On the Coexistence, Switching Bifurcation and FPGA Implementation for a Non-smooth Rayleigh-Duffing-like System

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Research Article

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Posted Date: March 14th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1422264/v1

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On the Coexistence, Switching Bifurcation and FPGA Implementation for a non-smooth Rayleigh-Duffing-like system

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Abstract: In this paper, the oscillation behaviors of a non-smooth Rayleigh-Duffing-like system with periodically external excitation are studied through the theory of discontinuous dynamical system. From the switching conditions of the system through the boundary, the mapping structures of Duffing-like system are presented, and the dynamical mechanism of the switching motions in the Rayleigh-Diffing-like-system are investigated. The bifurcations, phase planes and double-parameter mapping diagrams are developed, and the coexistence and multistability phenomena of the systems on the switching boundary are also studied. Finally, the field-programmable gate array (FPGA) hardware circuit experiment is applied to realize the Rayleigh-Duffing-like system, which is in consistent with the numerical simulation results.

Keywords: Switching flow, Coexistence, Rayleigh-Duffing-like system, FPGA implementation

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1. Introduction

Nonlinear science is a comprehensive discipline gradually developed on the basis of various branches of disciplines characterized by nonlinearity since the last century [1-3], which includes many complexities that cannot be ignored in practical problems. With the development of science and technology, people's desire to understand the nature of nonlinear phenomena [4] has become more and more urgent. The study of nonlinear systems has gradually attracted the attention of scholars, and has become one of the hot fields of current research. As a typical nonlinear system, Duffing systems have widely and rich dynamic characteristics [5-7], which show a variety of bifurcation phenomena and chaotic oscillation, and have great application value in engineering.

Since the Duffing system was first proposed in 1918 [8], more and more scientists have devoted themselves to the study of the Duffing system. In the literatures reported in the past, the analysis methods of Duffing system mainly include conventional numerical analysis [9], harmonic balance method [10-12], Melnikov function method [13-15] and so on. Scientists have investigate different electronic component on the basis of the classic Duffing models to construct a new system, including the famous Rayleigh-Duffing systems [16-19], van der Pol-Duffing systems [20-23], and memristor-based Duffing systems [24], and then analyzed the systems built by dynamic features. S.Sabarathinam [25] once proposed a memristive Duffing circuit system, and the dynamic behaviors of the constructed system were investigated through numerical analysis. To study the dynamics of a randomly excited nonlinear system with discontinuities, an adaptive variable time-step numerical integration method was proposed, and combined this method with the bisection method near the discontinuity, the correctness and effectiveness of the method were verified by the path numerical integration method finally [26]. Compared with smooth systems, non-smooth systems can not only produce richer nonlinear dynamic phenomena, but also describe the actual system more accurately. Therefore, non-smooth systems are more worthy of study. First of all, a new Duffing-like system with chaotic generator $x|x|$ was first constructed [27], subsequently, the signum function was also applied to the construction of non-smooth Duffing systems [28-29]. Base on discontinuous dynamics theory, Chen and Min studied a Duffing non-autonomous system with absolute function [30], and analyzed the complex behavior of non-smooth Duffing system by applying different parameter mappings. In
order to understand the kinematic complexity of non-smooth Duffing systems, Xu [20] studied the bifurcation properties of van der Pol-Duffing systems under different motion states by a semi-analytical method, and based on the eigenvalues, the richer nonlinear dynamics of the system were revealed. Finally, the semi-analytical results were verified by the midpoint integration method. For the analysis of non-smooth system, although numerical analysis can give the analytical solutions of the system, there has been a lack of theoretical analysis. Luo [31-32] proposed a new theory for non-smooth dynamical systems with connectable and accessible subdomains, and studied the local singularities and switchability of flows on separation boundaries from one domain to its neighboring domains by such a theory.

