Comparisons of inverse dynamics formulations in a spatial redundantly actuated parallel mechanism constrained by two point-contact higher kinematic pairs

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Comparisons of inverse dynamics formulations in a spatial redundantly actuated parallel mechanism constrained by two point-contact higher kinematic pairs

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Abstract A spatial redundantly actuated parallel mechanism (RAPM) constrained by two point-contact higher kinematic pairs (HKPs) has been designed, arising from the inspiration of mastication of human beings. In this paper, firstly, the constrained motions of the mechanism are described in detail, thereafter five models are formulated by the well-known Newton-Euler’s law, the Lagrangian equations and the principle of virtual work, to explore its rigid-body inverse dynamics. The symbolic results show that the model structures based on these approaches are quite different: the model via the Newton-Euler’s law well reflects the nature of the mechanism in terms of the constraint forces from HKPs, and the existence of reaction forces at the spherical joints in it is tightly dependent on the number of kinematic chains. In comparison, the constraint forces and the reaction forces at spherical joints do not appear in the four models from the latter two methods, where the actuating torques can be minimised in a closed-form by virtue of the pseudo-inverse method. Meanwhile, the torques are smaller than those from the Newton-Euler’s law. Further, by using the dynamics model of the non-redundantly actuated counterpart as the core in both the second models from the energy and virtual work related methods, they outperform the first models significantly in computational speed, respectively, being minimally laborious and offering directly the simplest possible algorithms. Finally, the comparisons between the dynamics models of the RAPM and its counterpart clarify that the HKP constraints greatly alter the model structures and raise the technical difficulties.

Keywords Inverse dynamics · Dynamics formulation · Higher kinematic pair · Spatial parallel mechanism · Redundant actuation

1 Introduction

In the human masticatory system, the mandible can perform masticatory behaviours in terms of motions and bite forces in the three-dimensional (3D) space, driven by the muscle contractions in different ways [1]. Simultaneously, it is always pivoted at the condyles via the left and right temporomandibular joints (TMJs), which are between the temporal bone of the skull and the mandibular condyles. From the viewpoint of mechanism, this system is inherently a spatial parallel mechanism (PM) with high actuation redundancies, since it has a greater number of mastication muscles than the degrees of freedom (DOFs) of the mandible. A spatial redundantly actuated parallel mechanism (RAPM) constrained by two higher kinematic pairs (HKPs) for mastication has been built in a bio-inspired manner, where the base is the skull, the six kinematic chains are the most primary chewing muscles, the end effector is the mandible, and the two HKPs are employed acting as TMJs at the two sides, respectively [2] [3]. A comprehensive review of the biomechanical findings about the human jaw structure and chewing muscles, masticatory robotics, and their applications was reported in Chapter 1 of [4].

PMs are superior than their serial counterparts in terms of larger stiffness, larger dynamic load carrying capacity, higher motion accuracy and lower inertia; as such, they are extensively employed in a variety of domains where these merits are of great interest [5] [6] [7] [8]. These advantages are further enhanced by actuation redundancy, which can eliminate singularities to expand the useful workspace, control antagonistic internal forces to alleviate backlashes, increase the energy efficiency, and raise the dynamic performance, etc., as shown in [9] [10] [11] [12] [13] [14]. The state of the art of their extensive
applications can be found in [15]. Generally, PMs with or with no redundant actuations are only composed of lower kinematic pairs, as in these above-mentioned publications; in this paper, nevertheless, the RAPM under study is characterised by two HKP constraints which bring two redundant actuations, being innovative and rare in the mechanism design.

The RAPM with two HKPs at hand has been designed to evaluate the time-varying dynamics of food textures in a biomimetic fashion during the chewing process as a robotic device in the food industry. Accordingly, in its practical applications, it is necessary to precisely reproduce the chewing behaviours of human beings using this mechanism, in terms of the 3D chewing motions and bite forces. As such, an accurate and computationally efficient inverse dynamics model is fundamental to its mechanical design, performance evaluation, real-time motion and/or force control.

The classical techniques including the Newton-Euler’s law, the Lagrangian formulation and the principle of virtual work proposed for general mechanical systems are broadly adopted to find the inverse dynamics solutions of RAPMs. In the first approach, the Newton’s law and Euler’s equation are directly applied to each isolated body in the mechanism after their free-body diagrams are drawn. By virtue of it, all the reaction wrenches in the joints are readily available. The latter two methods are related to the energy and the virtual work in the mechanisms, respectively. In the Lagrangian equation, the complete kinetic and potential energies are described by a set of generalised coordinates and their first-time derivatives; next a well-ordered procedure is followed to develop the final model. Unlike the Newton-Euler’s law, the elimination of all reaction wrenches is allowed since the beginning. In using the principle of virtual work, the sum of applied and inertia wrenches exerted at the mass centre of each movable body is firstly determined, and then the derived virtual displacement of each body being compatible with the closed-loop constraints are computed. In the following, the publications about this area of interest in their own right, i.e., rigid body inverse dynamics of RAPMs are briefly reviewed.

The dynamics of a planar 3-DOF RAPM was analysed based on the Newton-Euler’s law in [16]. Employing this law to each part in the manipulator would result in a tedious work because of the large number of movable mechanical parts. Instead, by utilising the left-right symmetry of the manipulator, the law was applied to the left and the right bodies, and the end effector, respectively, to obtain the dynamics model of the entire manipulator. The left and the right bodies were actually the parts integrated at the left and the right sides of the manipulator, respectively. Hence, a simplified dynamics model was established, by virtue of which the dynamic performance of redundant actuation was studied. The maximum dynamic load-carrying capacity of a planar 3-DOF RAPM was analysed also via the Newton-Euler’s law in [17]. Based on the dynamics model, the linear programming problem was formulated to explore the capacity and the role of actuation redundancy. In [18], the inverse dynamics of the 4RRR planar RAPM was solved to achieve the shake force/moment balancing. The underlined letter indicates that it is the active joint in the mechanism all through this paper. The required torque encompassed three individual parts: the motion of the end effector, the rotation of the link whose one end situated the actuator, and the rotation of the intermediate link that connected the end effector and the first link, respectively. The first part was derived using the instantaneous power equivalence between the actuator and the end effector; the second part was computed by the Euler’s equation, while the third part was quite sophisticated.

In [19], the Lagrange-D’Alembert formulation was employed to derive the dynamics equations of a planar 2RRR/RR RAPM. Thereafter, the model-based motion control was capable of being realised in the task space. In [20], the Lagrange formulation with unknown Lagrangian multipliers was used to address the dynamics of the same RAPM from [19]. The multipliers representing the magnitude of constraint forces were eliminated using the null space of the differential matrix of the closed-loop
constrained equation. The identical procedure was also utilised in a planar 2-DOF RAPM with parallelograms in the chains for the optimal design in [21]. In [22], firstly the Newton-Euler’s law was applied to derive the dynamics of the chain of a variable mass 3RRR planar RAPM, then the Lagrange multipliers technique was used to establish the dynamics model of the entire mechanism. The classical partitioning method was applied to compute the multipliers. Based on it, an effective proportional-derivative feedforward-feedback controller was designed. A 5UPS/PRPU RAPM with five DOFs has been designed to deal with the coordination of the driving forces in [23]. The Lagrangian equations of the second type without Lagrange multipliers was employed to solve the inverse dynamics. On this basis, the weighted optimisation principle was employed to distribute the driving force. In [24], the dynamics model was formatted for a 4PUS/PPPU RAPM with five DOFs and seven actuators using the Lagrange method first in the task space. Then by virtue of the generalized left inverse of the Jacobian matrix mapping the velocity of the end effector into that of the active joints, this model was ultimately built in the joint space. The computed-torque synchronization controller was designed based on it. To minimise the energy consumption of a planar PM, redundant actuations were employed in [14]. The dynamics model was established using the Lagrange method. The dynamics model-based experiments showed that actuation redundancy effectively reduced the energy consumption. With the same method, the dynamics equations of a 3-DOF spatial RAPM were written in [25], then the terminal sliding mode control was realised.

In [26], the principle of virtual work was used to build the model of the same planar 3-DOF RAPM from [16]. On this foundation, a position and force switching control strategy was then designed. Revealing the superiority of redundant actuations in the dynamic performance of an 8PSS spatial RAPM required a quantitative comparison with its non-redundantly actuated counterpart. To this end, in [12], this principle was employed to solve the inverse dynamics. Based on it, a series of dynamic performance indices were proposed. Numerical results showed that the mechanism with redundant actuations outperformed its counterpart in these indices. The principle of virtual work was used to derive the dynamics model of a 4PSS/PU spatial RAPM with three DOFs in [27], to measure the maximum angular and translational accelerations, respectively, brought by actuation redundancy. In [13], in order to evaluate the dynamics performance of planar PMs with actuation and kinematic redundancies, an inverse dynamics model was a necessity. Firstly, the Newton-Euler’s law was used to determine the resultant wrenches acting at each body. Then the principle of virtual work was used to find the input torques. This principle expressed by generalised coordinates was used to establish the dynamics model of RAPMs in [28]. On this basis, a neural network synchronous controller was designed to coordinate internal forces and driving forces. A 6PUS+UPU RAPM with 5 DOFs was employed as a case study. This principle in combination with the screw theory were together employed in a 4PPPS RAPM with twelve actuators and six redundant actuations in [29], to address the high redundancy efficiently.

Apart from the three conventional methods, the more recent Udwadia-Kalaba theory and the natural orthogonal complement method have been employed to solve the inverse dynamics of RAPMs in [30] [31] [32].

From this short review, it is remarked that the inverse dynamics solution is quite fundamental, and is incorporated into publications about the study of performance evaluation and control design of RAPMs mainly consisting of lower kinematic pairs. In our previous work [3], an initial attempt on the rigid body dynamics of the RAPM with two HKPs at hand has been made via the hybrid of the Lagrangian equations and the Newton-Euler’s law, which is quite complex and thus error-prone: the forces at HKPs without friction effects are ideal constraints which must not be in the Lagrangian formulation, whilst they must be considered in the Newton-Euler’s law. In fact, the model can be established by the two formulations independently and separately, which is the first motivation for writing this paper. Meanwhile, the
influence of the HKP constraints onto the modelling process, the final model structure, the numerical results and the computational cost has not been figured out clearly. The comprehensive study on these unknown areas of interest is able to facilitate the mechanical design, the performance evaluation and improvement, as well as real-time control doubtlessly. As a consequence, studying them requires an exploration of inverse dynamics using different methods in-depth, which is the second motivation for carrying out the related study.

In this paper, it is assumed that all the bodies including the HKP constraints, rotational and spherical joints are rigid, frictionless and free of clearances. The inertia of the spherical joints is quite small and then is not considered in the formulation. The sequel begins with a detailed description of the mechanism. Next, the constrained motion of the end effector and the kinematics of the chains are derived. Thereafter, five dynamics models are built analytically from the Newton-Euler’s law, the Lagrangian formulation and the principle of virtual work. Finally, numerical computations are conducted to verify and compare the rightness and computational efficiency of the models. The role of HKP constraints in the modelling is not only investigated by the two models from the latter two methods, respectively, but also comparatively examined with the 6RSS PM presented in Chapter 4 of [4], which does not have HKPs.

The main contributions in this paper are:

1. The inverse dynamics solution of the constrained RAPM has been explored deeply via the Newton-Euler’s law, the Lagrangian formulation and the principle of virtual work, respectively.

2. Under the latter two methods, using the dynamics model of the 6RSS PM as the core of the RAPM’s model can considerably alleviate the computational demands.

3. The insight into the influence of HKP constraints to the inverse dynamics has been provided clearly in terms of the model structure, the numerical results and the computational cost.

