde{u}_n$ gives $\tilde{v}_n^- \leq \tilde{u}_n$. Letting $n \to \infty$, we obtain $\tilde{v}^- \leq \tilde{u}$, where $\tilde{u}(x,t) := u_{(\tilde{k})}(x+\xi(t),t)$ and $\tilde{v}^-$ is a minimal solution of

$$\begin{cases} \tilde{v}_t^- = \dfrac{1}{1+\delta}\Delta \tilde{v}^- + \left\{\dfrac{c_1\lambda}{(1+\delta)|x|^2} - c_1 M\right\}\tilde{v}^-, & (x,t) \in A_\infty(0) \times (0, T_\delta), \\ \tilde{v}^-(x,0) = c_1 C_0 |x|^{-\tilde{k}}, & x \in A_\infty(0). \end{cases}$$

Since $\tilde{v}_n^-(\cdot,0)$ is radially symmetric, $\tilde{v}^-$ is also radially symmetric. Then we rewrite

$$\tilde{v}^-(x,t) = c_1 C_0 e^{-c_1 M t} v\left(r, \frac{t}{1+\delta}\right), \qquad r = |x|,$$

where v is a minimal solution of

$$\begin{cases} v_t = v_{rr} + \dfrac{N-1}{r} v_r + \dfrac{c_1\lambda}{r^2} v, & r > 0,\ t > 0, \\ v(r,0) = r^{-\tilde{k}}, & r > 0. \end{cases}$$

By $c_1\lambda < \lambda_c$, $\tilde{k} < \alpha_2(c_1\lambda)+2$, Lemma 2.3 and Theorem 1.3, the minimal solution $v = v(r,t;\tilde{k})$ can be written as

$$v(r,t;\tilde{k}) = \frac{r^{-\alpha_1(c_1\lambda)}}{\sigma(\tilde{k} - \alpha_1(c_1\lambda))} t^{-\frac{\tilde{k}-\alpha_1(c_1\lambda)}{2}} \varphi\left(\frac{r}{t^{1/2}}\right),$$

where σ and φ are given in Lemma 2.2 with $l = \tilde{k} - \alpha_1(c_1\lambda)$ and $d = N - 2\alpha_1(c_1\lambda)$. Here, by Lemma 2.2(iv), $v(r,t;\tilde{k}) \to \infty$ as $\tilde{k} \uparrow \alpha_2(c_1\lambda) + 2$ uniformly on any compact subset of $(0,\infty) \times (0,\infty)$. Hence

$$u_{(\tilde{k})}(x,t) = \tilde{u}(x-\xi(t),t) \geq \tilde{v}^-(x-\xi(t),t) = c_1 C_0 e^{-c_1 M t} v\left(x-\xi(t), \frac{t}{1+\delta}; \tilde{k}\right) \to \infty$$

as $\tilde{k} \uparrow \alpha_2(c_1\lambda)+2$ uniformly on any compact subset of $Q_{(0,\infty),(0,(1+\delta)^{-1}T_\delta)}$. This is a contradiction. □

7. Nonexistence in the supercritical case

In this section,we consider the supercritical case and show the non-existence of a solution. In order to prove Theorem 1.5, it suffices to study the case where there exist constants $\lambda_0 \in (\lambda_c, \lambda_c + 1)$, $r_0 \in (0,1)$ and $\bar{t} \in (0,1)$ such that

$$V(x,t) \geq \frac{\lambda_0}{|x-\xi(t)|^2}$$

for $x \in A_{r_0}(\xi(t))$ and $t \in [0,\bar{t}]$. Our idea for the proof is to use the lower estimate by Theorem 1.2, and derive a contradiction from the equivalent integral equation by assuming the existence of a solution of (1.1).

We prepare a lemma, and then prove Theorem 1.5.

