On triple correlation sums involving Fourier coefficients of cusp forms

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On triple correlation sums involving Fourier coefficients of cusp forms

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Abstract

Let \( g \) be a newform of trivial level, and \( f \) a newform of prime level \( p \). In this paper, we investigate the sum \( \sum_{m \geq 1} \sum_{n \geq 1} a_n \lambda_g(m) \lambda_f(m + pn) U(m/X) V(n/H) \) to determine the explicit dependence on the level, where \( a_n \in \mathbb{C} \) be any complex sequences. As a result, we prove a uniform bound with respect to the level parameter \( p \), and present that this type of sum is non-trivial for any given \( H, X \geq 2 \).

MSC: 11F67; 11F66

Keywords: Automorphic forms; Fourier coefficients; Triple correlation sums

1 Introduction

A basic but important problem in number theory is the triple correlation sums problem, which concerns the non-trivial bounds for

\[
\sum_{h \leq H} \sum_{n \leq X} a(n)b(l_1 n + l_2 h)c(h) \quad \text{or} \quad \sum_{h \leq H} \sum_{n \leq X} a(n)b(n + l_1 h)c(n + l_2 h).
\]

Here, \( a(n), b(n) \) and \( c(n) \) are three arithmetic functions, \( H, X \geq 2 \) and \( l_1, l_2 \in \mathbb{Z} \). These type of sums play the vital rôles of their own in many topics, such as the moments of \( L \)-functions (or zeta-functions), subconvexity, the Gauss circle problem and the Quantum Unique Ergodicity (QUE) conjecture, etc (see for instance [24, 13, 2] [6] [11] [10] [18] [16] [15] and the references therein). In the case of all the arithmetic functions being the divisor functions, Browning [5] found that

\[
\sum_{h \leq H, n \leq X} a(n)b(n + h)c(n + 2h) = \frac{11}{8} Y(h) \prod_p \left( 1 - \frac{1}{p} \right)^2 \left( 1 + \frac{2}{p} \right) HX \log^3 X + o(HX \log^3 X)
\]

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for an explicit multiplicative function $\Upsilon(h)$, provided that $H \geq X^{3/4 + \varepsilon}$. It is remarkable that Blomer [3] used the spectral decomposition for partially smoothed triple correlation sums to prove that

$$
\sum_{h \geq 1} \sum_{n \leq X} W\left(\frac{h}{H}\right) \tau(n) a(n+h) \tau(n+2h) = H \hat{W}(1) \sum_{n \leq X} a(n) \sum_{d \geq 1} S(2n,0;d) \frac{d}{d^2} \times (\log n + 2\gamma - 2 \log d)^2 + O\left(\left(\frac{H^2}{\sqrt{X}} + H X^{\frac{1}{3}} + \sqrt{XH} + \frac{X}{\sqrt{H}}\right) \|a\|_2^2\right).
$$

Here, the gamma constant $\gamma \approx 0.57721$, $W$ is a smooth function supported on $[1/2, 5/2]$ with bounded derivatives, $\hat{W}$ denotes the Mellin transform of $W$ and

$$
\|a\|_2 = \sqrt{\sum_{n \leq X} |a^2(n)|}
$$

is the $\ell^2$-norm. Notice that, the average over $H$ is reduced very substantially to $H \geq X^{1/3 + \varepsilon}$. Let $k,k' \geq 2$ be any ever integers. Let $f_1 \in B_k(1)$ and $f_2 \in B_{k'}(1)$ be two Hecke newforms on $GL_2$, with $\lambda_{f_1}$ and $\lambda_{f_2}$ being their $n$-th Hecke eigenvalues, respectively (see §2 for definitions). Subsequently, Lin [25] claimed that

$$
\sum_{h \geq 1, n \leq X} W\left(\frac{h}{H}\right) \lambda_{f_1}(n) a(n+h) \lambda_{f_2}(n+2h) \ll X^\varepsilon \left(\sqrt{XH} + \frac{X}{\sqrt{H}}\right) \|a\|_2^2,
$$

which is non-trivial, provided that $H \geq X^{2/3 + \varepsilon}$. Recently, Singh [35], however, were able to attain that, for any $f_1, f_2, f_3 \in B_k^*(1)$ (or $B_{k'}(1)$) and some constant $\eta > 0$,

$$
\sum_{h \geq 1, n \geq 1} W_1\left(\frac{h}{H}\right) W_2\left(\frac{n}{X}\right) \lambda_{f_1}(n) \lambda_{f_2}(n+h) \lambda_{f_3}(n+2h) \ll X^{1-\eta + \varepsilon} H,
$$

where $W_1, W_2$ are two smooth bump functions supported on the interval $[1/2, 5/2]$ with bounded derivatives. Until now, the best result is due to Lü-Xi [26, 27] who achieved that

$$
\sum_{h \geq 1, n \leq X} W\left(\frac{h}{H}\right) a(n)b(n+h) \lambda_{f_1}(n+2h) \ll X^\varepsilon \Delta_1(X,H) \|a\|_2 \|b\|_2,
$$

which allows one to take $H \geq X^{2/5 + \varepsilon}$. Here, the definition of $\Delta_1(X,H)$ can be referred to [27 Theorem 3.1]. More recently, Hulse et al. [17] successfully attained

$$
\sum_{h \geq 1} \sum_{n \geq 1} \lambda_{g_1}(n) \lambda_{g_2}(h) \lambda_{g_3}(2n - h) \exp\left(-\frac{h}{H} - \frac{n}{X}\right) \ll X^{\kappa - 1 + \theta + \frac{1}{2} + \varepsilon} X^\frac{k-1}{2} - \theta + \frac{1}{2} + \varepsilon, \quad (1.1)
$$
where $\vartheta < 7/64$ denotes the currently best approximation towards the Generalized Ramanujan Conjecture. Here, $\lambda_{g_1}(n)$, $\lambda_{g_2}(n)$ and $\lambda_{g_3}(n)$ denote the $n$-th non-normalized coefficients of holomorphic cusp forms $g_1$, $g_2$ and $g_3$, each of weight $\kappa \geq 2$, level $M \geq 2$ and trivial nebentypus.