2 The robotic mechanism

The kinematic diagram of the RAPM constrained by two HKPs is illustrated in Fig. 1. The maxilla (i.e., the base) is fixed on the ground and the movable mandible (i.e., the end effector) is connected to the base by six independent kinematic chains. The maxilla, to which the inertia frame \{S\} is assigned, is not shown in the figure for a clear exhibition of movable bodies. This frame consists of a horizontal \(X_S - Y_S\) plane perpendicular to the vertical \(Z_S\) axis. A frame \{M\} is established at the mass centre \(O_M\) of the end effector. The origins and orientations of \{S\} and \{M\} overlap when the mechanism is at the home position, that is, the maxilla and the mandible are in the occlusal state. The origin \(O_M\) is used as the reference point to describe the mandibular translations, and its orientations with respect to \{S\} are described by \(XYZ\) Euler angles, that is, \(\alpha, \beta\) and \(\gamma\) around the three axes of \{M\}. The layout of the six chains is in accordance with the six most primary masticatory muscles of human beings. Each chain contains a rotational actuator fixed onto the base, whose driving shaft connects a crank \(G_S, (i=1, \ldots, 6)\) with a rotational joint at \(G_i\), and a coupler \(S_M\) that joins the crank and the end effector via two spherical joints at its two ends \(S_i\) and \(M_i\), respectively. The rotation of the \(i\)th actuator with respect to \{S\} is described by the actuator frame \{C_i\} attached at \(G_i\). In it, the \(X_{C_i}\) axis is directed from \(G_i\) to \(S_i\), the \(Z_{C_i}\) axis runs through the driving shaft of the actuator, and the \(Y_{C_i}\) axis completes the frame, obeying the right-hand rule. A frame \{N_i\} is attached at the mass centre \(E_i\) of \(S_M\) to describe its motions with respect to \{S\}. The \(X_{N_i}\) axis points from \(S_i\) to \(M_i\), the \(Y_{N_i}\) axis is parallel to the cross product of two unit vectors defined along the \(X_{N_i}\) and \(X_S\) axes, and the \(Z_{N_i}\) axis is defined by the right-hand rule.
Two HKPs modelling the left and right TMJs are formed by the two condyle balls being in contact with the articular surfaces. In the mechanism prototype, the two contacts between the left and right condyles of the mandible and the maxilla at TMJs, are realised by the two condyle balls being always constrained within a curved condylar socket, as shown in Fig. 2. The width of the socket equals the diameter of the condyle ball; thus, it can always guarantee the point-contact during the movement of the mandible. By virtue of this design, the motion of the condyle ball centre is always constrained onto a surface, which is offset from the upper and lower surfaces of the socket by the ball radius. Thereupon, it is clear that the end effector is actuated by six chains and constrained by the environment at the two HKPs simultaneously.
3 Kinematics of the mechanism

3.1 Constrained motions of the end effector

A second-order surface has been designed according to [33], being used as the workspace of the centre of the condylar ball as in [2] [3] [34]. Its cross section is identical along the $Y_S$ axis in $\{S\}$, being in accordance with that in [1], and the range of the lateral movement of the lower jaw has been determined from [35]. However, on the one hand, when the mechanism tracks the real chewing trajectories of healthy human subjects, the condylar ball mainly slips along the curved surface surrounded by blue dashed lines as in Fig. 2, which can be easily approximated by a flat one. On the other hand, it is quite difficult to derive the analytical expressions of the parasitic motions in the RAPM under the second-order surface. At these regards, in this paper where the chewing system is explored from the viewpoint of mechanical dynamics and with an emphasis on the constrained dynamics of the end effector, the surfaces in $\{S\}$ where the left and right condyle ball centres $T_L$ and $T_R$ slide on are designed as flat (unit: mm):

\[
\begin{align*}
Z_L &= p_1X_L + p_2, \quad p_3 \leq X_L \leq p_4, \quad p_5 \leq Y_L \leq p_6, \\
Z_R &= p_1X_R + p_2, \quad p_3 \leq X_R \leq p_4, \quad -p_5 \leq Y_R \leq -p_6, \\
p_1 &= -1.1, \quad p_2 = 13.215, \quad p_3 = -27.65, \quad p_4 = -14.65, \quad p_5 = 69, \quad p_6 = 75
\end{align*}
\]
From the Kutzbach-Grübler criterion, the mechanism now has four DOFs, but the information on which four DOFs to choose is not given. They are to be derived from a rigorous computation below: the coordinates of $T_i (i=L,R)$ in $\{S\}$ can be expressed as

$$O_{S_i}T_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} = O_{S_M}T_i + \hat{S}R\cdot O_{M_i}T_i,$$  \hspace{1cm} (2)

where $O_{S_M} = [X \ Y \ Z]^T$ denotes the $3 \times 1$ position vector of $O_M$ in $\{S\}$, $\hat{S}R = R_x(\alpha) \cdot R_y(\beta) \cdot R_z(\gamma)$ is the rotation matrix from $\{S\}$ to $\{M\}$, and $R_x(\alpha), R_y(\beta), R_z(\gamma)$ are three rotation matrices about the $X_M, Y_M,$ and $Z_M$ axes by $\alpha, \beta,$ and $\gamma$, respectively. It is worth noting that in this paper, a matrix/vector/scalar in local frames owns a leading superscript on its left to denote the specific frame it refers to, but those in $\{S\}$ omit their superscripts for the sake of convenience and clarity. The six motion variables of the end effector are grouped and expressed as

$$X_{EE} = [X \ Y \ Z \ \alpha \ \beta \ \gamma]^T \hspace{1cm} (3)$$

From Eq.(2), one can obtain

$$X_L = X + \hat{S}R_{(1,1)} \cdot O_{M_i}T_i, \hspace{1cm} X_R = X + \hat{S}R_{(1,3)} \cdot O_{M_i}T_i,$$

$$Z_L = Z + \hat{S}R_{(3,1)} \cdot O_{M_i}T_i, \hspace{1cm} Z_R = Z + \hat{S}R_{(3,3)} \cdot O_{M_i}T_i \hspace{1cm} (4)$$

where $\hat{S}R_{(i,j)}$ is the $i$th row of $\hat{S}R$. Putting Eq.(4) into Eq.(1) produces

$$Z + \hat{S}R_{(3,1)} \cdot O_{M_i}T_L = p_1 \cdot (X + \hat{S}R_{(1,1)} \cdot O_{M_i}T_L) + p_2$$

$$Z + \hat{S}R_{(3,1)} \cdot O_{M_i}T_R = p_1 \cdot (X + \hat{S}R_{(1,3)} \cdot O_{M_i}T_R) + p_2 \hspace{1cm} (5)$$

In view of the left-right symmetry of $O_{M_i}T_L$ and $O_{M_i}T_R$ in $\{M\}$, a summation and a subtraction of the two equations in Eq.(5) sidewise yield

$$Z = p_1X + p_2 + \left(p_1 \cdot \hat{S}R_{(1,1)} - \hat{S}R_{(1,3)}\right) \cdot \begin{bmatrix} \mu O_{M_i}T_{L(i)} \\ \mu O_{M_i}T_{R(i)} \end{bmatrix}$$

$$\gamma = -a \tan\left(\frac{sa \psi}{p_1c\beta + csa\beta}\right)\hspace{1cm} (6)$$

where $\mu O_{M_i}T_{L(i)}$ and $\mu O_{M_i}T_{R(i)}$ are the first and third terms of $\mu O_{M_i}T_L$, respectively. From these computations, it is found $Z$ and $\gamma$ are transferred from DOFs to parasitic motions and they are functions of $q_{EE}$, which is a $4 \times 1$ vector by grouping four DOFs as

$$q_{EE} = [X \ Y \ \alpha \ \beta]^T \hspace{1cm} (7)$$

It actually constitutes the task space of the mechanism. To characterise the instantaneous configuration of the mechanism, Eq.(3), or both Eqs.(6) and (7) ad hoc are needed. In other words, it can still perform motions in six directions with four DOFs and two parasitic motion variables. Regarding this, redundant actuations in the mechanism are essentially caused by constraints from the base directly onto the end
effector, which is completely different from the two methodologies mentioned in [10]. It is also worth noting that though the workspace of the centre of the condylar ball is simplified as a flat surface as in Eq.(1), there is still a strongly nonlinear and sophisticated relationship between \( Z/\gamma \) and \( q_{EE} \) in Eq.(6).

After figuring out the DOFs of the end effector, its motions can be defined below. The angular velocity is

\[
\omega_{EE} = M_{ob} \cdot \dot{q}_{EE}
\]

where

\[
M_{ob} = M_{oa} \cdot M_j
\]

\[
M_{oa} = \begin{bmatrix} 0_3 & R_o \end{bmatrix}
\]

\[
R_o = \begin{bmatrix} 1 & 0 & s\beta \\ 0 & c\alpha & -s\alpha c\beta \\ 0 & s\alpha & -c\alpha c\beta \end{bmatrix}
\]

\( 0_3 \) is the \( 3 \times 3 \) zero matrix, and \( M_j \) denotes the \( 6 \times 4 \) Jacobian matrix between \( X_{EE} \) and \( q_{EE} \), namely

\[
M_j = \text{Jacobian}(X_{EE}, q_{EE})
\]

\( \dot{X}_{EE} = M_j \cdot \dot{q}_{EE} \)

The translational velocity of the end effector is expressed by

\[
V_{0a} = M_{ib} \cdot \dot{q}_{EE}
\]

where \( M_{ib} = M_{ia} \cdot M_j \), \( M_{ia} = \begin{bmatrix} E_3 & 0_3 \end{bmatrix} \), and \( E_3 \) is the \( 3 \times 3 \) identity matrix. The angular and translational accelerations of the end effector is found by the differentiation of Eqs.(8) and (10) with respect to time, respectively:

\[
\dot{\omega}_{EE} = \frac{\partial M_{ib}}{\partial q_{EE}} \cdot (q_{EE} \otimes E_3) \cdot \dot{q}_{EE} + M_{ob} \cdot \ddot{q}_{EE}
\]

\[
\dot{V}_{0a} = \frac{\partial M_{ib}}{\partial q_{EE}} \cdot (\dot{q}_{EE} \otimes E_3) \cdot \dot{q}_{EE} + M_{ib} \cdot \ddot{q}_{EE}
\]

where \( \otimes \) is the Kronecker product.

3.2 Kinematics of the \( i \)th chain

The inverse kinematics of the mechanism, i.e., \( \theta = \vartheta(q_{EE}) \) \( \theta = [\theta_1 \ldots \theta_i]^T \) that constitutes a system of six decoupled equations expressed by \( q_{EE} \), has already been derived in Section 3.2 of [34]. Nevertheless, the motions of the coupler \( SM_i (i=1, \ldots, 6) \) are still needed for its rigid-body dynamics. Due to the two spherical joints at \( S_i \) and \( M_i \), the coupler can rotate around the three orthogonal axes of \( \{N_i\} \). The rotation around \( X_{N_i} \) axis is a passive DOF for it is not controllable; this rotational range is quite small thanks to the physical restrictions from the used spherical joints in the mechanical design, however. At these regards, it is assumed that there is no axial rotation in the coupler. Two Euler angles \( \beta_i \) and \( \gamma_i \) around the \( Y_{N_i} \) and \( Z_{N_i} \) axes, respectively, are used to express the rotation of \( SM_i \) in \( \{S\} \) in terms of the two rotational matrices \( R_y(\beta_i) \) and \( R_z(\gamma_i) \). Thereafter, the coordinate vector of the coupler can be expressed
as

\[ S_i M_i = n_{sR} \cdot R_i(\beta_i) \cdot R_Z(\gamma_i) \cdot \begin{bmatrix} \|S_i M_i\| \\ 0 \\ 0 \end{bmatrix} \]  

(12)

where \( n_{sR} \) is the orientation of \( S_i M_i \) in \( \{S\} \) at the initial configuration of the mechanism at hand, and \( \|S_i M_i\| \) is the length of \( S_i M_i \). From the geometry of the mechanism in \( \{S\} \), the vector of the coupler can be found from the difference of the position vector of \( M_i \) and \( S_i \):

\[ S_i M_i = O_{si} O_M + O_M M_i - O_{si} G_i - G_i S_i \]

(13)

where

\( O_{si} M_i = \hat{S} R \cdot \hat{M} O_{si} M_i \) and \( \hat{M} O_{si} M_i \) is the coordinate vector of \( M_i \) in \( \{M\} \);

\( O_{si} G_i \) is the constant position vector of \( G_i \) in \( \{S\} \);

and \( G_i S_i = c_{\theta_i} R \cdot R_Z(\theta_i) \cdot \begin{bmatrix} \|G_i S_i\| \\ 0 \\ 0 \end{bmatrix} \), in which \( c_{\theta_i} R \) is the orientation of \( G_i S_i \) in \( \{S\} \) at the initial configuration of the mechanism, \( R_Z(\theta_i) \) is the rotation matrix about the \( Z_{\theta_i} \) axis by \( \theta_i \), and \( \|G_i S_i\| \) is the length of \( G_i S_i \).

Substituting Eq.(12) and \( \theta_i = \theta_i(q_{EE}) \) into Eq.(13) produces

\[ \beta_i = -a \tan \frac{n_{s3}}{n_{s2}} \]

\[ \gamma_i = a \sin n_{s3} \]

(14)

where \( \begin{bmatrix} n_{s3} \\ n_{s2} \end{bmatrix} = n_{sR}^{-1} \cdot \hat{S} M_i \). For the sake of convenience, a \( 3 \times 1 \) generalised vector defined as

\[ q_i = \begin{bmatrix} \theta_i \\ \beta_i \\ \gamma_i \end{bmatrix} \]

consists of the joint space of the \( i \)th chain and it is adopted to completely specify its configuration. It is highlighted that \( q_i \) is the function of \( q_{EE} \), i.e., \( q_i = q_i(q_{EE}) \).

To derive the relationship between the first-time derivative of \( q_i \) and \( q_{EE} \), Eq.(13) can be rewritten as

\[ O_{si} G_i + G_i S_i + S_i M_i = O_{si} O_M + O_M M_i \]

(15)

The left- and right-hand sides can be expressed by \( q_i \) and \( q_{EE} \), respectively. The first-time derivative of Eq.(15) yields

\[ M_{si} \cdot \dot{q}_i = M_{2i} \cdot \dot{q}_{EE} \]

(16)
where
\[
M_{1i} = \text{Jacobian } \left( O_{S_i} G_i + G_i S_i + S_i M_i, \quad q_i \right)
\]

\[
M_{2i} = \text{Jacobian } \left( O_{S_i} O_{M_i} + O_{M_i} M_i, \quad q_{EE} \right)
\]

Moreover, one can find that
\[
\dot{q}_i = M_{1i} \cdot \ddot{q}_{EE}
\]

where \( M_{3i} = M_{1i}^T \cdot M_{2i} \), and differentiating Eq.(16) produces that
\[
M_{1i} \cdot \dot{q}_i + M_{2i} \cdot \ddot{q}_i = M_{2i} \cdot \ddot{q}_{EE} + M_{2i} \cdot \dddot{q}_{EE}
\]

Upon substitution of Eq.(17) into Eq.(18), it gives rise to the second-time derivative of \( q_i \) as
\[
\ddot{q}_i = M_{1i} \cdot \dddot{q}_{EE} + M_{3i} \cdot \dddot{q}_{EE}
\]

where \( M_{4i} = M_{1i}^T \cdot (M_{3i} - M_{1i} \cdot M_{3i}) \). So far, \( \dot{q}_i \) and \( \ddot{q}_i \) have been derived as quantities intimately associated with the motion of the \( i \)th chain, rather than merely with its configuration.