Lemma 7.1. *Let $\underline{t} < \bar{t}$. Then there exist positive constants $C_1 = C_1(N, C_\xi, \gamma)$ and $C_2 = C_2(N)$ independent of $\underline{t}$ and $\bar{t}$ such that*

$$\int_{\underline{t}}^{t} G(z-\xi(s), t-s)ds \geq \frac{\Gamma(\frac{N-2}{2})}{4\pi^{N/2}}|z-\xi(t)|^{2-N} - C_1|z-\xi(t)|^{1+2\gamma-N} - C_2(t-\underline{t})^{-\frac{N-2}{2}}$$

for $z \in A_1(\xi(t))$ and $t \in (\underline{t}, \bar{t})$.

Proof. We consider the following I_1, I_2 and I_3:

$$\begin{aligned}\int_{\underline{t}}^{t} G(z-\xi(s), t-s)ds &= \int_{-\infty}^{t} G(z-\xi(t), t-s)ds - \int_{-\infty}^{\underline{t}} G(z-\xi(t), t-s)ds \\ &\quad + \int_{\underline{t}}^{t} (G(z-\xi(s), t-s) - G(z-\xi(t), t-s))ds \\ &=: I_1 - I_2 + I_3.\end{aligned}$$

One can easily show that

$$I_1 = \frac{\Gamma(\frac{N-2}{2})}{4\pi^{N/2}}|z-\xi(t)|^{2-N}, \qquad |I_2| \leq C(t-\underline{t})^{-\frac{N-2}{2}}.$$

Let us estimate I_3. We have

$$|I_3| \leq C\int_{\underline{t}}^{t} \frac{\left||z-\xi(t)|^2 - |z-\xi(s)|^2\right|}{(t-s)^{\frac{N}{2}+1}} e^{-\frac{|z-\xi(t)|^2}{4(t-s)}} \int_0^1 e^{\frac{\theta}{4(t-s)}\left||z-\xi(t)|^2-|z-\xi(s)|^2\right|} d\theta ds.$$

From (A1), it follows that

$$\begin{aligned}\left||z-\xi(t)|^2 - |z-\xi(s)|^2\right| &\leq 2|z-\xi(t)||\xi(t)-\xi(s)| + |\xi(t)-\xi(s)|^2 \\ &\leq \begin{cases} \frac{1}{2}|z-\xi(t)|^2 + 3C_\xi^2(t-s)^{2\gamma}, \\ 2C_\xi|z-\xi(t)|(t-s)^\gamma + C_\xi^2(t-s)^{2\gamma}. \end{cases}\end{aligned}$$

Then by $t-s \le \bar t < 1$ and $1/2 < \gamma \le 1$, we have

$$\begin{aligned}
|I_3| &\le C\int_{\underline{t}}^{t} \frac{|z-\xi(t)|(t-s)^{\gamma}+(t-s)^{2\gamma}}{(t-s)^{\frac{N}{2}+1}} e^{-\frac{|z-\xi(t)|^2}{8(t-s)}}ds \\
&\le C\int_{\underline{t}}^{t} \Big\{|z-\xi(t)|(t-s)^{-\frac{N}{2}-1+\gamma}+(t-s)^{-\frac{N}{2}-1+2\gamma}\Big\} e^{-\frac{|z-\xi(t)|^2}{8(t-s)}}ds \\
&\le C\int_{-\infty}^{t} \Big\{|z-\xi(t)|(t-s)^{-\frac{N}{2}-1+\gamma}+(t-s)^{-\frac{N}{2}-1+2\gamma}\Big\} e^{-\frac{|z-\xi(t)|^2}{8(t-s)}}ds \\
&\le C(|z-\xi(t)|^{1+2\gamma-N}+|z-\xi(t)|^{4\gamma-N}) \le C|z-\xi(t)|^{1+2\gamma-N}
\end{aligned}$$

for $z\in A_1(\xi(t))$. Thus the lemma follows. □

Proof of Theorem 1.5. Contrary to the conclusion, suppose that (1.1) has a nontrivial solution u for $t\in[0,\bar t)$. Fix α and ε such that

$$N-2-\frac{2}{N-2}\lambda_0 < \alpha < \frac{N-2}{2}, \qquad 0<4\varepsilon<\frac{2\lambda_0}{(N-2-\alpha)(N-2)}-1. \tag{7.1}$$