Motivated by the above discussions in this work, the theory of discontinuous dynamical system for nonlinear system will be adopted in this paper, the analytical conditions of the system motions on the separation boundary are analyzed to understand the complexity of motion switching. In addition, in order to further understand the motion state of the periodically driven non-smooth Rayleigh-Duffing-like-system, the conversion plane and mapping structure are defined respectively. The coexistence characteristics are studied through numerical simulations, and the system is realized by FPGA to verify the correctness of the numerical analysis. The main contents of the remaining chapters of this paper are as follows: Section 2 descripts a nonlinear system and figures out the switching conditions via the boundary. Furthermore, the mapping structures are introduced in section 2, too. Numerical simulations and results are presented in section 3. Then, in section 4 simulation results measured from Field-Programmable Gate Array is exhibited. Finally, section 5 concludes the paper.
2. Problem statement

2.1 System description

The periodically driven Rayleigh-Duffing-like system with the function $x|x|$ is considered as

$$\ddot{x} - ax + bx|x| = \varepsilon [\mu (1 - \dot{x}^2)\dot{x} + F \sin(\omega t)]$$  \hspace{1cm} (1)

where the spring coefficients $a$ and $b$ are positive, $\varepsilon$ is a small positive constant, $\mu$ is the damping coefficient, $F$ and $\omega$ are the amplitude and frequency of the periodically excitation, respectively. The Rayleigh-Duffing-oscillator[19] in Eq.(1) has Small horseshoes which was induced by the quadratic function through the analysis of Melnikov function method.

The system described in Eq.(1) belongs to non-smooth system due to the absolute function, and the switching flow exists in domains. As shown in Fig. (1), the concepts of regions and boundaries are formulated in a phase plane. According to the Duffing system with absolute function studied in Eq.(1), the phase plane is mainly divided into three parts: two distinct domains and a separation boundary. The dark area on the left represents $\Omega_2$, the tint area on the right is $\Omega_1$, and the boundary of the middle dotted line is $\partial \Omega_{\alpha\beta}$. Two arrows means the possible directions of the flow to the switching boundary.

![Phase plane divided by two regions and a boundary](image)

Therefore, two sub-domains $\Omega_\alpha (\alpha \in \{1,2\})$ and their boundary $\partial \Omega_{\alpha\beta}$, which are given the system separation, are described as
\[ \Omega_1 = \{ (x, y) \mid x \in (D, +\infty) \} \]
\[ \Omega_2 = \{ (x, y) \mid x \in (-\infty, D) \} \]

and
\[ \partial \Omega_{\alpha \beta} = \Omega_{\alpha} \cap \Omega_{\beta} = \{ (x, y) \mid \varphi_{\alpha \beta}(x, y) = 0 \} \]

It should be noted that the subscript \((\cdot)_{\alpha \beta}\) signifies the boundary from \(\Omega_{\alpha}\) to \(\Omega_{\beta}\) \((\alpha, \beta \in \{1, 2\}\) and \(\alpha \neq \beta\)).

Furthermore, the state variable \(x\) and the vector field \(F\) are given the definitions as follows:
\[ x = (x, \dot{x})^T = (x, y)^T \text{ and } F=(y,F)^T \]

Following the method in [33] and the above formula, the differential equation of Duffing system can be expressed as
\[ \dot{x} = F^{(\kappa)}_{\lambda}(x, t), (\kappa, \lambda \in \{0, 1, 2\}) \]

where
\[
\begin{align*}
F^{(\alpha)}_{\alpha}(x, t) &= (y, F^{(\alpha)}_{\alpha}(x, t))^T \text{ in } \Omega_{\alpha} (\alpha \in \{1, 2\}), \\
F^{(\beta)}_{\alpha}(x, t) &= (y, F^{(\beta)}_{\alpha}(x, t))^T \text{ in } \Omega_{\alpha} (\alpha \neq \beta \in \{1, 2\}); \\
F^{(0)}_{\alpha}(x, t) &= (y, 0)^T \text{ on } \partial \Omega_{\alpha \beta} \text{ for stick,} \\
F^{(0)}_{\alpha}(x, t) &= \{ F^{(\alpha)}_{\alpha}(x, t), F^{(\beta)}_{\alpha}(x, t) \} \text{ on } \partial \Omega_{\alpha \beta} \text{ for non-stick} \\
F^{(1)}_{1}(x, t) &= ax - bx|x| + \varepsilon[\mu(1-x^2)\dot{x} + F \sin(\omega t)] \\
F^{(2)}_{2}(x, t) &= ax + bx|x| + \varepsilon[\mu(1-x^2)\dot{x} + F \sin(\omega t)]
\end{align*}
\]

Through \(\kappa\) and \(\lambda\) in Eq. (5), the imaginary and real of the vector field in the two regions of system motion can be analyzed. Substituting into Eq. (6), the vector field \(F^{(\alpha)}_{\alpha}(x, t)\) is real in the region \(\Omega_{\alpha}\), while the vector field \(F^{(\beta)}_{\alpha}(x, t)\) is imaginary in the region \(\Omega_{\alpha}\), and is determined by the vector field in the region \(\Omega_{\beta}\). In addition, \(F^{(0)}_{\alpha}(x, t)\) represents the vector field on the boundary, and \(F_{\alpha}(x, t)\) is the scalar force in the region\(\Omega_{\alpha}\). The above concepts are all used to describe the discontinuity of the Rayleigh-Duffing-like system on the boundary.