In this paper, we shall consider the level aspect for the triple correlation sums. It is noticeable that, just lately, Munshi [33] obtained that, whenever $X^{1/3+\varepsilon} \leq p \leq X$, one has the inequality

$$\sum_{n \geq 1} \lambda_f(n)\lambda_f(n+pm) \ll p^{1/4}X^{3/4+\varepsilon}$$

for any fixed integer $m$ such that $|m| \leq X/p$. In the present paper, we shall go further to obtain the following quantitative estimate:

**Theorem 1.1.** Let $X, H \geq 2$, and $p$ be a prime such that $p \leq X$. Let $U, V$ be two smooth weight functions supported $[1/2, 5/2]$ with bounded derivatives. Then, for any any complex sequences $a = a_n$ and any newforms $g \in \mathcal{B}_k(1)$ (or $\mathcal{B}_\lambda^*_k(1)$) and $f \in \mathcal{B}_k(p)$ (or $\mathcal{B}_\lambda^*_k(p)$), we then have

$$\sum_{m \geq 1} \sum_{n \geq 1} a_n \lambda_g(m)\lambda_f(m+pn)U\left(\frac{m}{X}\right)V\left(\frac{n}{H}\right) \ll X^\varepsilon \max\left(\sqrt{XHp}, X\right)\|a\|_2,$$

where the implied constant depends only on the weight $k$ (or the spectral parameter $\lambda$) and $\varepsilon$.

**Remark 1.2.** If the sequences $a = a_n$ satisfy the mild property that $\|a\|_2^2 = H$ for any height $H \geq 2$, our main result (1.3) is non-trivial for any given parameters $X$ and $H$; particularly, for any automorphic cusp form $\pi$ of any rank $N \geq 2$, with $\lambda_\pi(n)$ being its $n$-th normoailzed Fourier coefficient, we find

$$\sum_{m \geq 1} \sum_{n \geq 1} \lambda_\pi(n)\lambda_g(m)\lambda_f(m+pn)U\left(\frac{m}{X}\right)V\left(\frac{n}{H}\right) \ll X^\varepsilon \max\left(\sqrt{XHp}, X\right)\|a\|_2$$

by the Rankin-Selberg’s bound, which says that $\sum_{n \leq X} |\lambda_\pi(n)|^2 \ll_{\pi,\varepsilon} X^{1+\varepsilon}$ (see for instance [8, Remark 12.1.8]).

**Remark 1.3.** The merits that comes from (1.3) is that the implied constant does not depend on the level parameter anymore. One may verify that our result (1.3), however, is exhibited to be a strengthened upper-bound whenever $H \geq \sqrt{X/p}$, compared with an application of Munshi’s estimate (1.2). Indeed, Munshi’s estimate implies the upper-bound $\ll p^{1/4}X^{3/4+\varepsilon}\sqrt{H}\|a\|_2$ for the triple sum above. One, on the other hand, wanders if some points can be employed to make a difference to the establishment of the non-trivial bounds for the scenarios where the cusp forms $f, g$ being of higher rank (see for instance [8, Chapter 5] for definitions); we shall discuss this cause on another occasion.
Notations. Throughout the paper, ε always denotes an arbitrarily small positive constant which might not be the same at each occurrence. \( n \sim X \) means that \( X < n \leq 2X \). For any integers \( m, n \), \((m, n)\) denotes the greatest common divisor of \( m \) and \( n \); \( \mu(n) \) is the Möbius function of \( n \). We use Landau’s \( f = O(g) \) and Vinogradov’s \( f \ll g \) as synonyms; thus, \( f(x) \ll g(x) \) for \( x \in \mathcal{X} \) (where the set \( \mathcal{X} \) must be specified either explicitly or implicitly) means that \( |f(x)| \leq Cg(x) \) for all \( x \in \mathcal{X} \) and some constant \( C > 0 \); we also use \( f \approx g \) to mean that both relations \( f \ll g \) and \( g \ll f \) hold, of course with possibly different implied constants. Finally, for any positive integers \( m, n \) and \( c \), \( S(m, n; c) = \sum_{\alpha \mod c} e((m\alpha + n\alpha)/c) \) is the 2-dimensional Kloosterman sum, where \( * \) indicates that the summation is restricted to \( (x, c) = 1 \), and \( \overline{x} \) is the inverse of \( x \) modulo \( c \).

2 Preliminaries

2.1 Automorphic forms

Let \( k \geq 2 \) be an even integer and \( N > 0 \) be an integer. We denote by \( S_k(N) \) the vector space of holomorphic cusp forms on \( \Gamma_0(N) \) with trivial nebentypus and weight \( k \). For any \( f \in S_k(N) \), one has a Fourier expansion

\[
  f(z) = \sum_{n \geq 1} \psi_f(n) n^{k-1/2} e(nz)
\]

for \( \text{Im}(z) > 0 \). Analogously, we denote by \( S_\lambda(N) \) the vector space of Maass forms on \( \Gamma_0(N) \) with trivial nebentypus, weight 0 and eigenvalue \( \lambda = 1/4 + r^2 > 1/4 \) (so that \( r \in \mathbb{R} \)). Then, for any \( f \in S_\lambda(N) \), one has a Fourier expansion