Let $\underline{t}\in(0,\bar t)$. By Theorem 1.2 and the comparison principle, there exist constants $r_1\in(0,r_0)$ and $c>0$ such that

$$u(x,t)\ge c|x-\xi(t)|^{-\alpha}, \qquad x\in A_{r_1}(\xi(t)),\ t\in[\underline{t},\bar t].$$

Then there exists a constant $r_2\in(0,r_1/2)$ such that

$$\frac{4\pi^{N/2}(\lambda_c+1)C_1(2r_2)^{2\gamma-1}}{\lambda_c\Gamma(\frac{N-2}{2})}<\varepsilon, \qquad \frac{2N\omega_N(\lambda_c+1)C_2}{N-2} r_2^{\frac{N-2}{2}}<\varepsilon \tag{7.2}$$

and that

$$c_0' < (1+\varepsilon)\,c_0, \tag{7.3}$$

where positive constants c_0 and c_0' are given by

$$\begin{aligned}
c_0 &:= \inf\{|x-\xi(t)|^{\alpha}u(x,t);\ |x-\xi(t)|<r_2,\ \underline{t}<t<\bar t\},\\
c_0' &:= \inf\{|x-\xi(t)|^{\alpha}u(x,t);\ |x-\xi(t)|<r_2,\ \underline{t}+r_2<t<\bar t\}.
\end{aligned}$$

Let (x,t) satisfy $|x-\xi(t)|<r_2$ and $\underline{t}+r_2<t<\bar t$. Then,

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^N} G(x-y,t)u_0(y)dy + \int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)V(y,s)u(y,s)dyds \\
&\ge \lambda_0 c_0 \int_{\underline{t}}^{t}\int_{|y-\xi(s)|<r_2} G(x-y,t-s)|y-\xi(s)|^{-\alpha-2}dyds \\
&= \lambda_0 c_0 \int_{|y|<r_2} |y|^{-\alpha-2}\left(\int_{\underline{t}}^{t} G(x-y-\xi(s),t-s)ds\right)dy.
\end{aligned}$$

This together with Lemma 7.1 and $t > \underline{t} + r_2$ shows that

$$
\begin{aligned}
u(x,t) &\geq \lambda_0 c_0 \frac{\Gamma(\frac{N-2}{2})}{4\pi^{N/2}} \int_{|y|<r_2} |y|^{-\alpha-2}|x-y-\xi(t)|^{2-N} dy \\
&\quad - \lambda_0 c_0 C_1 \int_{|y|<r_2} |y|^{-\alpha-2}|x-y-\xi(t)|^{1+2\gamma-N} dy \\
&\quad - \lambda_0 c_0 C_2 (t-\underline{t})^{-\frac{N-2}{2}} \int_{|y|<r_2} |y|^{-\alpha-2} dy \\
&\geq c_0 \left\{ \lambda_0 \frac{\Gamma(\frac{N-2}{2})}{4\pi^{N/2}} - \lambda_0 C_1 (2r_2)^{2\gamma-1} \right\} \int_{|y-\xi(t)|<|x-\xi(t)|} \frac{|x-y|^{2-N}}{|y-\xi(t)|^{2+\alpha}} dy \\
&\quad - \frac{N\omega_N \lambda_0 C_2}{N-2-\alpha} c_0 r_2^{\frac{N-2}{2}-\alpha}.
\end{aligned} \tag{7.4}
$$

Set $r = |x-\xi(t)|$. We claim that

$$
\int_{|y-\xi(t)|<r} \frac{|x-y|^{2-N}}{|y-\xi(t)|^{2+\alpha}} dy = \frac{8\pi^{N/2}}{(N-2-\alpha)(N-2)\Gamma(\frac{N-2}{2})} r^{-\alpha}. \tag{7.5}
$$