2.2 Switching conditions

From Luo[32,33], the necessary and sufficient conditions for the system crossing the boundary \(\partial \Omega_{\alpha \beta}\) are
\[ G^{(0,\alpha)}(x_m, t_{m-}) = n_x^{\alpha\beta} \cdot F^{(\alpha)}(x_m, t_{m-}) < 0, \]
\[ G^{(0,\beta)}(x_m, t_{m+}) = n_y^{\alpha\beta} \cdot F^{(\beta)}(x_m, t_{m+}) < 0, \]
for \( n_x^{\alpha\beta} \rightarrow \Omega_\alpha \)
\[ G^{(0,\beta)}(x_m, t_{m-}) = n_y^{\alpha\beta} \cdot F^{(\beta)}(x_m, t_{m-}) > 0, \]
\[ G^{(0,\alpha)}(x_m, t_{m+}) = n_x^{\alpha\beta} \cdot F^{(\alpha)}(x_m, t_{m+}) > 0 \]
for \( n_y^{\alpha\beta} \rightarrow \Omega_\beta \) \hspace{1cm} (8)

where \( \alpha, \beta \in \{1, 2\} \) and \( \alpha \neq \beta \) with
\[ n_{\alpha\beta} = \nabla \phi_{\alpha\beta} = (\partial_x \phi_{\alpha\beta}, \partial_y \phi_{\alpha\beta})^T \] \hspace{1cm} (9)

Note that \( \nabla = (\partial_x, \partial_y)^T \) is the Hamilton operator with \( \partial_x(\cdot) = \partial(\cdot) / \partial x \) and \( \partial_y(\cdot) = \partial(\cdot) / \partial y \). Moreover, \( t_m \) is the switching time in the switching set, and \( t_{m \pm} = t_m \pm 0 \) means motions in domains close to the boundary rather than on the boundary. Therefore, according to Eq.\( (8) \), \( t_{m-} \) and \( t_{m+} \) represent the time of approaching and departing from the boundary, respectively.

The analytical conditions to perform grazing motion on boundary \( \partial \Omega_{\alpha\beta} \) are given by
\[ G^{(0,\alpha)}(x_m, t_{m z}) = n_x^{\alpha\beta} \cdot F^{(\alpha)}(x_m, t_{m z}) = 0, \]
\[ G^{(1,\alpha)}(x_m, t_{m z}) = n_x^{\alpha\beta} \cdot DF^{(\alpha)}(x_m, t_{m z}) > 0 \]
in domain \( \Omega_\alpha \) \hspace{1cm} (10)

for \( n_x^{\alpha\beta} \rightarrow \Omega_\alpha \)

and
\[ G^{(0,\beta)}(x_m, t_{m z}) = n_y^{\alpha\beta} \cdot F^{(\beta)}(x_m, t_{m z}) = 0, \]
\[ G^{(1,\beta)}(x_m, t_{m z}) = n_y^{\alpha\beta} \cdot DF^{(\beta)}(x_m, t_{m z}) < 0 \]
in domain \( \Omega_\beta \) \hspace{1cm} (11)

for \( n_y^{\alpha\beta} \rightarrow \Omega_\beta \)

where
\[ DF^{(\alpha)}(x, t) = (F_\alpha(x, t), \nabla F^{(\alpha)}(x, t) \cdot F^{(\alpha)}(x, t) + \hat{\partial}_t F_\alpha(x, t))^T \] \hspace{1cm} (12)

According to Eq.\( (3) \) and Eq.\( (9) \), the normal vector can be obtained as follows,
\[ n_{\alpha\beta} = n_{\alpha\beta} = (1, 0)^T \] \hspace{1cm} (13)