The rotational velocity of \( S_i M_i \) is
\[
\omega_{S,M_i} = s_n R \cdot R_{nS,M_i} \cdot \dot{q}_i = s_n R \cdot R_{nS,M_i} \cdot M_{3i} \cdot \ddot{q}_{EE}
\]

where \( R_{nS,M_i} = \begin{bmatrix} 0 & 0 & s \beta_i \\ 0 & 1 & 0 \\ 0 & 0 & c \beta_i \end{bmatrix} \), and its rotational acceleration is
\[
\ddot{\omega}_{S,M_i} = s_n R \left( \frac{\partial R_{nS,M_i}}{\partial \dot{q}_i} \left( \dot{q}_i \otimes E_i \right) \cdot \dot{q}_i + R_{nS,M_i} \cdot \ddot{q}_i \right)
\]

Putting Eqs.(17) and (19) into Eq.(21a), \( \ddot{\omega}_{S,M_i} \) can also be expressed by \( q_{EE} \cdot \dot{q}_{EE} \cdot \ddot{q}_{EE} \) as
\[
\ddot{\omega}_{S,M_i} = s_n R \left( \frac{\partial R_{nS,M_i}}{\partial \dot{q}_i} \left( (M_{3i} \cdot \dddot{q}_{EE}) \otimes E_i \right) \cdot (M_{3i} \cdot \dddot{q}_{EE}) + R_{nS,M_i} \cdot (M_{3i} \cdot \dddot{q}_{EE} + M_{4i} \cdot \dddot{q}_{EE}) \right)
\]

which is more complicated with more symbols and arithmetic operations than Eq.(21a).

The coordinate vector of the mass centre \( E_i \) in \( \{ S \} \) is
\[
O_{S} E_i = O_{S} G_i + G_i S_i + S_i E_i = O_{S} G_i + s_n^2 R \cdot R_{x} (\theta) \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{\partial \omega_{S,M_i}}{\partial \dot{q}_i} \end{bmatrix} + s_n R \cdot R_{x} (\beta_i) \cdot R_{y} (\gamma_i) \cdot \begin{bmatrix} 0.5 \cdot \omega_{S,M_i} \end{bmatrix}
\]

The translational velocity and acceleration of \( E_i \) can then be found by the differentiation of Eq.(22) with respect to time once and twice, respectively
\[
V_{E_i} = J_{E_i} \cdot \dot{q}_i
\]

\[
\dot{V}_{E_i} = J_{E_i} \cdot \ddot{q}_i + J_{E_i} \cdot \dddot{q}_i
\]
where

\[ \mathbf{J}_E = \text{Jacobian}(\mathbf{O}_E, \mathbf{q}_E) \]

\[ \dot{\mathbf{J}}_E = \frac{\partial \mathbf{J}_E}{\partial \mathbf{q}_E} \left( \dot{\mathbf{q}}_E \otimes \mathbf{E}_3 \right) \]

Upon substitution of Eqs. (17) and (19) into Eq. (23), it yields

\[ \mathbf{V}_E = \mathbf{J}_E \cdot \mathbf{M}_{3i} \cdot \dot{\mathbf{q}}_{EE} \]

\[ \dot{\mathbf{V}}_E = \left( \mathbf{J}_E \cdot \mathbf{M}_{3i} + \mathbf{J}_E \cdot \mathbf{M}_{4i} \right) \dot{\mathbf{q}}_{EE} + \mathbf{J}_E \cdot \mathbf{M}_{3i} \cdot \ddot{\mathbf{q}}_{EE} \]  

(24)

With this kinematics background provided so far in the mechanism of interest, one is in position to write equations of motion based on the three inverse dynamics methods below.

4 Newton-Euler formulation

The Newton-Euler’s law is first applied to each body including the end effector, the coupler and the crank in each chain. The equations of motion are subsequently ordered suitably, arriving at the dynamics model of the entire mechanism at hand.

4.1 The end effector

The free-body diagram of the end effector is shown in Fig. 3. The forces acting on it include:

the constrained forces \( \mathbf{F}_{rL} \) and \( \mathbf{F}_{rR} \) at the left and right condylar balls, respectively;

the six reaction forces \( \mathbf{F}_{M1} (i = 1, \ldots, 6) \) which are exerted from the six couplers;

its gravitational force \(-m_{EE} \cdot \mathbf{g}\) at the mass centre \( \mathbf{O}_{M} \), in which \( m_{EE} \) is its mass and \( \mathbf{g} = [0 \ 0 \ 9800] \text{mm/s}^2 \) is the gravitational acceleration vector;

and the bite force \( \mathbf{F}_B \) at the point \( B \) on the right molar which is predefined in inverse dynamics.
Thanks to the specific design for the workspace of the condylar ball centres, $F_{L}$ and $F_{R}$ can be expressed as

$$F_{L} = M_{H} \cdot F_{L} \quad F_{R} = M_{H} \cdot F_{R}$$

where $M_{H} = \begin{bmatrix} -p_{1} & 0 & 1 \end{bmatrix}'$, and $F_{i}$ ($i = L, R$) is the unknown magnitude of the constrained force at HKPs along $Z_{S}$ axis. If $F_{Z}$ is positive/negative, it means that the condylar ball is receiving constrained forces from the lower/upper surface of the condylar socket. The sagittal view of the condylar ball and the constrained force are given in Fig. 4, indicating the flat lower condylar surface is exerting a constraint force onto the condylar ball at that time instant.

The Newton-Euler’s law of the end effector gives

$$\begin{align*}
\sum_{i=1}^{6} F_{mi} + F_{b} - m_{EE} \cdot g + M_{H} \cdot (F_{xL} + F_{xR}) &= m_{EE} \cdot \vec{V}_{O_{M}} \\
\sum_{i=1}^{6} O_{M} M_{i} \times F_{mi} + O_{M} B \times F_{b} + O_{M} T_{L} \times M_{H} \cdot F_{xL} + O_{M} T_{R} \times M_{H} \cdot F_{xR} &= I_{EE} \cdot \vec{\alpha}_{EE} + \alpha_{EE} \times (I_{EE} \cdot \vec{\omega}_{EE})
\end{align*}$$

where $I_{EE} = S_{M} R \cdot M_{EE} \cdot S_{M} R'$, and $M_{EE}$ is the 3x3 moment of inertia taken about the mass centre $O_{M}$ and expressed in $\{M\}$. Eq.(26) can be rewritten in a compact form as

$$M_{3b} \cdot F_{M_{i,4}} = M_{3b} - M_{4b} \cdot F_{Z}$$

where

$$M_{3b} = \begin{bmatrix}
E_{3} & \cdots & E_{3} \\
O_{M} M_{1} \times & \cdots & O_{M} M_{6} \times
\end{bmatrix}_{6 \times 18} \quad F_{M_{i,4}} = \begin{bmatrix}
F_{mi} \\
\vdots \\
F_{M_{6}}
\end{bmatrix}_{18 \times 1}$$

$$M_{3b} = \begin{bmatrix}
I_{EE} \cdot \vec{\alpha}_{EE} + \alpha_{EE} \times (I_{EE} \cdot \vec{\omega}_{EE}) \\
E_{3} \cdot O_{M} B \times F_{b} \cdot M_{4b} = \begin{bmatrix}
E_{3} & \cdots & E_{3} \\
O_{M} T_{L} \times & \cdots & O_{M} T_{R} \times
\end{bmatrix}_{6 \times 18} (E_{2} \otimes M_{H}) \cdot F_{Z} = \begin{bmatrix}
F_{Z} \end{bmatrix}_{18 \times 1}
\end{bmatrix}$$

Fig. 4 Sagittal view of the condylar ball and the constrained force
4.2 The coupler $S_M_i$

The forces acting at the $i$th coupler $S_M_i$ is shown in Fig. 5, and its Newton-Euler’s formulation is

$$F_S - F_M = m_{S,M_i} \cdot \dot{V}_{E_i} + m_{S,M_i} \cdot g$$
$$E_S \times F_S + E_M \times (- F_M) = I_{S,M_i} \cdot \dot{\omega}_{S,M_i} + \omega_{S,M_i} \times (I_{S,M_i} \cdot \omega_{S,M_i})$$ (28)

where $m_{S,M_i}$ is the mass of $S_M_i$, $I_{S,M_i} = \frac{S}{N} \cdot R_i\cdot \dot{W}_{S,M_i}^T$, $N$ is the inertia tensor of $S_M_i$ with respect to its mass centre $E_i$ and expressed in $\{N_i\}$, and $\frac{S}{N} = \frac{S}{N} \cdot R_i \cdot \beta_i \cdot R_j \cdot \gamma_i$.

Combining the two equations in Eq.(28) yields

$$M_{S_i} \times F_{M_i} = E_{S,M_i}$$ (29)

where $E_{S,M_i} = I_{S,M_i} \cdot \dot{\omega}_{S,M_i} + \omega_{S,M_i} \times (I_{S,M_i} \cdot \omega_{S,M_i}) + 0.5 \cdot m_{S,M_i} \cdot S_M_i \times (\dot{V}_{E_i} + g)$. Among the three equations in Eq.(29), arbitrarily only two are independent. The first two are chosen for the following computation. For the six couplers one can write that

$$M_{5b} \cdot F_{M_{5,6}} = M_{4b}$$ (30)

where

$$M_{5b} = \begin{bmatrix} (M_{S_i} \times)_{(1,2),i} \\ \vdots \\ (M_{S_i} \times)_{(12,1,6)} \end{bmatrix}, \quad M_{4b} = \begin{bmatrix} E_{S,M_i,(1,2)} \\ \vdots \\ E_{S,M_i,(1,2)} \\ \vdots \end{bmatrix}_{12,i}$$

In it, $(M_{S_i} \times)_{(i,2)}$ denotes the first two rows of the skew-symmetric matrix $M_{S_i} \times$, and $E_{S,M_i,(1,2)}$ is a $2 \times 1$ vector containing the first two rows in $E_{S,M_i}$. For the six couplers, from the Newton’s equation in Eq.(28), it can also be obtained that
\[ F_{S_{i,a}} = F_{M_{i,a}} + M_{S_{i,b}} \]  

(31)

where \( F_{S_{i,a}} \) and \( M_{S_{i,b}} \) are given by

\[
\begin{bmatrix}
F_{S_1} \\
\vdots \\
F_{S_{18}}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
m_{S_{1,a}}(\dot{\mathbf{V}}_{S_{i}} + \mathbf{g}) \\
\vdots \\
m_{S_{18,a}}(\dot{\mathbf{V}}_{S_{i}} + \mathbf{g})
\end{bmatrix}
\]

\[ i = 1, \ldots, 18 \]

4.3 The crank \( G_{S_i} \)

The crank \( G_{S_i} \) can only rotate around the \( Z_i \) axis of frame \( \{C_i\} \). For the sake of convenience, its one-dimensional rigid body dynamics is analysed in frame \( \{C_0\} \), which is \( \{C_i\} \) at the initial configuration of the mechanism. From Fig. 5, the equation of motion of \( G_{S_i} \) is

\[
\tau_i + \left( c_i G_{S_i} \times \right)_{(3,3)} c_i F_{S_i} = I_{G_{S_i}} \cdot \ddot{\theta}_i
\]  

(32)

where \( \tau_i \) is the torque offered by the \( i \)-th actuator, \( c_i G_{S_i} \) is the \( 3 \times 1 \) vector of \( G_{S_i} \) in \( \{C_0\} \), \( \left( c_i G_{S_i} \times \right)_{(3,3)} \) is the third line of the skew matrix \( c_i G_{S_i} \times \), \( c_i F_{S_i} \) is the reaction force exerted by the coupler at the crank and measured in \( \{C_0\} \), and \( I_{G_{S_i}} \) is the rotational inertia of \( G_{S_i} \) that is a scalar. \( c_i G_{S_i} \) and \( c_i F_{S_i} \) can be computed as

\[
c_i G_{S_i} = R_x(\theta) \cdot \begin{bmatrix} G_{S_i} \end{bmatrix}
\]

\[
c_i F_{S_i} = c_i R^T \cdot (-F_{S_i})
\]  

(33)

By substituting Eq.(33) into Eq.(32), it can be rewritten as

\[
\tau_i = M_{6a} \cdot F_{S_{i,a}} + I_i \cdot \ddot{\theta}_i
\]  

(34)

where \( M_{6a} = \left( c_i G_{S_i} \times \right)_{(3,3)} c_i R^T \). For the six cranks it can be found that

\[
\boldsymbol{\tau} = M_{6a} \cdot F_{S_{i,a}} + M_{6b}
\]  

(35)

where

\[
\tau = \begin{bmatrix}
\tau_1 \\
\vdots \\
\tau_6
\end{bmatrix},
M_{6a} = \begin{bmatrix}
M_{6a1} & \cdots & \vdots \\
\vdots & \ddots & \vdots \\
M_{6a18} & \cdots & M_{6a18}
\end{bmatrix},
M_{6b} = \begin{bmatrix}
I_1 \cdot \ddot{\theta}_1 \\
\vdots \\
I_6 \cdot \ddot{\theta}_6
\end{bmatrix}
\]

4.4 The entire mechanism

Combining Eqs.(27) and (30), i.e., the Newton-Euler’s formulation of the end effector and the six couplers produces

\[
\begin{bmatrix}
M_{2b} \\
M_{3b}
\end{bmatrix} \cdot F_{M_{i,a}} = \begin{bmatrix}
M_{3b} \\
M_{4b}
\end{bmatrix} - \begin{bmatrix}
M_{4b} \\
0_{12 \times 2}
\end{bmatrix} \cdot F_Z
\]  

(36)
The reaction forces at \( M_i \) can be computed that
\[
F_{M_{i,a}} = M_{10b} - M_{11b} \cdot F_Z
\]
where
\[
M_{10b} = \begin{bmatrix}
M_{2b} \\
M_{5b}
\end{bmatrix}^{-1} \begin{bmatrix}
M_{3b}
M_{6b}
\end{bmatrix} \cdot M_{11b} = \begin{bmatrix}
M_{2b} \\
M_{5b}
\end{bmatrix}^{-1} \begin{bmatrix}
0_{12 	imes 2}
\end{bmatrix}
\]
Putting it into Eq.(31) yields
\[
F_{S_{i,a}} = M_{12b} - M_{11b} \cdot F_Z
\]
where \( M_{12b} = M_{3b} + M_{10b} \). Substituting Eq.(38) into Eq.(35) produces the dynamics model sought as
\[
\tau = M_{13b} - M_{14b} \cdot F_Z
\]
where
\[
M_{13b} = M_{4b} \cdot M_{12b} + M_{9b} \\
M_{14b} = M_{4b} \cdot M_{11b}
\]
It is shown that the inverse dynamics problem of the mechanism at hand can be reduced to solving a system of six linear equations in eight unknowns, which means Eq.(39) is indeterminate. This phenomenon is engendered by the constrains from HKPs to the end effector, which introduces two unknown constraint forces/redundant actuations/parasitic motions and eliminates two DOFs simultaneously.