For $y \in B_r(\xi(t))$, we take $\rho > 0$ and $\theta \in (-\pi/2, \pi/2)$ as

$$
\rho = |x-y|, \qquad |y-\xi(t)|^2 = \rho^2 + r^2 - 2\rho r \cos\theta,
$$

and then

$$
\begin{aligned}
\int_{B_r(\xi(t))} \frac{|x-y|^{2-N}}{|y-\xi(t)|^{2+\alpha}} dy &= 2\int_0^{\pi/2} \int_0^{2r\cos\theta} \frac{\rho^{2-N}}{(\rho^2+r^2-2\rho r\cos\theta)^{1+\frac{\alpha}{2}}} \cdot \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \cdot \rho(\rho\sin\theta)^{N-2} d\rho d\theta \\
&= \frac{4\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^{\pi/2} \sin^{N-2}\theta \int_0^{2r\cos\theta} \frac{\rho}{\left\{(\rho - r\cos\theta)^2 + r^2\sin^2\theta\right\}^{1+\frac{\alpha}{2}}} d\rho d\theta.
\end{aligned}
$$

By the change of variables $\rho = r\cos\theta + r\sin\theta\tan\phi$, the last integral is written as

$$
\int_0^{2r\cos\theta} \frac{\rho}{\left\{(\rho - r\cos\theta)^2 + r^2\sin^2\theta\right\}^{1+\frac{\alpha}{2}}} d\rho = r^{-\alpha} \frac{\cos\theta}{\sin^{1+\alpha}\theta} \int_{-(\frac{\pi}{2}-\theta)}^{\frac{\pi}{2}-\theta} \cos^\alpha\phi d\phi.
$$

This together with Fubini's theorem shows

$$
\begin{aligned}
\int_{B_r(\xi(t))} \frac{|x-y|^{2-N}}{|y-\xi(t)|^{2+\alpha}} dy &= \frac{8\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} r^{-\alpha} \int_0^{\pi/2} \cos^\alpha\phi \int_0^{\frac{\pi}{2}-\phi} \sin^{N-3-\alpha}\theta\cos\theta d\theta d\phi \\
&= \frac{8\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} r^{-\alpha} \times \frac{\pi^{\frac{1}{2}}\Gamma(\frac{N-1}{2})}{2(N-2-\alpha)\Gamma(\frac{N}{2})},
\end{aligned}
$$

so that (7.5) follows.

By the estimate (7.4) together with (7.5), (7.1), $\lambda_0 < \lambda_c + 1$ and (7.2), we have

$$\begin{aligned} |x-\xi(t)|^\alpha u(x,t) \geq c_0 & \left\{ \frac{2\lambda_0}{(N-2-\alpha)(N-2)} - \frac{8\pi^{N/2}\lambda_0 C_1(2r_2)^{2\gamma-1}}{(N-2-\alpha)(N-2)\Gamma(\frac{N-2}{2})} \right\} \\ & - \frac{N\omega_N \lambda_0 C_2}{N-2-\alpha} r_2^{\frac{N-2}{2}} c_0 \\ & \geq c_0 \left\{ 1 + 4\varepsilon - \frac{4\pi^{N/2}(\lambda_c+1)C_1(2r_2)^{2\gamma-1}}{\lambda_c \Gamma(\frac{N-2}{2})} \right\} - \frac{2N\omega_N(\lambda_c+1)C_2}{N-2} r_2^{\frac{N-2}{2}} c_0 \\ & \geq c_0(1+2\varepsilon). \end{aligned}$$

Then, taking the infimum over $\{(x,t); |x-\xi(t)| < r_2,\ \underline{t} + r_2 < t < \overline{t}\}$, we obtain

$$c_0' \geq (1+2\varepsilon)\, c_0.$$

However, this contradicts (7.3). □

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