\[
  f(z) = 2\sqrt{|y|} \sum_{n \neq 0} \psi_f(n) K_{ir}(2\pi|ny|) e(nx),
\]

where \( z = x + iy \) and \( K_{ir} \) denotes the \( K \)-Bessel function. \( S_k(N) \) and \( S_\lambda(N) \) are finite dimensional Hilbert spaces which can be equipped with the Petersson inner products

\[
  \langle f_1, f_2 \rangle = \int_{\Gamma_0(N) \backslash \mathbb{H}} f_1(z) \overline{f_2(z)} y^{k-2} dx dy
\]

and

\[
  \langle f_1, f_2 \rangle = \int_{\Gamma_0(N) \backslash \mathbb{H}} f_1(z) \frac{\overline{f_2(z)}}{y^2} dx dy,
\]

respectively. We recall the Hecke operators \( \{T_n\} \) with \( (n, N) = 1 \) which satisfy the multiplicativity relation

\[
  T_n T_m = \sum_{d \mid (n, m)} T_{nm}^{\frac{d^2}{d}}, \quad (2.4)
\]
The adjoint of $T_n$ with respect to the Petersson inner products is itself, hence, $T_n$ is normal. One can find an orthogonal basis $\mathcal{B}_k(N)$ ($\mathcal{B}_\lambda(N)$ respectively) of $\mathcal{S}_k(N)$ ($\mathcal{S}_\lambda(N)$ respectively) consisting of common eigenfunctions of all the Hecke operators $T_n$ with $(n, N) = 1$. For each $f \in \mathcal{B}_k(N)$ or $\mathcal{B}_\lambda(N)$, denote by $\lambda_f(n)$ the $n$-th Hecke eigenvalue which satisfies

$$T_n f(z) = \lambda_f(n) f(z)$$

for all $(n, N) = 1$. From (2.4), one has

$$\psi_f(m) \lambda_f(n) = \sum_{d|n,m} \psi_f \left( \frac{mn}{d^2} \right)$$

for any $m, n > 1$ with $(n, N) = 1$. In particular, $\psi_f(1) \lambda_f(n) = \psi(n)$, if $(n, N) = 1$. Therefore,

$$\lambda_f(m) \lambda_f(n) = \sum_{d|(n,m)} \lambda_f \left( \frac{mn}{d^2} \right), \quad (2.5)$$

if $(mn, N) = 1$.

The Hecke eigenbasis $\mathcal{B}_k(N)$ ($\mathcal{B}_\lambda(N)$ respectively) also contains a subset of newforms $\mathcal{B}_k^*(N)$ ($\mathcal{B}_\lambda^*(N)$ respectively), those forms which are simultaneous eigenfunctions of all the Hecke operators $T_n$ for any $n \geq 1$ and normalized to have first Fourier coefficient $\psi_f(1) = 1$. The elements of $\mathcal{B}_k^*(N)$ and $\mathcal{B}_\lambda^*(N)$ are usually called primitive forms. Regarding the individual bounds for the Fourier coefficients $\lambda_f(n)$, for general $n \geq 1$, we have

$$\lambda_f(n) \ll (nN)^{\varepsilon}, \quad (2.6)$$

whenever $f \in \mathcal{B}_k(N)$ (or $\mathcal{B}_\lambda(N)$).

We will need the following general Voronoi-type summation formula which is Theorem A.4 [23].

**Lemma 2.1.** Let $k \geq 2$ be an even integer and $N > 0$ be an integer. Let $f \in \mathcal{B}_k(N)$ (or $\mathcal{B}_\lambda(N)$) be a newform. For $(a, q) = 1$ set $N_2 := N/(N, q)$. If $h \in C^\infty(\mathbb{R}^{x,+})$ is a Schwartz class function vanishing in a neighborhood of zero, then there exists a complex number $\varpi$ of modulus one, which depends on $a$, $q$ and $f$, and a newform $f^* \in \mathcal{B}_k(N)$ (or $\mathcal{B}_\lambda(N)$) such that

$$\sum_n \lambda_f(n) e \left( \frac{an}{q} \right) h \left( \frac{n}{X} \right) = \frac{2\pi \varpi}{q \sqrt{N_2}} \sum_n \lambda_{f^*}(n) e \left( -\frac{aN_2n}{q} \right) \mathcal{H}^\vartheta \left( \frac{nX}{q^2N_2}; h \right)$$

$$+ \frac{2\pi \varpi}{q \sqrt{N_2}} \sum_n \lambda_{f^*}(n) e \left( \frac{aN_2n}{q} \right) \mathcal{H}^\varphi \left( \frac{nX}{q^2N_2}; h \right),$$

where

$$\mathcal{H}^\vartheta(x; h) = \int_0^\infty h(\xi) J_f \left( 4\pi \sqrt{x\xi} \right) d\xi, \quad \text{and} \quad \mathcal{H}^\varphi(x; h) = \int_0^\infty h(\xi) K_f \left( 4\pi \sqrt{x\xi} \right) d\xi.$$

In this formula,
• if $f$ is holomorphic of weight $k$ then
  
  \[ J_f(x) = 2\pi i^k J_{k-1}(x), \quad K_f(x) = 0. \]

• if $f$ is a Maass form with eigenvalue $\lambda = 1/4 + r^2$ then
  
  \[ J_f(x) = \frac{-\pi}{\sin(\pi r)} (J_{2ir}(x) - J_{-2ir}(x)), \quad K_f(x) = 4 \cosh(\pi r) K_{2ir}(x). \]