5 Lagrangian formulation

In writing the Lagrange equations of the RAPM, it must be emphasised that \( F_{r_i} \) and \( F_{r_e} \) are ideal constraint forces, as such, they do not appear in the formulation. There are two methods to build the models in this section: in Model 1, the equations are directly formatted to the RAPM; while in Model 2, intuitively, the dynamics model of the 6RSS PM without HKPs is build first, thereafter, the two HKP constraints are modelled, to achieve the model of the RAPM of interest.

5.1 Model 1

At first, the mechanism is virtually cut at \( M_i \), then the dynamics of the end effector and the chains are formulated in the task space and the joint space independently. Next, in view of the closed-loop constraints in reality, the constraint forces using the Lagrange multipliers are added to generate the complete dynamics model of the RAPM. Finally, these multipliers are eliminated by virtue of the null-space method, reaching the dynamics model sought.

5.1.1 The end effector of the RAPM

In view of the translational and the angular velocities of the end effector in Eqs.(8) and (10), respectively, its kinetic energy is
\[
T_{EE} = \frac{1}{2} \begin{bmatrix} V_{o_{EE}} \\ \omega_{EE} \end{bmatrix}^T \begin{bmatrix} m_{EE} \cdot \omega_{EE} \\ I_{EE} \end{bmatrix} \begin{bmatrix} V_{o_{EE}} \\ \omega_{EE} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \dot{q}_{EE} \end{bmatrix}^T M_{EE} \cdot \begin{bmatrix} \dot{q}_{EE} \end{bmatrix} \]

15
where $\mathbf{M}_{EE}$ is its $4 \times 4$ mass matrix that can be expressed in a closed-form as

$$
\mathbf{M}_{EE} = \begin{bmatrix} 
\mathbf{M}_{ib}^T & \mathbf{m}_{EE} \cdot \mathbf{E}_3 
\mathbf{I}_{EE} & \mathbf{M}_{ib}
\end{bmatrix}
$$

(41)

Its potential energy is

$$
P_{EE} = m_{EE} \cdot \mathbf{g} \cdot \mathbf{Z}
$$

(42)

Regarding these, its Lagrange function is

$$
L_{EE} = T_{EE} - P_{EE} = \frac{1}{2} \dot{\mathbf{q}}_{EE}^T \cdot \mathbf{M}_{EE} \cdot \dot{\mathbf{q}}_{EE} - m_{EE} \cdot \mathbf{g} \cdot \mathbf{Z}
$$

(43)

One can derive that

$$
\frac{\partial L_{EE}}{\partial \mathbf{q}_{EE}} = \mathbf{M}_{EE} \cdot \dot{\mathbf{q}}_{EE}
$$

$$
\frac{d}{dt} \left( \frac{\partial L_{EE}}{\partial \mathbf{q}_{EE}} \right) = \mathbf{M}_{EE} \cdot \ddot{\mathbf{q}}_{EE} + \frac{\partial \mathbf{M}_{EE}}{\partial \mathbf{q}_{EE}^T} \cdot (\dot{\mathbf{q}}_{EE} \otimes \mathbf{E}_4) \cdot \dot{\mathbf{q}}_{EE}
$$

(44)

$$
\frac{\partial L_{EE}}{\partial \mathbf{q}_{EE}} = \frac{1}{2} (\mathbf{E}_4 \otimes \dot{\mathbf{q}}_{EE})^T \cdot \frac{\partial \mathbf{M}_{EE}}{\partial \mathbf{q}_{EE}} \cdot \dot{\mathbf{q}}_{EE} - m_{EE} \cdot \mathbf{g} \cdot \mathbf{Z}
$$

by virtue of which the Lagrangian equation of the end effector is

$$
\frac{d}{dt} \left( \frac{\partial L_{EE}}{\partial \mathbf{q}_{EE}} \right) - \frac{\partial L_{EE}}{\partial \mathbf{q}_{EE}} = \mathbf{M}_{EE} \cdot \ddot{\mathbf{q}}_{EE} + \mathbf{C}_{EE} \cdot \dot{\mathbf{q}}_{EE} \cdot \dot{\mathbf{q}}_{EE} + \mathbf{G}_{EE} \cdot \dot{\mathbf{q}}_{EE} = \mathbf{F}_{EE}
$$

(45)

where

$$
\mathbf{C}_{EE} = \frac{\partial \mathbf{M}_{EE}}{\partial \mathbf{q}_{EE}^T} \cdot (\dot{\mathbf{q}}_{EE} \otimes \mathbf{E}_4) - \frac{1}{2} (\mathbf{E}_4 \otimes \dot{\mathbf{q}}_{EE})^T \cdot \frac{\partial \mathbf{M}_{EE}}{\partial \mathbf{q}_{EE}}
$$

$$
\mathbf{G}_{EE} = m_{EE} \cdot \mathbf{g} \cdot \mathbf{Z}
$$

are the $4 \times 4$ Coriolis and centrifugal force matrix, and the $4 \times 1$ gravitational force vector of the end effector, respectively. $\mathbf{F}_{EE}$ is the $4 \times 1$ generalised force vector corresponding to the four DOFs of the end effector. Because the $3 \times 1$ bite force $\mathbf{F}_B$ is measured in \{S\}, it must be transferred into the directions of the four DOFs as follows:

The instantaneous power exerted by $\mathbf{F}_B$ is

$$
\left[ \begin{bmatrix} \mathbf{E}_3 \\ \mathbf{O}_M \mathbf{B} \times \mathbf{a}_{OE} \end{bmatrix} \cdot \mathbf{F}_B \right]^T \cdot \left[ \begin{bmatrix} \mathbf{V}_{OE} \\ \mathbf{O}_M \mathbf{B} \times \mathbf{a}_{OE} \end{bmatrix} \right] = \mathbf{F}_B^T \cdot \left[ \begin{bmatrix} \mathbf{E}_3 \\ \mathbf{O}_M \mathbf{B} \times \mathbf{a}_{OE} \end{bmatrix} \right]^T \cdot \left[ \begin{bmatrix} \mathbf{M}_{ib} \\ \mathbf{M}_{ib} \end{bmatrix} \right] \cdot \dot{\mathbf{q}}_{EE} = \mathbf{F}_{EE}^T \cdot \dot{\mathbf{q}}_{EE}
$$

(46)

Hence, the generalized force vector can be mapped by $\mathbf{F}_B$ as

$$
\mathbf{F}_{EE} = \left[ \begin{bmatrix} \mathbf{E}_3 \\ \mathbf{O}_M \mathbf{B} \times \mathbf{a}_{OE} \end{bmatrix} \right]^T \cdot \left[ \begin{bmatrix} \mathbf{M}_{ib} \\ \mathbf{M}_{ib} \end{bmatrix} \right] \cdot \mathbf{F}_B
$$

(47)

Now the dynamics model of the end effector has been formulated straightforwardly in the task space.
5.1.2 The \(i\)th chain

For the coupler \(S,M_i\), its kinetic energy can be computed as

\[
T_{S,M_i} = \frac{1}{2} \begin{bmatrix} V_{S_i}^T & m_{S,M_i} \cdot E \end{bmatrix} \begin{bmatrix} \omega_{S,M_i} \\ \omega_{S,M_i} \end{bmatrix} 
\]

Upon substitution of Eqs.(20) and (23) into Eq.(48), it produces

\[
T_{S,M_i} = \frac{1}{2} \dot{q}_i^T \cdot \dot{q}_i
\]

where \(M_{S,M_i}\) is the mass matrix of \(S,M_i\), and can be expressed as

\[
M_{S,M_i} = \begin{bmatrix} J_{E_i} \\ N_i \cdot R \cdot R_{o\in S,M_i} \end{bmatrix}^T \begin{bmatrix} m_{S,M_i} \cdot E \end{bmatrix} \begin{bmatrix} J_{E_i} \\ N_i \cdot R \cdot R_{o\in S,M_i} \end{bmatrix}
\]

The potential energy of \(S,M_i\) is

\[
P_{S,M_i} = m_{S,M_i} \cdot g \cdot O_i E_i(3)
\]

where \(O_i E_i(3)\) is the third term of \(O_i E_i\).

The crank \(G,S\), only rotates around the \(Z_{c_i}\) axis of \(\{C_i\}\), thus its kinetic energy is

\[
T_{G,S} = \frac{1}{2} I_{G,S} \cdot \dot{\theta}_i^2 = \frac{1}{2} \dot{q}_i^T \cdot M_{G,S} \cdot \dot{q}_i
\]

where \(M_{G,S} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \cdot I_{G,S} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix}\) is its \(3\times3\) mass matrix. The mass centre of the crank lies on the \(Z_{c_i}\) axis, thus its potential energy \(P_{G,S}\) is a constant. Regarding these, the kinetic energy, the potential energy and the Lagrangian function of the \(i\)th chain are computed sequentially as

\[
T_{G,S,M_i} (q_i \cdot \dot{q}_i) = T_{S,M_i} + T_{G,S} = \frac{1}{2} \dot{q}_i^T \cdot M_{G,S,M_i} (q_i) \cdot \dot{q}_i
\]

\[
P_{G,S,M_i} (q_i) = P_{S,M_i} + P_{G,S}
\]

\[
L_i = T_{G,S,M_i} (q_i \cdot \dot{q}_i) - P_{G,S,M_i} (q_i)
\]

As such one can compute that

\[
\frac{\partial L_i}{\partial q_{i}} = M_{G,S,M_i} (q_i) \cdot \dot{q}_i
\]

\[
\frac{d}{dt} \left( \frac{\partial L_i}{\partial q_{i}} \right) = \frac{\partial L_i}{\partial q_{i}} + \frac{\partial M_{G,S,M_i}}{\partial q_{i}} \left( \dot{q}_i \otimes E \right) \cdot \dot{q}_i + \frac{\partial M_{G,S,M_i}}{\partial q_{i}} \left( \dot{q}_i \otimes E \right) \cdot \dot{q}_i
\]

\[
\frac{\partial L_i}{\partial q_{i}} = \frac{1}{2} \left( E \otimes \dot{q}_i \right)^T \frac{\partial M_{G,S,M_i}}{\partial q_{i}} \cdot \dot{q}_i - m_{S,M_i} \cdot g \cdot \frac{\partial O_i E_i(3)}{\partial q_{i}}
\]

The Lagrangian formulation of the \(i\)th chain is
\[ M_{G,S,M} \left( \mathbf{q}_i \right) \dot{\mathbf{q}}_i + C_{G,S,M} \left( \mathbf{q}_i, \dot{\mathbf{q}}_i \right) \ddot{\mathbf{q}}_i + G_{G,S,M} \left( \mathbf{q}_i \right) = F_{G,S,M} \left( \tau_i \right) \]  \hfill (54)

where \( M_{G,S,M} \left( \mathbf{q}_i \right) \), \( C_{G,S,M} \left( \mathbf{q}_i, \dot{\mathbf{q}}_i \right) \), \( G_{G,S,M} \left( \mathbf{q}_i \right) \), and \( F_{G,S,M} \left( \tau_i \right) \) are the 3\times3 mass matrix, the 3\times3 Coriolis and centrifugal force matrix, the 3\times1 gravitational force vector, and the 3\times1 generalized force vector, respectively. They can be expressed sequentially as

\[ M_{G,S,M} \left( \mathbf{q}_i \right) = M_{S,M} + M_{G,S} \]
\[ C_{G,S,M} \left( \mathbf{q}_i, \dot{\mathbf{q}}_i \right) = \frac{\partial M_{G,S,M}}{\partial \mathbf{q}_i} \left( \dot{\mathbf{q}}_i \otimes E_3 \right) - \frac{1}{2} \left( E_3 \otimes \dot{\mathbf{q}}_i \right)^T \cdot \frac{\partial M_{G,S,M}}{\partial \mathbf{q}_i} \]
\[ G_{G,S,M} \left( \mathbf{q}_i \right) = m_{S,M} \cdot \frac{\partial \mathbf{O}_i E_{(3)}}{\partial \mathbf{q}_i} \]
\[ F_{G,S,M} \left( \tau_i \right) = \begin{bmatrix} \tau_i \\ 0_{2 \times 1} \end{bmatrix} \]

Now the model of the \( i \)th chain has been conveniently built in its joint space.