The zero and first order G-functions for \( \alpha \in \{1, 2\} \) are given by
From Eq.(8) and Eq.(14), the analytical conditions for the passable motion of
the Rayleigh-Duffing-like system via the boundary $\partial \Omega_{\alpha \beta}$ are shown as

\begin{align}
G^{(0,1)}(x_m, t_{m\pm}) &= y_{m-} < 0, \quad \text{from } \Omega_1 \to \Omega_2 \\
G^{(0,2)}(x_m, t_{m\pm}) &= y_{m+} < 0
\end{align} 

and

\begin{align}
G^{(1,1)}(x_m, t_{m\pm}) &= F_1(x_m, t_{m\pm}) > 0, \\
G^{(1,2)}(x_m, t_{m\pm}) &= F_2(x_m, t_{m\pm}) > 0
\end{align} 

According to Eq.(10) and Eq.(11), the analytical conditions for the grazing
motion at boundary $\partial \Omega_{\alpha \beta}$ are given by

\begin{align}
G^{(0,1)}(x_m, t_{m\pm}) &= y_{m\pm} = 0, \quad \text{in domain } \Omega_1 \\
G^{(1,1)}(x_m, t_{m\pm}) &= F_1(x_m, t_{m\pm}) > 0
\end{align} 

and

\begin{align}
G^{(0,2)}(x_m, t_{m\pm}) &= y_{m\pm} = 0, \quad \text{in domain } \Omega_2 \\
G^{(1,2)}(x_m, t_{m\pm}) &= F_2(x_m, t_{m\pm}) < 0
\end{align} 

2.3 Switching sets and mapping structures

In order to further understand the switching state of the Rayleigh-Duffing-like
system in the process of motion, this section mainly presents the switching set and
the mapping structure, and describes the corresponding mapping structure in the
periodic state and the chaotic state, as shown in Fig.2. When the motion trajectory
of the system is close to the left boundary, the switching set is $\Xi^+$, and when the
motion trajectory is close to the right boundary, the switching set is $\Xi^-$. The
specific concept is shown in the following formula

\begin{align}
\Xi^+ &= \{(y_k, \omega t_k) \mid x_\parallel = D^+ \}, \\
\Xi^- &= \{(y_k, \omega t_k) \mid x_\parallel = D^- \}
\end{align} 

The switching conditions represented by the mapping structure $P_\alpha (\alpha \in \{1,2\})$
in Fig. 2 are as follows:

\[ P_1 : \Xi^+ \to \Xi^+, \quad P_2 : \Xi^- \to \Xi^- \quad (21) \]

Combining the above two equations, the mapping structure \( P_a (\alpha \in \{1, 2\}) \) can also be changed to the following

\[
\begin{align*}
P_1 &: (D^+ , y_k, \omega t_k) \to (D^+ , y_{k+1}, \omega t_{k+1}) \\
P_2 &: (D^- , y_k, \omega t_k) \to (D^- , y_{k+1}, \omega t_{k+1})
\end{align*}
\]

(22)

The mapping structure of the system in the periodic motion state can be expressed by

\[
P = (p^{(k_{i2})}_{2} \circ p^{(k_{i1})}_{1}) \circ \ldots \circ (p^{(k_{i1})}_{2} \circ p^{(k_{i1})}_{1})
\]

(23)

where \( k_{i1} \in \{0, 1\} \) for \( l \in \{1, 2, \ldots , m\} \) and \( \lambda \in \{1, 2\} \). \( P^{(0)}_{\lambda} = 1 \) and \( P^{(k)}_{\lambda} = P_{\lambda} \circ P^{(k-1)}_{\lambda} \).

Moreover, Eq.(23) can be further simplified as

\[
P = \left( \circ_{l=1}^{m} p^{(k_{i1})}_{l} \right)
\]

(24)

Substituting \( m = k_{12} = k_{11} = 1 \) into Eq.(23), the trajectory of periodic motion crosses the switching boundary, and the specific mapping structure is as follows:

\[ P_{2l} = P_2 \circ P_1 : \Xi^+ \to \Xi^- \quad (25) \]

And

\[
\begin{align*}
P_1 &: (D, y_k, t_k) \to (D, y_{k+1}, t_{k+1}) \\
P_2 &: (D, y_{k+1}, t_{k+1}) \to (D, y_{k+2}, t_{k+2})
\end{align*}
\]

(26)

Similarly, the mapping structure of the system in other periodic motions can be deduced according to the similar method.

Fig.2. Switching sets and mappings
3. System dynamical behaviors

3.1 Bifurcation and phase analysis

For a better understanding of dynamical behaviors in the Rayleigh-Duffing-like system, the switching bifurcations with varying spring coefficients and amplitude will be presented, and the coexisting attractors through the switching boundary will also be drawn with different mapping structures.