One may write

\[ J_{k-1}(x) = x^{\frac{k}{2}} (F^+_k(x)e(x) + F^-_k(x)e(-x)) \] (2.7)

for some smooth functions $F^\pm_k$ satisfying that

\[ x^j F^\pm_k(x) \ll_{k,j} \frac{x}{(1+x)^{\frac{j}{2}}} \]

for any $j \geq 0$; the existence being guaranteed for instance by [36, Section 6.5] if $x < 1$ and [36, Section 3.4] if $x \geq 1$. One thus sees that, for $\mathcal{H}^\flat$, $n$ is essentially truncated at $n \ll q^2 N_2 / X^{1-\varepsilon}$, by repeated integration by parts. Furthermore, notice that, by Appendix of [4],

\[ K_{2ir}(x) \ll_{r, \varepsilon} \begin{cases} 
(1 + |r|)^\varepsilon, & 0 < x \leq 1 + \pi |r|, \\
\exp(-x) x^{-\frac{1}{2}}, & x > 1 + \pi |r|;
\end{cases} \] (2.8)

one will find the $n$-variable enjoys the analogous truncation range for $\mathcal{H}^\sharp$ with that for $\mathcal{H}^\flat$.

### 2.2 The Wilton-type bound

We have the following Wilton-type bound which shall be used in the proof of the main theorem; the resource, however, may be referred to [22], together with [12] and [33].

**Lemma 2.2.** Let $X \geq 2$ and $W$ be a smooth function, compactly supported on $[1/2, 5/2]$ with bounded derivatives. Then, for any $\alpha \in \mathbb{R}$ and any newform $f \in \mathcal{B}_k(N)$ (or $\mathcal{B}_k^*(N)$), we have

\[ \sum_{n \geq 1} \lambda_f(n) e(n\alpha) W\left( \frac{n}{X} \right) \ll \sqrt{X} N^{\frac{1}{2} + \varepsilon}, \] (2.9)

where the implied constant depends only on the weight $k$ (or the spectral parameter $\lambda$) and $\varepsilon$. 

6
2.3 The delta method

The $\delta$-symbol method was developed in [6, 7] as variant of the circle method. Further development and applications can be found in Jutila [20, 21], Heath-Brown [14], Munshi [31], and more recently [1] to name a few. We will now briefly recall a version of the circle method which is due to Duke, Friedlander and Iwaniec (see for instance [19, Chapter 20]).

Lemma 2.3. Let $Q \geq 1$ be a large parameter to be chosen later. For any $n \leq X$, one has

$$
\delta(n) = \frac{1}{Q} \sum_{q \leq Q} \frac{1}{q} \sum_{a \mod q \ (a,q)=1} e\left(\frac{an}{q}\right) \int_{\mathbb{R}} g(q, \tau) e\left(\frac{n\tau}{qQ}\right) d\tau,
$$

where

$$
g(q, \tau) = 1 + h(q, \tau) \quad \text{with} \quad h(q, \tau) = O\left(\frac{1}{qQ} \left(\frac{1}{Q} + \frac{q}{Q}\right)\right),
$$

$$
\tau^j \frac{\partial^j}{\partial \tau^j} g(q, \tau) \ll \log Q \min\left(\frac{Q}{q}, \frac{1}{|\tau|}\right),
$$

and $g(q, \tau) \ll |\tau|^{-A}$ for any sufficiently large $A$. In particular, the effective range of the $\tau$-integral is $[-X^{\varepsilon}, X^{\varepsilon}]$.

2.4 Some estimates involving Kloosterman sums

We will have a need of the following lemmas which will be applied in §3.

Lemma 2.4. Let $Q \geq 2$. Let $F(x,y)$ be a smooth function supported on $[1/2, 5/2] \times [1/2, 5/2]$ with partial derivatives satisfying

$$
X^i Y^j \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} F\left(x, \frac{y}{X}, \frac{y}{Y}\right) \ll_{i,j} 1
$$

for any $X, Y \geq 1$ and integers $i, j \geq 0$. Then, for any $c \in \mathbb{Z}$, complex sequences $a = \{a_n\}$ and newform $f \in B_k(1)$ (or $B_k^*(1)$), there holds that

$$
\sum_{q \geq 1} \sum_{n \geq 1} a_n S(n, c; q) F\left(\frac{n}{X}, \frac{q}{Q}\right) \ll X^{\varepsilon} \left(\sqrt{XQ} + 1_{Q^2 \geq X} Q^2\right) \|a\|_2,
$$

where the symbol $1_{\mathbb{P}}$ equals 1 if the assertion $\mathbb{P}$ is true, and 0 otherwise.

Proof. First, by the Cauchy-Schwarz inequality, we might evaluate the double sum as

$$
\leq \left( \sum_{q_1, q_2 \geq 1} \sum_{n \geq 1} S(n, c; q_1) S(n, c; q_1) F\left(\frac{n}{X}, \frac{q_1}{Q}\right) F\left(\frac{n}{X}, \frac{q_2}{Q}\right) \right)^{1/2} \|a\|_2.
$$
It thus follows from the Weil bound that the non-generic terms \( q_1 = q_2 \) shall contribute a upper-bound \( \ll X^{1+\varepsilon}\mathcal{Q}^2 \) to the parentheses above, which gives the term \( \sqrt{X\mathcal{Q}}\|a\|_2 \) on the RHS of (2.10); while, for the generic terms \( q_1 \neq q_2 \), if one writes \( q_1 = \tilde{q}_1\delta, \ q_2 = \tilde{q}_2\delta \) with \( \delta = (q_1, q_2) \) satisfying that \( (\delta, \tilde{q}_1) = 1 \), Poisson summation formula to the sum over \( n \) (with the modulus \( \tilde{q}_1\tilde{q}_2\delta \)) thus might produce the following bound for the triple sum that

\[
\ll \sum_{\delta \leq \mathcal{Q}} \sum_{\tilde{q}_1, \tilde{q}_2 \leq \delta} \sup_{0 < \|l\| \ll \tilde{q}_1\tilde{q}_2\delta/x^{1-\varepsilon}} \left| \sum_{\alpha \equiv \tilde{q}_1\tilde{q}_2\delta \bmod \tilde{q}_1\tilde{q}_2\delta} S(\alpha, c; \tilde{q}_1\delta)S(\alpha, c; \tilde{q}_2\delta) e\left(\frac{\alpha l}{\tilde{q}_1\tilde{q}_2\delta}\right) \right|,
\]