### 5.1.3 The entire mechanism

For the complete mechanism, it can be formatted that

\[ M_i \left( \mathbf{q}_i \right) \dot{\mathbf{q}}_i + C_i \left( \mathbf{q}_i, \dot{\mathbf{q}}_i \right) \ddot{\mathbf{q}}_i + G_i \left( \mathbf{q}_i \right) = F_i \left( \tau, \mathbf{q}_{EE} \right) \]  \hfill (56)

where

\[ \mathbf{q}_i = \begin{bmatrix} q_{i \left( \mathbf{q}_{EE} \right)} \\ q_{i \left( \mathbf{q}_{EE} \right)} \\ \vdots \\ q_{i \left( \mathbf{q}_{EE} \right)} \end{bmatrix}_{22 \times 1} \]
\[ M_i = \begin{bmatrix} M_{G,S,M} \left( \mathbf{q}_i \right) \\ \vdots \\ M_{G,S,M} \left( \mathbf{q}_i \right) \end{bmatrix}_{22 \times 22} \]
\[ C_i = \begin{bmatrix} C_{G,S,M} \left( \mathbf{q}_i, \dot{\mathbf{q}}_i \right) \\ \vdots \\ C_{G,S,M} \left( \mathbf{q}_i, \dot{\mathbf{q}}_i \right) \end{bmatrix}_{22 \times 22} \]
\[ G_i = \begin{bmatrix} G_{G,S,M} \left( \mathbf{q}_i \right) \\ \vdots \\ G_{G,S,M} \left( \mathbf{q}_i \right) \end{bmatrix}_{22 \times 1} \]
\[ F_i = \begin{bmatrix} F_{G,S,M} \left( \tau_i \right) \\ \vdots \\ F_{G,S,M} \left( \tau_i \right) \end{bmatrix}_{22 \times 1} \]

are the generalised coordinate vector, the mass matrix, the Coriolis and centrifugal force matrix, the gravitational force vector, and the generalised external force vector free of closed-loop constraints, respectively. Their sizes are given in the subscripts on the right. Physically, there are 18 holonomic constraint equations in total from the closed-loops by the kinematic chains and the end effector, and they are determined as

\[ \Phi = \begin{bmatrix} O_{S_i} G_i + G_i S_i + S_i M_i - O_{S_i} O_{M_i} - O_{M_i} M_i = 0_{3 \times 1} \\ \vdots \\ O_{S_h} G_h + G_h S_h + S_h M_h - O_{S_h} O_{M_h} - O_{M_h} M_h = 0_{3 \times 1} \end{bmatrix} \]  \hfill (57)

The complete dynamics model is built as

\[ M_i \left( \mathbf{q}_i \right) \dot{\mathbf{q}}_i + C_i \left( \mathbf{q}_i, \dot{\mathbf{q}}_i \right) \ddot{\mathbf{q}}_i + G_i \left( \mathbf{q}_i \right) + \Phi_\theta \cdot \lambda = F_i \left( \tau, \mathbf{q}_{EE} \right) \]  \hfill (58)
where $\Phi_0 = \text{Jacobian}(\Phi, q_i)$ is the $18 \times 22$ constraint Jacobian matrix, $\lambda_i$ is the $18 \times 1$ unknown Lagrangian multiplier vector, which means the magnitudes of the generalized constraint forces $\Phi_0' \cdot \lambda_i$ in the mechanism at hand.

From Section 3.2, the 22 terms in $q_i$ are not all independent: $q_i$ ($i = 1, \ldots, 6$) is the function of $q_{EE}$ whose four elements are independent, thus $q_i$ cannot serve as an independent generalised coordinate vector. In the meantime, from Eqs.(17) and (19), one can find that

$$\dot{q}_i = M_{p1} \cdot \dot{q}_{EE}$$
$$\ddot{q}_i = M_{p1} \cdot \ddot{q}_{EE} + M_{p2} \cdot q_{EE}$$

where

$$M_{p1} = \begin{bmatrix} M_{31} \\ \vdots \\ M_{36} \\ E_4 \end{bmatrix}_{22 \times 4}, M_{p2} = \begin{bmatrix} M_{41} \\ \vdots \\ M_{46} \\ 0 \end{bmatrix}_{22 \times 4}$$

Substituting Eq.(59) into Eq.(58) yields

$$M_i(\dot{q}_i) \cdot \left( M_{p1} \cdot \dot{q}_{EE} + M_{p2} \cdot q_{EE} \right) + C_i(q_i, \dot{q}_i) \cdot M_{p1} \cdot \dot{q}_{EE} + G_i(q_i) + \Phi_0' \cdot \lambda_i = F_i(\tau, q_{EE})$$

where $q_i$ and $\dot{q}_i$ in $M_i(q_i), C_i(q_i, \dot{q}_i)$ and $G_i(q_i)$ can be numerically computed by $q_{EE}$ and $\dot{q}_{EE}$. To avoid computing $\lambda_i$, the null space method is used: differentiating the constraint equations in Eq.(57) with respect to time gives rise to

$$\dot{\Phi} = \Phi_0 \cdot \dot{q}_i = 0_{18 \times 1}$$

Upon substitution of the first equation in Eq.(59) into Eq.(61), it results in

$$\dot{\Phi} = \Phi_0 \cdot M_{p1} \cdot \dot{q}_{EE} = 0_{18 \times 1}$$

In it, $\dot{q}_{EE}$ is the first-time derivative of the independent generalised coordinate vector $q_{EE}$ and it can be assigned arbitrarily. However, while doing so, it must hold that

$$\Phi_0 \cdot M_{p1} = 0_{4 \times 4}$$

or equivalently, $M_{p1}' \cdot \Phi_0^T = 0_{4 \times 18}$. Thereby, multiplying both sides of Eq.(60) by $M_{p1}'$ from the left gives rise to

$$M_{p1}' \cdot M_i(q_i) \cdot \left( M_{p1} \cdot \dot{q}_{EE} + M_{p2} \cdot q_{EE} \right) + M_{p1}' \cdot C_i(q_i, \dot{q}_i) \cdot M_{p1} \cdot \dot{q}_{EE} + M_{p1}' \cdot G_i(q_i) = M_{p1}' \cdot F_i(\tau, q_{EE})$$

In this manner, it is not necessary to compute $\lambda_i$. This equation can be rewritten as

$$M_{p1}' \cdot \dot{q}_{EE} + C_{r1} \cdot \dot{q}_{EE} + G_{r1} = J_{r1}' \cdot \tau + F_{EE1}$$

where
\[ M_{t_1} = M^T_{t_1} \cdot M_1(\dot{q}_1) \cdot M_{t_1} \]
\[ C_{t_1} = M^T_{t_1} \cdot \left( M_1(\dot{q}_1) \cdot M_{t_2} + C_1(\dot{q}_1, \dot{\dot{q}}_1) \cdot M_{t_1} \right) \]
\[ G_{t_1} = M^T_{t_1} \cdot G_1(\dot{q}_1) \]
\[ J_{\theta I} = \begin{bmatrix} M_{11(\theta)} \\ \vdots \\ M_{66(\theta)} \end{bmatrix} \]

and \( J_{\theta I} \) actually denotes the \( 6 \times 4 \) Jacobian matrix between \( \theta \) and \( q_{EE} \), i.e.,
\[
J_{\theta I} = \text{Jacobian} (\theta, q_{EE}) \]
\[
\dot{\theta} = J_{\theta I} \cdot \dot{q}_{EE} \quad (66)
\]

In Eq.(65), there are four equations and six unknowns, indicating the 4-DOF mechanism is actuated by six actuators.

5.2 Model 2

It is recalled that the conversion of two DOFs \( Z \) and \( \gamma \) into parasitic motions in the RAPM is caused by the two HKP constraints onto the end effector as in Section 3.1. An alternative though intuitive and simple approach is presented in this section: the idea is to first model the inverse dynamics of the 6RSS PM, then on this basis, the HKP constraints are modelled to arrive at the solution of the RAPM at hand.

5.2.1 The end effector of the 6RSS PM

In the 6RSS PM, the six DOFs of the end effector can be expressed in Eq.(3), and its angular velocity, the translational velocity of the mass centre \( O_M \) can be computed as
\[
\omega_{EE} = M_{0a} \cdot \dot{X}_{EE} \\
V_{0a} = M_{0a} \cdot \dot{X}_{EE} \quad (67)
\]
where \( M_{0a} \) and \( M_{a} \) are recalled in Eqs.(8) and (10), respectively. Its kinetic energy is
\[
T_{EE2} = \frac{1}{2} \cdot \dot{X}_{EE}^T \cdot M_{EE} \cdot \dot{X}_{EE} \quad (68)
\]
where \( M_{EE2} = \begin{bmatrix} M_{aa} & M_{a} \\ M_{a}^T & M_{0a} \end{bmatrix} \) denotes its \( 6 \times 6 \) mass matrix. Its potential energy is
\[
P_{EE2} = m_{EE} \cdot g \cdot Z \quad (69)
\]
It is remarked that though the expressions of Eqs.(42) and (69) are exactly identical, \( Z \) in Eq.(42) is the function of \( q_{EE} \) in the RAPM, while in Eq.(69), \( Z \) is a DOF of the 6RSS PM. The Lagrange function is
\[
L_{EE2} = T_{EE2} - P_{EE2} = \frac{1}{2} \cdot \dot{X}_{EE}^T \cdot M_{EE2} \cdot \dot{X}_{EE} - m_{EE} \cdot g \cdot Z \quad (70)
\]
By virtue of it, one can obtain that
\[
\frac{\partial L_{EE2}}{\partial X_{EE}} = M_{EE2}(X_{EE}) \cdot \dot{X}_{EE}
\]
\[
\frac{d}{dt} \left( \frac{\partial L_{EE2}}{\partial X_{EE}} \right) = M_{EE2}(X_{EE}) \cdot \ddot{X}_{EE} + \frac{\partial M_{EE2}}{\partial X_{EE}} \cdot (X_{EE} \otimes E_o) \cdot \dot{X}_{EE}
\]
\[
\frac{\partial L_{EE2}}{\partial X_{EE}} = \frac{1}{2} \left( E_o \otimes X_{EE} \right)^T \cdot \frac{\partial M_{EE2}}{\partial X_{EE}} \cdot \dot{X}_{EE} - m_{EE} \cdot g \cdot \frac{\partial Z}{\partial X_{EE}}
\]

The instantaneous power exerted by the bite force \( F_B \) is

\[
F_B^T \cdot V_{\alpha_B} + \left( O_M B \times F_B \right)^T \cdot \omega_{EE} = F_{EE2}^T \cdot \dot{X}_{EE}
\]

where \( F_{EE2} \) is the generalised force vector corresponding to \( X_{EE} \) and it is computed as

\[
F_{EE2} = M_{2a}^T \cdot F_B
\]

where \( M_{2a} = \left[ E_o \quad (O_M B \times)^T \right] \cdot \left[ M_{a} \quad M_{0a} \right] \). From Eqs.(71) and (73), the dynamics model of the end effector can be written as

\[
M_{EE2}(X_{EE}) \cdot \ddot{X}_{EE} + C_{EE2}(X_{EE} \cdot \dot{X}_{EE}) \cdot \dot{X}_{EE} + G_{EE2}(X_{EE}) = M_{2a}^T \cdot F_B
\]

where

\[
C_{EE2}(X_{EE}, \dot{X}_{EE}) = \frac{\partial M_{EE2}}{\partial X_{EE}} \cdot (X_{EE} \otimes E_o) - \frac{1}{2} \left( E_o \otimes X_{EE} \right)^T \cdot \frac{\partial M_{EE2}}{\partial X_{EE}}
\]

\[
G_{EE2}(q_{EE}) = m_{EE} \cdot g \cdot \frac{\partial Z}{\partial X_{EE}}
\]

are the \( 6 \times 6 \) Coriolis and centrifugal force matrix and the \( 6 \times 1 \) gravitational force vector of the end effector, respectively.