Notice, here, the inner-most sum vanishes, if \( l = 0 \), and it is necessary that \( \mathcal{Q}^2 > X \) as well. At this point, on applying Chinese remainder theorem, the sum over \( \alpha \) turns out to be

\[
\tilde{q}_1\tilde{q}_2\delta e\left(-\frac{\delta l\tilde{q}_2 c}{\tilde{q}_1}\right) \sum_{s \equiv \delta \bmod \delta} e\left(\frac{\tilde{q}_1 c \cdot (\tilde{q}_2 s + l)}{\tilde{q}_2\delta}\right)
\]

with \( \delta \equiv 1 \bmod \tilde{q}_1, \ \|l\| \equiv 1 \bmod \tilde{q}_1, \ \bar{\tilde{q}_1}\tilde{q}_1 \equiv 1 \bmod \delta \) and \( \bar{\tilde{q}_2} s + l(\tilde{q}_2 s + l) \equiv 1 \bmod \tilde{q}_2\delta \); trivially evaluating everything thus exactly leads to the term \( \mathcal{Q}^2\|a\|_2 \) in (2.10). \( \square \)

**Lemma 2.5.** Let the parameters \( X, \mathcal{Q}, c \), the form \( f \) and the complex sequences \( a = \{a_n\} \) be as in Lemma 2.4. Let \( W(x, y, z) \) be a smooth function supported on \([1/2, 5/2] \times [1/2, 5/2] \times [1/2, 5/2] \) with partial derivatives satisfying

\[
X^i Y^j Z^k \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} \frac{\partial^k}{\partial z^k} W\left(\frac{x}{X}, \frac{y}{Y}, \frac{z}{Z}\right) \ll_{i,j,k} 1
\]

for any \( X, Y, Z \geq 1 \) and integers \( i, j, k \geq 0 \). We then have

\[
\sum_{q \geq 1} \sum_{n \geq 1} a_n \sum_{m \geq 1} \frac{\lambda_f(m)}{\sqrt{m}} S(m - np, c; q) W\left(\frac{m}{X}, \frac{n}{Y}, \frac{q}{\mathcal{Q}}\right) \ll X^{\varepsilon} \left(\sqrt{\mathcal{Q}} + Q^2\right)\|a\|_2. \tag{2.11}
\]

**Proof.** To show the lemma, the initial procedure is to invoke the Cauchy-Schwarz inequality; we are thus led to evaluating

\[
\sum_{q_1, q_2 \geq 1} \sum_{n \geq 1} \sum_{m_1, m_2 \geq 1} \frac{\lambda_f(m_1)\lambda_f(m_2)}{\sqrt{m_1 m_2}} S(m_1 - np, c; q_1)
\]

\[
\times \bar{S}(m_2 - np, c; q_2) W\left(\frac{m_1}{X}, \frac{n}{Y}, \frac{q_1}{\mathcal{Q}}\right) W\left(\frac{m_2}{X}, \frac{n}{Y}, \frac{q_2}{\mathcal{Q}}\right). \tag{2.12}
\]

(1) First, let us begin with considering the generic terms \( q_1 = q_2 = q \), say. An application of Poisson summation to the \( n \)-sum with the modulus \( q \) yields an alternative form for (2.12):

\[
H \sum_{l_1 \in \mathbb{Z}} \sum_{q \sim \mathcal{Q}} \frac{1}{q} \sum_{m_1, m_2 \geq 1} \frac{\lambda_f(m_1)\lambda_f(m_2)}{\sqrt{m_1 m_2}} \mathcal{Y}_0(l_1, m_1, m_2, p, c; q, h) \mathcal{I}_0(l_1, m_1, m_2, h),
\]
where

\[ \mathcal{Y}^\dagger(l, m_1, m_2, p, c; q) = \sum_{\alpha \bmod q} S(m_1 - \alpha p, c; q) \overline{S(m_2 - \alpha p, c; q)} e \left( -\frac{\alpha l}{q} \right), \]

and

\[ \mathcal{I}_0(l, m_1, m_2) = \int_{\mathbb{R}} W\left( \frac{m_1}{X}, \xi, \frac{q}{Q} \right) W\left( \frac{m_2}{X}, \xi, \frac{q}{Q} \right) e \left( \frac{iH\xi}{q} \right) d\xi. \]

Notice that the exponential sum modulo \( q \) asymptotically equals

\[ q e \left( -\frac{m_1 pl}{q} \right) \sum_{\alpha \bmod q}^* e \left( \frac{(m_1 - m_2)\alpha + n \cdot \alpha - \overline{pl} - n\overline{p}}{q} \right), \]

where we have employed the relation involving Ramanujan sum that

\[ S(n, 0; q) = \sum_{ab \equiv q} \mu(a) \sum_{\beta \bmod q} e \left( \frac{\beta n}{b} \right). \quad (2.13) \]

Upon combining with Lemma 2.2, one thus sees that the zero-frequency shall contribute a bound by \( \ll H Q X^\varepsilon \) for any \( \varepsilon > 0 \). While, on the other hand, if \( Q > H \), one may find the non-zero frequencies will be indispensable to contribute a magnitude to (2.12). It can be demonstrated that, in this situation, the contribution, however, is estimated as \( \ll Q^2 X^\varepsilon \); this gives totally a quantity by

\[ \ll H Q X^\varepsilon + 1_{Q > H} Q^2 X^\varepsilon. \quad (2.14) \]

(2) Now, we are left with the non-generic case where \( q_1 \neq q_2 \). One writes \( q_1 = q'_1 h, q_2 = q'_2 h \), with \( (q_1, q_2) = h \) and \( (q'_1, q'_2) = 1 \). Notice that \( h \) is co-prime with one of factors \( q'_1, q'_2 \); without loss of generality, we assume \( (h, q'_1) = 1 \). The expression in (2.12) thus becomes

\[ \sum_{h \leq Q} \sum_{q_1, q_2 \leq Q/h} \sum_{n \geq 1} \sum_{m_1, m_2 \geq 1} \frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1 m_2}} S(m_1 - np, c; q'_1 h) \]

\[ \times \overline{S(m_2 - np, c; q'_2 h)} W\left( \frac{m_1}{X}, \frac{n}{H}, \frac{q'_1 h}{Q} \right) W\left( \frac{m_2}{X}, \frac{n}{H}, \frac{q'_2 h}{Q} \right). \quad (2.15) \]

Upon applying Poisson summation twice to the \( n \)-sum above, we thus arrive at

\[ H \sum_{l_2 \in \mathbb{Z}} \sum_{h \leq Q} \sum_{q_1, q_2 \leq Q/h} \frac{1}{q'_1 q'_2 h} \sum_{m_1, m_2 \geq 1} \frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1 m_2}} \]

\[ \times \mathcal{Y}^\dagger(l_2, m_1, m_2, p, c; q'_1, q'_2, h) \mathcal{I}^\dagger(l_2, m_1, m_2, h), \]
where the exponential sum $Y^\dagger$ and the resulting integral $I^\dagger$ are defined as

$$Y^\dagger(l, m_1, m_2, p, c; q'_1, q'_2, h) = \sum_{\alpha \mod q'_1q'_2h} S(m_1 - \alpha p, c; q'_1h)$$

$$\times S(m_2 - \alpha p, c; q'_2h) e\left(-\frac{\alpha l}{q'_1q'_2h}\right),$$

and

$$I^\dagger(l, m_1, m_2, h) = \int_{\mathbb{R}} W\left(\frac{m_1}{X}, \xi, \frac{q'_1h}{Q}\right) W\left(\frac{m_2}{X}, \xi, \frac{q'_2h}{Q}\right) e\left(\frac{lH\xi}{q'_1q'_2h}\right) d\xi.$$ 

Here, one finds that the zero-frequency $l_2 = 0$ does exist anymore. Indeed, upon recalling that $(h,q'_1) = 1$, one writes $\alpha = q'_1q'_2x + q'_2h\alpha y$, with $x \mod q'_2h$ and $y \mod q'_1$ such that $(x, q'_2h)$ and $(y, q'_1) = 1$; applying Chinese remainder theorem and (2.13), the sum over $\alpha$ thus essentially turns out to be

$$q'_1q'_2h e\left(\frac{m_1q'_2pl + q'_2h\alpha}{q'_1} + \frac{m_2q'_1pl}{q'_2h}\right)$$

$$\times \sum_{\alpha \mod h}^* e\left(\frac{(m_1 - m_2)q'_1q'_2\alpha}{q'_2h} - nq'_1\cdot q'_2\alpha - \frac{l}{q'_1} + nq'_2\alpha\right),$$

It thus follows from Lemma 2.2 that the display (2.15) is dominated by $\ll \varepsilon X^\varepsilon Q^4$, upon opening the Kloosterman sum. This, together with (2.14), shows the desired estimates in the parentheses of (2.11).

\[\square\]

### 3 Proof of theorem 1.1

This section is devoted to the proof of Theorem 1.1. We shall first manage to separate the variables $n, m$ by applying Lemma 2.3; in this paper, we shall employ an important new input - the ‘conductor lowering mechanism’ (see [29], [30] or the survey [32]). One may see that actually there holds the following

$$\delta(n) = \frac{1}{pQ} \sum_{q \leq Q} q \sum_{a \mod qp \atop (a, q) = 1} e\left(\frac{an}{qp}\right) \int_{\mathbb{R}} g(q, \tau) e\left(\frac{n\tau}{gQp}\right) d\tau; \quad (3.16)$$

while, the parameter $Q$ shall be taken as $Q = \sqrt{X/p}$. Now, for three smooth functions $U, V, R$, supported $[1/2, 5/2]$ with bounded derivatives, we shall detect the shift $l = m + pn$
via (3.16), which yields an alternative form for the triple sum in (1.3) as follows

\[
S(X, p, H) = \frac{X\sqrt{H}}{pQ} \int_{\mathbb{R}} \sum_{q \leq Q} \frac{g(q, \tau)}{q} \sum_{\gamma \mod pq \; l \geq 1}^{*} \frac{a_l}{\sqrt{l}} \exp \left( -\frac{\gamma l}{q} \right) V_\tau \left( \frac{l}{H} \right) \sum_{m \geq 1} \frac{\lambda_g(m)}{\sqrt{m}} \\
\times \exp \left( \frac{m\gamma}{pq} \right) U_\tau \left( \frac{m}{X} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} \exp \left( -\frac{n\gamma}{q} \right) R_\tau \left( \frac{n}{X} \right) \, d\tau,
\]

where

\[
U_\tau(m) = U(m) \exp \left( \frac{mX\tau}{pqQ} \right), \quad R_\tau(n) = R(n) \exp \left( -\frac{nX\tau}{pqQ} \right), \quad V_\tau(u) = V(u) \exp \left( -\frac{uH\tau}{qQ} \right).
\]