### 5.2.2 The entire mechanism

The dynamics model of the \( i \)th (\( i = 1, \ldots, 6 \)) chain is computed in an identical manner as that in Section 5.1.2, but it must be remembered that now \( q_i, \dot{q}_i, \ddot{q}_i \) are functions of \( X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE} \), i.e.,

\[
\begin{align*}
q_i &= q_i(X_{EE}) \\
\dot{q}_i &= \dot{q}_i(X_{EE}, \dot{X}_{EE}) \\
\ddot{q}_i &= \ddot{q}_i(X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE})
\end{align*}
\]

Differentiating Eq.(15) with respect to time again results in

\[
M_{iu} \cdot \dot{q}_i = \bar{M}_{2i} \cdot \dot{X}_{EE}
\]

where \( \bar{M}_{2i} = \text{Jacobian}(O_S O_M + O_M M_i, \quad X_{EE}) \) and its size is \( 3 \times 6 \). Thereby, it can be further found that

\[
\begin{align*}
\dot{q}_i &= \bar{M}_{iu} \cdot \dot{X}_{EE} \\
\ddot{q}_i &= \bar{M}_{iu} \cdot \ddot{X}_{EE} + \dddot{q}_i(X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE})
\end{align*}
\]
The dynamics model of the mechanism free of closed-loop constraints can be built as

\[
M_2(q_2) \ddot{q}_2 + C_2(q_2, \dot{q}_2) \dot{q}_2 + G_2(q_2) = F_2
\]

(77)

where

\[
q_2 = \begin{bmatrix} q_{X_E} \\ M_{EE}(X_E) \end{bmatrix}_{2n+1}, \\
M_2 = \begin{bmatrix} M_{G,S,M_1}(q_i) \\ \vdots \\ M_{G,S,M_6}(q_i) \end{bmatrix}, \\
C_2(q_2, \dot{q}_2) = \begin{bmatrix} C_{G,S,M_1}(q_i, \dot{q}_i) \\ \vdots \\ C_{G,S,M_6}(q_i, \dot{q}_i) \end{bmatrix}, \\
G_2 = \begin{bmatrix} G_{G,S,M_1}(q_i) \\ \vdots \\ G_{G,S,M_6}(q_i) \end{bmatrix}, \\
F_2 = \begin{bmatrix} F_{G,S,M_1}(r_1) \\ \vdots \\ F_{G,S,M_6}(r_6) \end{bmatrix}
\]

(78)

The constraint equation vector is as that in Eq.(57), and the complete dynamics model of the mechanism with closed-loop constraints is

\[
M_2(q_2) \ddot{q}_2 + C_2(q_2, \dot{q}_2) \dot{q}_2 + G_2(q_2) + \Phi_{p}^T \lambda_2 = \Phi_{p} \dot{X}_{EE}
\]

(79)

where \( \Phi_{p} = \text{Jacobian}(\Phi, \dot{q}_2) \) is the \( 18 \times 24 \) constraint Jacobian matrix, \( \lambda_2 \) is the \( 18 \times 1 \) unknown Lagrangian multiplier vector which means the magnitudes of the generalised constraint forces \( \Phi_{p}^T \lambda_2 \) in the 6RSS PM.

By virtue of the identical manner as in Section 5.1.3, one can derive that

\[
\ddot{\varphi}_2 = \dddot{\varphi}_2 = \dddot{\varphi}_2 = \dddot{\varphi}_2
\]

(79)

where \( \dddot{\varphi}_2 = \dddot{\varphi}_2 = \dddot{\varphi}_2 = \dddot{\varphi}_2 \).

Upon substitution of Eq.(79) into Eq.(78), it gives rise to

\[
M_2(q_2) \ddot{q}_2 + C_2(q_2, \dot{q}_2) \dot{q}_2 + G_2(q_2) + \Phi_{p}^T \lambda_2 = \Phi_{p} \dot{X}_{EE}
\]

(80)

Identically, using the null space method as in Section 5.1.3, the Lagrangian multiplier vector \( \lambda_2 \) is eliminated by multiplying Eq.(80) with \( \dddot{\varphi}_2 \), and it produces that
\[ M_{r2} \cdot \ddot{X}_{EE} + C_{r2} \cdot \dot{X}_{EE} + G_{r2} = M_{\theta1}^T \cdot F_2 = J_{\theta2}^T \cdot \tau + F_{EE2} \]  

(81)

where

\[ M_{r2} = \tilde{M}_{r2}^T \cdot M_{\theta1} \cdot (\theta) \cdot \tilde{M}_{r1} \]

\[ C_{r2} = \tilde{M}_{r2}^T \cdot (M_{r2} \cdot \tilde{M}_{r2} + C_2 (\theta) \cdot \tilde{M}_{r1}) \]

\[ G_{r2} = \tilde{M}_{r2}^T \cdot G_2 (\theta) \]

\[ J_{\theta2} = \begin{bmatrix} \tilde{M}_{36(1)} \\ \vdots \\ \tilde{M}_{36(L)} \end{bmatrix}_{6 \times 6} \]

and \( J_{\theta2} \) actually denotes the 6×6 Jacobian matrix between \( \theta \) and \( X_{EE} \), i.e.,

\[ \dot{\theta} = J_{\theta2} \cdot \dot{X}_{EE} \]

(82)

In Eq.(81), there are six equations and six unknowns. By adding the two HKP constraints, the DOFs have been transferred from \( X_{EE} \) to \( q_{EE} \), thereby, using Eq.(9), one can derive that

\[ \ddot{X}_{EE} = M_{J} \cdot \dot{q}_{EE} + M_{J} \cdot \ddot{q}_{EE} \]

(83)

where \( \dot{M}_{J} = \frac{\partial M_{J}}{\partial \theta} (q_{EE} \otimes E_4) \). Substituting Eqs.(9) and (83) into Eq.(81) generates

\[ M_{r2} \cdot M_{J} \cdot \ddot{q}_{EE} + (M_{r2} \cdot M_{J} + C_{r2} \cdot M_{J}) \cdot \dot{q}_{EE} + G_{r2} - F_{EE2} = J_{\theta2}^T \cdot \tau \]

(84)

Multiplying both sides of Eq.(84) with \( M_{J}^T \) yields

\[ M_{J}^T \cdot M_{r2} \cdot M_{J} \cdot \ddot{q}_{EE} + M_{J}^T \cdot (M_{r2} \cdot M_{J} + C_{r2} \cdot M_{J}) \cdot \dot{q}_{EE} + M_{J}^T \cdot (G_{r2} - F_{EE2}) = (J_{\theta2} \cdot M_{J})^T \cdot \tau \]

(85a)

in which the 4×4 generalised mass matrix \( M_{J}^T \cdot M_{r2} \cdot M_{J} \) is square. The size of \( J_{\theta2} \cdot M_{J} \) is 6×4, thus there are six unknowns and four equations. It is worth mentioning that all the matrices and vectors in Eq.(85a) are ultimately functions of \( q_{EE}, \dot{q}_{EE}, \ddot{q}_{EE} \), rather than \( X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE} \).

Or, equivalently and more conveniently, \( M_{J}^T \) can be directly multiplied to the two sides of Eq.(81)

\[ M_{J}^T \cdot (M_{r2} \cdot \ddot{X}_{EE} + C_{r2} \cdot \dot{X}_{EE} + G_{r2} - F_{EE2}) = (J_{\theta2} \cdot M_{J})^T \cdot \tau \]

(85b)

It is known that \( X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE} \) are functions of \( q_{EE}, \dot{q}_{EE}, \ddot{q}_{EE} \), and the numerical values of the former can be computed by those of the latter, as obtained and will be shown in the last six subplots in Fig. 6. In this manner, all the matrices and vectors in the bracket of the left-hand side of Eq.(85b) can be directly computed by \( X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE} \) rather than \( q_{EE}, \dot{q}_{EE}, \ddot{q}_{EE} \), in spite of the fact that they are ultimately functions of \( q_{EE}, \dot{q}_{EE}, \ddot{q}_{EE} \). In other words, the dynamics model of the RAPM can be conveniently generated by formatting the model of the 6RSS PM as its core firstly; next in modelling the two HKP constraints, only the numerical multiplication of \( M_{J}^T \) is needed. In the practical application of Model 2, Eq.(85b) would be directly used rather than Eq.(85a). It will be shown in Section 7.4.3 how improvements in computational
efficiency can be affected by using the dynamics model of the 6RSS PM as the core as exhibited in Eq.(85b) to formulate explicit equations of motion.

6 Principle of virtual work

6.1 Model 1

In using the principle of virtual work, it is also must be emphasised that $F_{T_i}$ and $F_{R_i}$ at the two HKPs are ideal constraint forces that are not involved with virtual work, thus they must not appear in the model. Thereby, the $6 \times 1$ resultants of applied and inertia wrenches exerted at the centre of the end effector are

$$W_{EE} = -M_{lb}$$  \hfill (86)

where $M_{lb}$ can be found in Eq.(27). The resultant wrench acting at $SM_i$ is

$$W_{S,M_i} = - \left[ m_{S,M_i} \cdot \left( \dot{V}_k + g \right) \right]$$  \hfill (87)

The one-dimensional resultant torque acting at the crank $GS_i$ in the direction of $\theta_i$ is

$$W_{G,S_i} = \tau_i - J_i \cdot \dot{\theta}_i$$  \hfill (88)

The virtual displacements in each body of the mechanism must be compatible with the kinematic constraints by the joints. They must be related to a set of independent generalised virtual displacements that is served by $\delta q_{EE}$. Through Eqs.(8) and (10), the virtual displacements of the end effector can be expressed as

$$\delta \chi_{EE} = \begin{bmatrix} M_{lb} \\ M_{lb} \end{bmatrix} \cdot \delta q_{EE}$$  \hfill (89)

By Eqs.(20) and (24), the virtual displacements of the coupler $SM_i$ are

$$\delta \chi_{S,M_i} = J_{S,M_i} \cdot \delta q_{EE}$$  \hfill (90)

where $J_{S,M_i} = \begin{bmatrix} J_{EE} \\ J_{S,M_i} \end{bmatrix} \cdot M_{bi}$. Finally, from Eq.(66), the one-dimensional virtual displacement of the crank $GS_i$ is related to $\delta q_{EE}$ as

$$\delta \chi_{G,S_i} = J_{\theta(i)} \cdot \delta q_{EE}$$  \hfill (91)

where $J_{\theta(i)}$ \hspace{1pt} ($i=1, \ldots, 6$) is the $i$th row of $J_{bi}$. The principle of virtual work for the inverse dynamics problem of the overall mechanism can be stated as

$$\delta \chi_{EE}^T \cdot W_{EE} + \sum_{i=1}^{6} \delta \chi_{S,M_i}^T \cdot W_{S,M_i} + \sum_{i=1}^{6} \delta \chi_{G,S_i}^T \cdot \left( \tau_i - J_i \cdot \dot{\theta}_i \right) = 0$$  \hfill (92)

Substituting Eqs.(86)–(91) into Eq.(92) results in
\[ \delta q_{EE} = \left[ \begin{array}{c} M_{lb} \\ M_{oa} \end{array} \right] \cdot \delta M_{lb} + \sum_{i=1}^{6} J_{S,M_i}^T \cdot W_{S,M_i} + J_{\theta_1}^T \cdot \tau - J_{\theta_1}^T \cdot M_{gb} = 0 \] (93)

where \( M_{gb} \) can be found in Eq.(35). Since Eq.(93) is valid for any \( \delta q_{EE} \), it follows that

\[ J_{\theta_1}^T \cdot \tau = \left[ \begin{array}{c} M_{lb} \\ M_{oa} \end{array} \right] \cdot \delta M_{lb} + \sum_{i=1}^{6} J_{S,M_i}^T \cdot W_{S,M_i} + J_{\theta_1}^T \cdot M_{gb} \] (94)

which contains four equations and six unknowns, showing the mechanism is redundantly actuated.

6.2 Model 2

Analogous to the second model in Section 5.2, the dynamics model of the 6RSS PM is firstly built via the principle of virtual work, and next the two HKP constraints onto the effector are modelled. From Eqs.(67) and (82), the virtual displacements of the end effector and the crank \( G, S_i \) are computed sequentially as

\[ \delta X_{EE} = \left[ \begin{array}{c} M_{ib} \\ M_{oa} \end{array} \right] \cdot \delta X_{EE} \]

\[ \delta X_{G,S_i} = J_{\theta_2(i)} \cdot \delta X_{EE} \] (95)

From Eqs.(20),(23) and (76), the virtual displacements of the \( i \)th coupler \( S_i M_i \) are

\[ \delta X_{S_i,M_i} = J_{S_i,M_i} \cdot \delta X_{EE} \] (96)

where \( J_{S_i,M_i} = \left[ \begin{array}{c} J_{E_i} \\ N_i \cdot R \cdot R_{oa,M_i} \end{array} \right] \cdot \delta M_{bi} \). Using the principle of virtual work, one can find that

\[ \delta X_{EE} \cdot W_{EE} + \sum_{i=1}^{6} \delta X_{S_i,M_i} \cdot W_{S_i,M_i} + \sum_{i=2}^{6} \delta X_{G,S_i} \cdot \left( \tau_i - I_i \cdot \dot{\theta}_i \right) = 0 \] (97)

Upon substitution of Eqs.(86)–(88), (95) and (96) into Eq.(97), it results in

\[ \delta X_{EE} = \left[ \begin{array}{c} M_{lb} \\ M_{oa} \end{array} \right] \cdot \delta M_{lb} + \sum_{i=1}^{6} J_{S,M_i}^T \cdot W_{S,M_i} + \sum_{i=2}^{6} \delta X_{G,S_i} \cdot \left( \tau_i - I_i \cdot \dot{\theta}_i \right) = 0 \] (98)

Now the two HKP constraints are added to the end effector. From Eq.(9), it yields that

\[ \delta X_{EE} = M_j \cdot \delta q_{EE} \] (99)

Putting it into Eq.(98) and omitting \( \delta q_{EE} \) produces

\[ M_j \left[ \begin{array}{c} M_{lb} \\ M_{oa} \end{array} \right] \cdot \delta M_{lb} - \sum_{i=1}^{6} J_{S,M_i}^T \cdot W_{S,M_i} + J_{\theta_2}^T \cdot M_{gb} = \left( J_{\theta_2} \cdot M_j \right)^T \cdot \tau \] (100)

which completes the procedure with six unknowns and four equations. Identically, it also must be kept in mind that in spite of the fact that all the terms in this equation are now functions of \( q_{EE}, \dot{q}_{EE}, \ddot{q}_{EE} \) rather than \( X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE} \), and the numerical values of the latter can be computed by the former and then fed into Eq.(100), as illustrated in Section 5.2.2. Using the dynamics model of the 6RSS PM as the core can
considerably alleviate the computational demands of the RAPM’s model, which will be exhibited in Section 7.4.3.

Fig. C6 Motions of the end effector in 3D space and time history.

7 Numerical computations and comparisons

As an illustrative example, the mechanism is commanded to follow the first 5 seconds of a real incisor trajectory of a healthy human subject, as shown in Fig. 6 of [34]. The trajectory viewed from different perspectives is given in the first four subplots in Fig. 6 in this paper. The reason why only the first 5 seconds are adopted is, through the complete time interval of 10 seconds, the chewing trajectory almost experiences an analogous profile. The experimental setup and procedure to obtain the mastication movements in human subjects can be found in Chapter 6 of [4]. The corresponding mandibular motions in the four elements of $q_{EE}$ as a function of time are also provided in the following subplots, using the same method in Eq.(38) of [2]: For the trajectory is defined along the three axes of $\{S\}$, only three scalar equations along these directions can be formulated; however, the RAPM has four DOFs, whereupon it is kinematically redundant to carry out this task, and one free DOF can be used to optimise the kinematic performance. In this regard, to minimise the two-norm sum of the track error is set as a simple optimal
goal. In this process, the physical constraints imposed by the mechanism like the inverse kinematics
equations and the workspace should be respected. Next, using Eq.(6), the numerical values of \(Z\) and \(\gamma\)
with respect to \(q_{EE}\) can be computed. Finally, the first- and second-time derivatives of these six motion
variables are also computed via Eqs.(9) and (83). In tracking this prescribed chewing trajectory, neither
the RAPM nor the 6RSS PM is at or near singular configurations. Correspondingly, the first 5 seconds
of an experimentally measured 3D bite force in \(\{S\}\) on peanuts by a healthy human subject on the molars
as in Fig. 7, is exerted onto the right molar of the end effector. Numerical computations are performed to
justify these models with a time step 0.1s. For the 6RSS PM without redundant actuations used as a
benchmark in Sections 7.4.2 and 7.4.4, its six DOFs and their first- and second-time derivatives as a
function of time are identical as those in Fig. 6. Correspondingly, the six chains in the two PMs undergo
identical motions in terms of the numerical values of \(q_i, \dot{q}_i, \ddot{q}_i \ (i = 1, \ldots, 6)\).

![Fig. 7 3D bite force profiles on peanuts [4].](image)

7.1 Newton-Euler’s approach

In this approach, the analytical expressions of the actuating torques \(\tau\), HKP constraint forces \(F_z\), and
reaction forces at the spherical joints \(F_{M_i}\) and \(F_{S_i}\) all have been formulated. There are six equations and
eight unknowns in Eq.(39), thus neither the torques nor the constrained forces can be determined uniquely
or independently. Two aims related to the biomechanics of interest are set to optimally resolve the
redundancy as

\[
A_1 = \min \| \mathbf{r} \| \\
A_2 = \min \| F_z \| 
\]

which correspond to the minimum efforts of the chewing muscles and minimum loads at TMJs,
respectively. The following constraints

\[
\begin{align*}
| \mathbf{r} | & \leq \tau_{\max} \\
\text{the equality constraint in Eq.(39)}
\end{align*}
\]

must be obeyed, where \(\tau_{\max}\) is the maximum torque that can be generated by the actuators. The classical
optimisation algorithm Sequential Quadratic Programming, which is characterised by its super-linear
convergence, is adopted to address the problem. Under the first aim, the input torques and the constraint forces versus time along the chewing trajectory are exhibited in Fig. 8. Under the second objective they have an analogous profile, thus are not exhibited for the sake of brevity. It is highlighted that after obtaining the constraint forces $F_Z$ in view of Eqs.(37) and (38), the reaction forces at all the spherical joints can be computed with very modest additional computational demands.

Specifically, if $F_Z = 0$ is assumed, which means $A_2$ is ideally realised, Eq.(39) has six equations with six unknowns in $\tau$ and they can be directly computed in a closed-form as

$$\tau = M'_{\tau b}$$

(103)

The physical meaning of Eqs.(103) is clear: when the torques satisfy this equation, there are no constraint forces from HKPs acting at the end effector. In this case, the torques are uniquely determined.

Fig. 8 Actuating torques and constraint forces under Aim 1

7.2 Lagrangian formulation

In Eq.(65), the constraint forces at the HKPs do not appear, and the only unknowns are the input torques which can be optimised as

$$\tau = \left( J_{\beta l}^T \right)^+ \cdot \left( M_{11} \cdot \dot{q}_{EE} + C_{11} \cdot \ddot{q}_{EE} + G_{1} - F_{EE1} \right)$$

(104)

where $\left( J_{\beta l}^T \right)^+$ is the pseudo-inverse of $J_{\beta l}^T$, and its physical meaning is to minimise $\tau$. While in the second model by the Lagrangian formulation, the torques can be minimised using the pseudo-inverse method again as

$$\tau = \left( J_{\beta 2} \cdot M_J \right)^+ \cdot M_J^T \cdot \left( M_{22} \cdot \ddot{X}_{EE} + C_{22} \cdot \dot{X}_{EE} + G_{2} - F_{EE2} \right)$$

(105)

7.3 Principle of virtual work

Analogous to the Lagrangian equation in Eq.(65), the input torques from the two models based on the principle of virtual work can be minimised sequentially as
\[
\tau = (J_{01}^T M_{0b}) \cdot M_{0b} - \sum_{i=1}^{n_6} J_{0i}^T \cdot W_{0i} + J_{01}^T \cdot M_{210} \\
\tau = (J_{02} \cdot M_{b})^T \cdot M_{b} - \sum_{i=1}^{n_6} J_{i}^T \cdot W_{i} + J_{02}^T \cdot M_{210}
\]

(106)

It is interesting to note that by putting the numerical values of \( \tau \) obtained from these energy methods into Eq.(39), \( F_Z \) are overdetermined and it cannot reach a precise solution, for there exist six independent equations but only two unknowns. The pseudo-inverse method can be employed again; hence two stages of optimisation and two inverse dynamics models are needed, which is not only extremely cumbersome but also error-prone. Moreover, the physical meaning is quite obscure: a set of actuating torques corresponds to a certain value of \( F_Z \) as in Eq.(39), hence they must be optimised simultaneously, rather by two stages of optimization.

7.4 Discussions

7.4.1 Model structures

The number of unknowns and equations in the inverse dynamics from different methods is summarised in Table 1, where NE means the model by the Newton-Euler’s law, LA1 and LA2 means Model 1 and Model 2 formulated by the Lagrangian equations, respectively, PVW1 and PVW2 means Model 1 and Model 2 formulated by the principle of virtual work, respectively.

<table>
<thead>
<tr>
<th></th>
<th>NE</th>
<th>LA1</th>
<th>LA2</th>
<th>PVW1</th>
<th>PVW2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of equations</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Number of unknowns</td>
<td>8</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

The similarity in this table is quite clear, i.e., there are always two unknowns in excess of equations in the five models. Compared with the dynamics models of other RAPMs from the literature, it can also be found that the number of unknowns is always larger than that of equations. Specifically, in this RAPM at hand, from the Newton-Euler’s law, there are two more constraint forces, while in the four models from the latter two methods, there are two more actuating torques. However, the nature of this RAPM is only able to be clearly illustrated in Eq.(39), which explicitly exhibits both the input torques and the HKP constraint forces. In other words, they must be optimised simultaneously in this model. Meanwhile, it is revealed that actuation redundancy is caused by the two HKP constraints: if \( F_Z \) is directly set as zero in Eq.(39), actuating torques can be directly obtained without optimisation.

In comparison, from the latter two methods, the information about the HKP constraint forces is not provided, and the only unknowns are input torques that can be computed directly. Nevertheless, the computation of \( F_Z \) is critical: the pressure is high at both the condylar ball and the surface of the condylar socket due to the point-contact. As a consequence, it is prone to cause wearing and then clearances. Meanwhile, the link attached by the condylar ball shown in Fig. 2 is easily suffered from breaking if the constraint force is large. Apart from the fact that the number of unknowns is in excess of equations from the four models, one cannot find the difference in the model structures between this RAPM at hand and others in publications. From these two aspects, the model from the Newton-Euler’s law is superior than those from the latter two methods in revealing the nature of the RAPM.

Besides, determining how the number of the RSS kinematic chains varies the model structure is briefly presented in the following:

The end effector has 4 DOFs, thus it is assumed that there exist \( n(n \geq 4) \) chains in the mechanism. In this regard, using the Newton-Euler’s law in Section 4, \( F_{M_{1b}} \) and \( F_{S_{1b}} \) are in lieu of \( F_{M_{1b}} \) and \( F_{S_{1b}} \).
respectively, and their sizes are both $3n \times 1$; $\mathbf{r}$ is a $n \times 1$ vector. The size of the matrices $\mathbf{M}_{3\mathbf{b}}$ in Eq.(27), $\mathbf{M}_{5\mathbf{b}}$ and $\mathbf{M}_{6\mathbf{b}}$ in Eq.(30), $\mathbf{M}_{7\mathbf{b}}$ in Eq.(31), $\mathbf{M}_{8\mathbf{b}}$ and $\mathbf{M}_{9\mathbf{b}}$ in Eq.(35), are now $6 \times 3n$, $2n \times 3n$, $2n \times 1$, $3n \times 1$, $n \times 3n$ and $n \times 1$, sequentially. Eq.(36) is now written as

$$M_{1b} \mathbf{1}_n - M_{3b} \mathbf{0}_{n \times 2} - M_{4b} \mathbf{0}_{(6+2n) \times 1} + \mathbf{F}_{M_{1\mathbf{r}}}, z = M_{16b} \mathbf{F}_Z$$

(107)

It is easy to find that only when $6+2n=3n$, i.e., $n=6$, the precise values of $\mathbf{F}_{M_{1\mathbf{r}}}, z$ can be obtained as from Eq.(37); otherwise, combining Eqs.(31), (35) and (36) gives rise to

$$M_{1b} \mathbf{E}_n - M_{3b} \mathbf{0}_{(6+2n) \times n} - M_{4b} \mathbf{0}_{n \times 2} = M_{16b} \mathbf{F}_Z$$

(108)

where $\mathbf{E}_n$ is the $n \times n$ identity matrix, and

$$M_{15b} = \begin{bmatrix} M_{2b} \\ M_{3b} \end{bmatrix}, M_{16b} = \begin{bmatrix} M_{4b} \\ M_{5b} \end{bmatrix}, M_{17b} = \begin{bmatrix} M_{6b} \\ M_{7b} \end{bmatrix}, M_{18b} = M_{6b} + M_{8b} \cdot M_{7b}$$

### Table 2 Number of unknowns and equations with $n(n \geq 4)$ kinematic chains

<table>
<thead>
<tr>
<th></th>
<th>NE($n=6$)</th>
<th>NE ($n \neq 6$)</th>
<th>LA1</th>
<th>LA2</th>
<th>PVW1</th>
<th>PVW2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of equations</td>
<td>6</td>
<td>$6 + 3n$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Number of unknowns</td>
<td>8</td>
<td>$4n + 2$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

From this process, the reaction forces at $M_i$ are now optimised together with $\mathbf{r}$ and $\mathbf{F}_Z$: the number of kinematic chains only influences the existence of the reaction forces at $M_i$ while the HKP constraint forces always exist in the final dynamics equation Eq.(108). In other words, the existence of HKP constraint forces $\mathbf{F}_Z$ is independent of the number of kinematic chains in the final equation. Proceeding in a like manner in the four models by the latter two methods, there would be four equations and $n$ unknowns, if the RAPM has $n$ chains. The number of unknowns and equations is summarised in Table 2. From the Newton-Euler’s law, the number of unknowns is in excess of equations by $4n + 2 - (6 + 3n) = n - 4$, as well as that from the latter two methods. Accordingly, Table 1 is just a summary for a specific case study compared with the generalised situation expressed in Table 2.

### 7.4.2 Numerical results

To justify and compare the five models numerically, two indices are set as

$$F_1 = \frac{1}{N} \sum_{i=1}^{N} \| \mathbf{r} \|, F_2 = \frac{1}{N} \sum_{i=1}^{N} \| \mathbf{F}_Z \|$$

(109)

where $N=51$ is the number of sampling points along the chewing trajectory. It is evident that $F_2$ is only used specifically in the model via the Newton-Euler’s law.

### Table 3 Indices from different models

<table>
<thead>
<tr>
<th></th>
<th>NE</th>
<th>LA1</th>
<th>LA2</th>
<th>PVW1</th>
<th>PVW2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$ (N.m)</td>
<td>0.1985</td>
<td>0.1985</td>
<td>0.1985</td>
<td>0.1985</td>
<td>0.1985</td>
</tr>
<tr>
<td>$F_2$ (N)</td>
<td>7.7028</td>
<td>4.1477e-09</td>
<td>-</td>
<td>0.1985</td>
<td>0.1985</td>
</tr>
</tbody>
</table>

Their values under different optimal aims and models are listed in Table 3. One can clearly find that $F_1$ from the Newton-Euler’s approach under Aim 1, and the Lagrangian formulation and the principle of virtual work is identical, even though there are two differences between the Newton-Euler’s approach and the latter two methods: the first is the model structures are different as stated in Section 7.4.1, and the second is, in the Newton-Euler’s law, a closed-form solution of $\mathbf{r}$ and $\mathbf{F}_Z$ is not available; whilst in
the latter two methods, a closed-form solution of \( \mathbf{r} \) exists. Thereby, it is sufficient to demonstrate the
rightness and accuracy of the developed models. Besides, it is also noted that under Aim 2 from the
Newton-Euler’s law, the index \( F_2 \) is nearly zero, and \( F_1 \) is identical with four decimal places as its
counterpart in the case when \( F_Z \) is directly set to zero.