We shall proceed to distinguish whether \((\gamma, p) = 1\) or not in the analysis, so that we are led to three parts, i.e., the non-degenerate term \(S_{\text{Non-de.}}\), the degenerate term \(S_{\text{Deg.}}\), and the error term \(S_{\text{Err.}}\), which are respectively given by

\[
S_{\text{Non-de.}}(X, p, H) = \frac{X\sqrt{H}}{pQ} \int_{\mathbb{R}} \sum_{q \leq Q} \frac{g(q, \tau)}{q} \sum_{\gamma \mod pq \; l \geq 1}^{*} \frac{a_l}{\sqrt{l}} \exp \left( -\frac{l\gamma}{pq} \right) V_\tau \left( \frac{l}{H} \right) \\
\times \sum_{m \geq 1} \frac{\lambda_g(m)}{\sqrt{m}} \exp \left( \frac{m\gamma}{pq} \right) U_\tau \left( \frac{m}{X} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} \exp \left( -\frac{n\gamma}{q} \right) R_\tau \left( \frac{n}{X} \right) \, d\tau,
\]

\[
S_{\text{Deg.}}(X, p, H) = \frac{X\sqrt{H}}{pQ} \int_{\mathbb{R}} \sum_{q \leq Q} \frac{g(q, \tau)}{q} \sum_{\gamma \mod q \; l \geq 1}^{*} \frac{a_l}{\sqrt{l}} \exp \left( -\frac{l\gamma}{q} \right) V_\tau \left( \frac{l}{H} \right) \\
\times \sum_{m \geq 1} \frac{\lambda_g(1, m)}{\sqrt{m}} \exp \left( \frac{m\gamma}{q} \right) U_\tau \left( \frac{m}{X} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} \exp \left( -\frac{n\gamma}{q} \right) R_\tau \left( \frac{n}{X} \right) \, d\tau,
\]

and

\[
S_{\text{Err.}}(X, p, H) = \frac{X\sqrt{H}}{pQ} \int_{\mathbb{R}} \sum_{q \leq Q \; p|q} \frac{g(q, \tau)}{q} \sum_{\gamma \mod pq \; l \geq 1}^{*} \frac{a_l}{\sqrt{l}} \exp \left( -\frac{l\gamma}{pq} \right) V_\tau \left( \frac{l}{H} \right) \\
\times \sum_{m \geq 1} \frac{\lambda_g(m)}{\sqrt{m}} \exp \left( \frac{m\gamma}{pq} \right) U_\tau \left( \frac{m}{X} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} \exp \left( -\frac{n\gamma}{pq} \right) R_\tau \left( \frac{n}{X} \right) \, d\tau,
\]

One might see that, here, it suffices to consider \(S_{\text{Non-de.}}\); the same argument works for \(S_{\text{Err.}}\) which serves as a noisy term and for which we save more. We shall now now begin with \(S_{\text{Non-de.}}\); the analysis of the term \(S_{\text{Deg.}}\) will be postponed to the end of this paper.
3.1 Treatment of $S^{\text{Non-de.}}$

In this part, let us concentrate on the analysis of $S^{\text{Non-de.}}$. For any $\iota, \nu, \upsilon, \rho \in \mathbb{R}$, write

$$\mathcal{W}_\tau (\iota, \nu, \upsilon, \rho) = U_\tau (\iota) R_\tau (\nu) V_\tau (\upsilon) \eta_Q (\rho Q),$$

where $\eta_Q$ is a smooth function supported on $[1/2, 5/2]$ with bounded derivatives. One might find that the quantity we are focusing on is the following

$$X^{\sqrt{H}} \sup_{\tau \ll X^\varepsilon} \sup_{Q \leq Q} \sum_{q \geq 1 \atop (q, p) = 1} \frac{g(q, \tau)}{q} \sum_{l \geq 1} \frac{a_l}{\sqrt{l}} \sum_{m \geq 1} \frac{\lambda_l(m)}{\sqrt{m}} \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}}$$

$$\times \sum_{l \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} e \left( -\frac{n \gamma}{pq} \right) \mathcal{W}_\tau \left( \frac{m}{X}, \frac{n}{X}, \frac{l}{H}, \frac{q}{Q} \right).$$

(3.21)

We intend to apply the Voronoi formula, Lemma 2.1; we thus arrive at

$$X^{\sqrt{H}} \sup_{\tau \ll X^\varepsilon} \sup_{Q \leq Q} \sum_{q \geq 1 \atop (q, p) = 1} \frac{g(q, \tau)}{q} \sum_{l \geq 1} \frac{a_l}{\sqrt{l}} \sum_{m \geq 1} \frac{\lambda_l(m)}{\sqrt{m}} \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}}$$

$$\times S(n - m, pl; pq) \left\{ \mathcal{W}_{\tau}^\ast \left( \frac{m X}{p Q^2}, \frac{n X}{p Q^2}, \frac{l}{H}, \frac{q}{Q} \right) + \mathcal{W}_{\tau}^\ast \left( \frac{m X}{p Q^2}, \frac{n X}{p Q^2}, \frac{l}{H}, \frac{q}{Q} \right) \right\},$$

(3.22)

where, for any $\ast, \ast \in \{\varphi, \sharp\}$, each integral $\mathcal{W}_{\tau}^\ast \ast$ is defined as

$$\mathcal{W}_{\tau}^\ast \ast (\iota, \nu, \upsilon, \rho) = \eta_Q (\rho Q) V_\tau (\upsilon) \mathcal{H}_\ast \left( \frac{Q^2 l}{q^2}, \frac{m X}{p Q^2}, \frac{n X}{p Q^2}, \frac{l}{H}, \frac{q}{Q} \right) \mathcal{H}_\ast \left( \frac{Q^2 l}{q^2}, \frac{m X}{p Q^2}, \frac{n X}{p Q^2}, \frac{l}{H}, \frac{q}{Q} \right).$$

It is remarkable that, here, from (2.7) and (2.8), we have the identical crude estimate that $\mathcal{W}_{\tau}^\ast \ast \ll X^\varepsilon$ for any $\varepsilon > 0$; in this sense, it suffices to deal only with $\mathcal{W}_{\tau}^{\varphi, \varphi}$, upon noticing that the argument of the other terms (i.e., $\mathcal{W}_{\tau}^{\varphi, \ast}$, $\mathcal{W}_{\tau}^{\ast, \varphi}$ and $\mathcal{W}_{\tau}^{\ast, \ast}$) can follow similarly with it. One, however, on the other hand, sees that the inner-most sum modulo $pq$ can be converted into

$$pS(\overline{p} (n - m), l; q)$$

with $n \equiv m \mod p$. Now, if one writes $n = m + pk$ with $k \ll X^\varepsilon$, we finds that (3.22) is no more than

$$\frac{X^{\sqrt{H}}}{Q} \sup_{\tau \ll X^\varepsilon} \sup_{Q \leq Q} \sum_{m \ll p X^\varepsilon \atop k \ll X^\varepsilon} \sum_{q \geq 1 \atop (q, p) = 1} \frac{g(q, \tau)}{q} \sum_{l \geq 1} \frac{a_l}{\sqrt{l}} \sum_{m \geq 1} \frac{\lambda_l(m)}{\sqrt{m}}$$

$$\times \sum_{l \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} S(l, k; q) \mathcal{W}_{\tau}^{\varphi, \varphi} \left( \frac{m X}{p Q^2}, \frac{n X}{p Q^2}, \frac{l}{H}, \frac{q}{Q} \right).$$
At this point, an application of Lemma 2.3 shows that the RHS of the expression above is bounded by

\[ \ll \frac{X^{1+\varepsilon}}{Q} \sup_{\tau \ll \varepsilon} \sup_{Q \leq Q} \left( \sqrt{H} + 1_{Q^2 \gg H} \right) \|a\|_2 \]

\[ \ll X^\varepsilon \max \left( \sqrt{XH}, X \right) \|a\|_2, \tag{3.23} \]

upon recalling the value of \( Q \).

### 3.2 Treatment of \( S^{\text{Deg.}} \)

Now, let us have a look at the multiple-sum \( S^{\text{Deg.}} \). One might verify that \( S^{\text{Deg.}} \) is of the form

\[ \frac{X\sqrt{H}}{pQ} \sup_{\tau \ll \varepsilon} \sup_{Q \leq Q} \sum_{q \geq 1} \frac{g(q, \tau)}{q} \sum_{l \geq 1} a_l \sum_{m \geq 1} \frac{\lambda_2(m)}{\sqrt{m}} \]

\[ \sum_{n \geq 1} \left( -\frac{lp\gamma}{q} \right) \sum_{m \geq 1} \lambda_2(m) \]

\[ \sum_{q \geq 1} \left( -\frac{n\gamma}{q} \right) \mathcal{W}_\tau \left( \frac{n}{X}, m, \frac{u}{Q} \right) \]

with \( \mathcal{W}_\tau \) being as before. We will now proceed by applying the Voronoi formula, Lemma 2.1, again to transform the sums over \( n \) into the dualized form, so that we infer that the expression above should be controlled by

\[ \frac{X\sqrt{H}}{pQ} \sup_{\tau \ll \varepsilon} \sup_{Q \leq Q} \sup_{n \ll pQ^2 X^{-1+\varepsilon}} \sum_{q \geq 1} \frac{g(q, \tau)}{q} \sum_{l \geq 1} a_l \sum_{m \geq 1} \frac{\lambda_2(m)}{\sqrt{m}} \]

\[ \times S(m - lp, n; q) \left\{ \tilde{\mathcal{W}}_\tau^\flat \left( \frac{m}{X}, \frac{nX}{pQ^2}, \frac{l}{H}, \frac{q}{Q} \right) + \tilde{\mathcal{W}}_\tau^\sharp \left( \frac{m}{X}, \frac{nX}{pQ^2}, \frac{l}{H}, \frac{q}{Q} \right) \right\} , \]

where, for \( \ast \in \{ \flat, \sharp \} \), each integral transform \( \tilde{\mathcal{W}}_\tau^\ast \) is given by

\[ \tilde{\mathcal{W}}_\tau^\ast (\iota, \nu, \upsilon, \rho) = \eta_Q (\rho Q) V_\tau (\nu) U_\tau (\upsilon) \mathcal{M}_* \left( \frac{Q^2 \nu}{q^2}; R_\tau \right) \cdot \]

By Lemma 2.5 we thus deduce that

\[ S^{\text{Deg.}}(X, p, H) \ll \frac{X^{1+\varepsilon}}{pQ} \sup_{\tau \ll \varepsilon} \sup_{Q \leq Q} \left( \sqrt{H} + Q \right) \|a\|_2 \]

\[ \ll X^\varepsilon \left( \sqrt{\frac{XH}{p}} + \frac{X}{p} \right) \|a\|_2. \]
This leads to the estimates we would like to prove in Theorem 1.1 upon combining with (3.23).

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