7.4.3 Computational cost

In order to assess the suitability of the models for real-time control, a reliable quantitative measure of the
computational load is useful. The kinematic and dynamic parameters involved in the numerical
computation can be found in [3]. The time consumption is summarised in Table 4. The procedures under
each approach are all divided into symbolic and numeric computations, and have been implemented in
programs written in Matlab, using an Intel(R) Core(TM) i7-8700K CPU@3.70GHz and 64GB of RAM.
By virtue of the symbolic computations, it is quite convenient to obtain the functions of Jacobian matrix,
and the mass matrix, and the Coriolis and centrifugal force matrix, etc. in these methods. They can be
conveniently called in the following numerical computations. Note that in LA2 and PVW2, the time for
the numerical computation of \( X_{EE}, \hat{X}_{EE}, \ddot{X}_{EE} \) by \( \mathbf{q}_{EE}, \dot{\mathbf{q}}_{EE}, \ddot{\mathbf{q}}_{EE} \) has been incorporated into the numerical
time.

<table>
<thead>
<tr>
<th>Table 4 Computational time of the RAPM (unit: s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aim 1</td>
</tr>
<tr>
<td>NE</td>
</tr>
<tr>
<td>Numeric</td>
</tr>
</tbody>
</table>

Under the Newton-Euler’s law, the computation time for the reaction forces at the spherical joints \( S_i \) and
\( M_i \) has been included. Regarding it, this model is quite economic, thanks to the availability of many
intermediate variables that can be used in the mechanical design, such as sizing the HKP-related
mechanical parts, the couplers, the cranks and the spherical joints. Besides, the cost in the numeric
computation for Aim 2 is nearly half of that for Aim 1, and it is almost equivalent to that when \( F_Z = 0 \)_{Z,s}.

Model 1 by the Lagrangian formulation is the most computationally intensive. The time consumption of
Model 1 by the principle of virtual work is less than those via the Newton-Euler’s law and the Lagrangian
formulation. This discovery is in a good agreement with the guess from [36]. The economy of
computation for the RAPM seems to be quite remarkable.

By comparing the cost between the two models by the Lagrangian formulation, the latter one results in a
considerably more economic algorithm for the solution of actuating torques, which is about 6 times that
of the first model. A similar comparison can be discovered between the two models under the principle
of virtual work, though the second model is only about 2.73 times that of the first one. In addition, due
to the fastest speed in the numerical computation in the second one, it has the potential to be readily
employed in the model-based real-time motion and/or force control.

The main reason why the two second models from the energy methods are greatly more efficient is
explained as follows: the resulting complexity of the two first models is a consequence of using the
complex symbolic expressions of \( Z \) and \( \gamma \) in Eq.(6), and the Jacobian matrix \( M_j \) for \( \mathbf{q}_{EE}, \dot{\mathbf{q}}_{EE}, \ddot{\mathbf{q}}_{EE} \) are used
directly. By contrast, in the two second models, even though the matrices and vectors in Eqs.(85b) and
(100) are ultimately functions of \( \mathbf{q}_{EE}, \dot{\mathbf{q}}_{EE}, \ddot{\mathbf{q}}_{EE} \), they can be computed by the numerical values of
\( X_{EE}, \hat{X}_{EE}, \ddot{X}_{EE} \) directly, which can be available from \( \mathbf{q}_{EE}, \dot{\mathbf{q}}_{EE}, \ddot{\mathbf{q}}_{EE} \) at the beginning of the formulation with
a relatively minor cost, and then are fed into the left-hand side of Eqs.(85b) and (100). In this manner,
the two second models in the latter two methods are free from the symbolic computation of \( Z, \gamma \) and \( M_j \).

Thereupon, the symbolic mathematical functions, which are called and used in the numerical computations, are less complex than those in the first two models. As a consequence, they offer the simplest possible computational algorithms.

A similar case is taken as an example. To acquire the numerical values of the angular acceleration \( \dot{\omega}_{S,M} \) of the coupler \( S,M_i \) and the translational acceleration \( \dot{V}_{E} \), Eqs.(21a) and (23) would be preferred over Eqs.(21b) and (24), because now \( \ddot{q}_i \) and \( \dddot{q}_i \) act as intermediate variables and can be obtained numerically and then fed into Eqs.(21a) and (23). This process is no doubt faster than Eqs.(21b) and (24) in which \( \dddot{q}_{EE} \) and \( \dddot{q}_{EE} \) are used with more symbolic computations.

In fact, the nature of the two models from the latter two methods is actually conceptually equivalent, respectively, which is to be explained from the perspective of symbolic computation: in the first model, \( q_{EE}, \dot{q}_{EE}, \ddot{q}_{EE} \) are directly used; by contrast, in the second one, \( X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE} \) are employed and they can ultimately be expressed in terms of \( \dddot{q}_{EE}, \dddot{q}_{EE} \). Specifically, thanks to the simplicity in Eq.(106) based on the principle of virtual work, one can easily observe that the two models are identical: from Eqs.(82) and (9), it can be found that

\[
\dot{\theta} = J_{q_2} \cdot \ddot{X}_{EE} = J_{q_2} \cdot M_j \cdot \dddot{q}_{EE} 
\]

(110)

Comparing it with Eq.(66) yields

\[
J_{q_1} = J_{q_2} \cdot M_j 
\]

(111)

Similarly, from Eqs.(76) and (9), it can be found

\[
\dddot{q}_i = \dddot{M}_h \cdot \dddot{X}_{EE} = \dddot{M}_h \cdot M_j \cdot \dddot{q}_{EE} 
\]

(112)

With Eq.(17), it yields

\[
M_h = \dddot{M}_h \cdot M_j 
\]

(113)

By multiplying \( M_j \) into the bracket on the left-hand side of Eq.(100), the two equations in Eqs.(94) and (100) are symbolically equivalent, namely, the two equations in Eq.(106) are identical.

### 7.4.4 Influence of HKP constraints

The dynamics model of the 6RSS PM without HKPs is also built via the above-mentioned three methodologies, as far as the role of the HKP constraints in the numerical results and the computational cost is concerned. The modelling process is not listed for the sake of clarity. The 6RSS PM also tracks the predefined chewing trajectory as in Fig. 6. It is found that based on these approaches, its inverse dynamics problem is reduced to solving a system of six linear equations in six unknowns. The input torques can be uniquely determined and there is no optimisation as in the RAPM. \( F_1 \) always equals to 0.2727N.m, which is also identical to that under the Newton-Euler’s law when \( F_x = 0 \). The reason why this occurs is quite self-explanatory: the six DOFs and their first- and second-time derivatives in the 6RSS PM are numerically equivalent to \( X_{EE}, \dot{X}_{EE}, \ddot{X}_{EE} \) of the RAPM. \( F_1 \) is larger than those of the RAPM under Aim 1 from the three methods. It means that redundant actuation is able to minimise the input torques, which is a well-developed opinion and has been proved in a number of publications.
The computational time under the three methods is given in Table 5, from which one can also find that the computational burden of the Newton-Euler’s law and the principle of virtual work is quite equivalent, and it is smaller than that from the Lagrange equations by a factor of about 2.2. By comparing Tables 4 and 5, one can see that the two HKP constraints greatly increase the computational complexity. The computational cost in the RAPM under LA2 and PVW2, is quite equivalent to that in the 6RSS PM under the Lagrange equations and the principle of virtual work, respectively. It indicates that using the second models from the energy and virtual work related methods renders the computational demand comparable to that of the 6RSS PM. The findings above are self-explanatory: As the central feature in the two second models from the latter two methods, the core of the dynamics of the 6RSS PM has been employed in Eqs.(85b) and (100), and the computational time of the numerical $M_f^J$ is quite minor. As such, a computational very efficient formulation without accuracy deterioration in the dynamics model has been provided in Sections 5.2 and 6.2.

### Table 5 Computational time of the 6RSS PM (unit: s)

<table>
<thead>
<tr>
<th></th>
<th>NE</th>
<th>LA</th>
<th>PVW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbolic</td>
<td>4.074</td>
<td>7.716</td>
<td>4.071</td>
</tr>
<tr>
<td>Numeric</td>
<td>1.301</td>
<td>4.053</td>
<td>1.062</td>
</tr>
<tr>
<td>Total</td>
<td>5.375</td>
<td>11.769</td>
<td>5.133</td>
</tr>
</tbody>
</table>

From these comparisons, the HKP constraints greatly enhance the computational complexity in the model from the Newton-Euler’s law and the two first models based on the energy and virtual work related methods, due to the complex expressions of $Z$ and $\gamma$, and the Jacobian matrix $M_J$, etc.

In the following, how the shape of the condylar socket raises the computational demands is discussed. Clearly, based on the derivation in Section 3.1, the complexity of the workspace of the left and right centres of the condylar balls in Eq.(1) directly determines that of $Z$ and $\gamma$, and the Jacobian matrix $M_J$. The simplest geometric constraints could possibly be a horizontal or a vertical plane that limits the motions of the condylar ball centre. For instance, a horizontal plane

$$
Z_l = C_Z, \quad p_3 \leq X_l \leq p_4, \quad p_5 \leq Y_l \leq p_6
$$

$$
Z_h = C_Z, \quad p_3 \leq X_h \leq p_4, \quad -p_5 \leq Y_h \leq -p_6
$$

is set in $\{S\}$ as the workplace of the centres of the condylar balls, where $C_Z$ is a constant. From the identical procedure in Section 3.1, one can compute that the two parasitic motions are

$$
Z = C_Z - M_{\text{O}_M T_{l(1)}}[^s R_{s,1},] M_{\text{O}_M T_{l(3)}}
$$

$$
\gamma = -a \tan \frac{t\alpha}{s\beta}
$$

which are still quite sophisticated. A similar conclusion can be reached when the workplace is constrained on a vertical plane in $\{S\}$. If the surface is in a higher order, it can be easily imagined that the expressions of $Z$ and $\gamma$ are more complex, giving rise to an increased nonlinearity in $M_J$ and $M_J^f$, and increasing the computational time ultimately.

Besides, in [2] [3] [34], the origin $O_M$ of frame $\{M\}$ is located at the middle of $T_l T_r$, rather at the mass centre of the end effector. In this manner, only the expression of $Z$ could be a little simpler, but the dynamics of the end effector has to be analysed with more computational and symbolic efforts, which is similarly stated and discovered in Section 4.2 of [5], and it is also expected to be more error-prone.
Though an accurate computation and quantitative measurement should be made to precisely judge how the location of the origin $O_M$ of $\{M\}$ alters the computational intensity, these discussions as abovementioned are sufficient to clearly clarify that the HKP constraints considerably complicate the task of formulating equations of motion.

8 Conclusion

The inverse dynamics of a spatial RAPM constrained by two HKPs was solved systematically in five models via the conventional Newton-Euler’s law, the Lagrangian equation and the principle of virtual work. These models were thoroughly studied and compared. Generally, studies on inverse dynamics concentrate on RAPMs with only lower kinematic pairs. As a consequence, the scientific contribution of this paper is the deep study of inverse dynamics in a spatial RAPM with both lower kinematic pairs and HKPs, in terms of revealing the difficulties by HKP constraints. Specifically, the following conclusions can be drawn:

1. The structure of the dynamics model via the Newton-Euler’s law is quite different from those by the latter two algorithms using the energy and the virtual work, respectively. The nature of the mechanism, i.e., the constraint forces from HKPs onto the end effector, is revealed by this law quite well. By contrast, they cannot be discovered via the latter two algorithms directly. In addition, the number of kinematic chains influences the existence of reaction forces at the spherical joints in the final dynamics equation from the Newton-Euler’s law, while they always do not appear in the models from the latter two methods.

2. The model by the Newton-Euler’s law under the first objective aim, and the four models from the latter two methods using the pseudo-inverse solution produce identical numerical results. In addition, by the Newton-Euler’s law, under the second aim and the case when the constraint forces at HKPs are directly set as zero, the actuating torques are equivalent.

3. The efficiency of the model from the Newton-Euler’s law is quite acceptable, while the first model by the Lagrangian formulation is extremely cumbersome among the five models, indicating it involves the greatest number of arithmetic operations. The two models from the principle of virtual work are the most computationally economic. In the two models by the Lagrangian formulations, the second one is much faster; an identical conclusion is also effective to the two models by the principle of virtual work. The computational cost from both the second models in the latter two methods approximately equals that of the 6RSS PM, mainly because the dynamics model of the 6RSS PM has been used as the core in these two second models. As a consequence, they are characterised by a straightforward and intuitive implementation, a computationally sound form, and an impressive improvement of the computational demand.

4. By comparing the computational requirements between the RAPM and the 6RSS PM, and investigating the influences brought by the shape of the condylar socket, it is discovered that the HKP constraints greatly raise the complexity in the mechanism.

As a benchmark developed for future studies, the models in this paper are quite fundamental, based on which some more recent and advanced methods such as Kane’s equations and Udwadia-Kalaba theory in [7] [8] [30] [31] etc., would be utilised to formulate simpler and faster algorithms in the RAPM, facilitating its real-time control in the future work. The modelling process is also readily applicable to other RAPMs whose end effectors are constrained by the environment directly.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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[34] C. Cheng, B. Liu, Y. Li, Z. Liu, S. Yang, Y. Wang, Elastodynamic performance of a spatial redundantly actuated parallel mechanism constrained by two point-contact higher kinematic pairs via a model reduction technique, Mechanism and Machine Theory, 167 (2022) 104570.