Firstly, take initial values \( x_0 = -0.1 \), \( y_0 = -0.1 \) and the system parameters \( a = 1, \varepsilon = 0.1, F = 12, \omega = 1, \mu = 6 \), draw the bifurcation diagram of the system with the parameter \( b \) changing, where "PD" represents "Period-doubling Bifurcation", "SN" represents "Saddle-node Bifurcation", and "PF" represents "Pitchfork Bifurcation". As shown in Fig.3 (a) below, the system undergoes a period-doubling bifurcation, temporarily enters a chaotic state when \( b = 0.485 \), and then changes back to the periodic state for \( b = 0.567 \). For \( b = 0.764 \), the system undergoes a pitchfork bifurcation, and the system has a saddle-node bifurcation at \( b = 0.852 \), subsequently. The coexistence state of the system is depicted by Fig.3 (b) with different initial conditions. By reducing the step size, the bifurcation interval is enlarged, and the characteristics of each bifurcation type in the interval are more clearly observed. In addition, when different bifurcation types occur in the system, the corresponding parameter values and the coexistence mapping structure under the corresponding parameter range are shown in Table 1. With different mapping structures, the coexistence behaviors are drawn in Fig.4. The coexisting chaotic motions for the mapping structures \( P_{21}^{(21)} \) with \( P_{21}^{(21)} \) for \( b = 0.495 \) are plotted in Fig.4(b), and the coexisting periodical motions are presented the mapping structures \( P_{21} \) with \( P_{21} \) for \( b = 0.45 \) in Fig.4(a), \( P_{21}^{(21)} \) with \( P_{21}^{(21)} \) for \( b = 0.77 \) in Fig.4(c), \( P_{21} \) with \( P_{21} \) b = 0.86 in Fig.4(d).
Fig. 3 Switching scenario with varying parameter b: (a) switching velocity; (b) the coexistence bifurcation with different initial conditions

Table 1 Bifurcation type, mapping structure and corresponding value of b

<table>
<thead>
<tr>
<th>Bifurcation type</th>
<th>Value of b</th>
<th>Ranges of b</th>
<th>Coexisting mapping structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>PD</td>
<td>0.485</td>
<td>(0.380,0.485)</td>
<td>$P_{21}$ with $P_{21}$</td>
</tr>
<tr>
<td>SN</td>
<td>0.567</td>
<td>(0.485,0.490)</td>
<td>$P_{(21)^2}$ with $P_{(21)^2}$</td>
</tr>
<tr>
<td>PF</td>
<td>0.764</td>
<td>(0.490,0.500)</td>
<td>$P_{(21)^n}$ with $P_{(21)^n}$</td>
</tr>
<tr>
<td>SN</td>
<td>0.852</td>
<td>(0.764,0.775)</td>
<td>$P_{(21)^n}$ with $P_{(21)^n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.852,0.875)</td>
<td>$P_{21}$ with $P_{21}$</td>
</tr>
</tbody>
</table>
Fig. 4 Coexistence mapping structure diagram corresponding to different parameters b:

(a) $b=0.45$, $P_{21}$ with $P_{21}$; (b) $b=0.495$, $P_{(21)}$ with $P_{(21)}$; (c) $b=0.77$, $P_{(21)}$ with $P_{(21)}$; 
(d) $b=0.86$, $P_{21}$ with $P_{21}$.

Next, take the parameter $a=1, b=1, \varepsilon=0.1, \omega=1, \mu=6$ with the initial value $x_0 = -0.001, y_0 = -0.001$. When the amplitude $F$ of the system changes in the interval $(0, 16)$ with a step size of $\Delta F = 0.002$, the bifurcation diagram of the system with the amplitude $F$ as the change parameter is shown in Fig. 5 (a). The system goes from a chaotic state to a saddle-node bifurcation at $F=2.67$ and then enters a periodic state and a chaotic state in turn. It can be observed that there are many periodic windows in the chaotic state, and finally the system enters a stable periodic state. The partial period window of the system is enlarged as shown in Fig. 5 (b), and the coexistence bifurcation diagrams of the system in this part are drawn. The coexistence mapping structure of the system in the range of amplitude $[3.8, 7.6]$ and the corresponding amplitude range are listed in Table 2. In Fig. 6(a), the coexisting phase diagrams of $F=4.07$ with the mapping structure $P_{(21)}$ and $P_{(21)}$ are drawn, which is in consistent with the bifurcation diagram in Fig. 5 (b). The coexistence of chaos and periodicity for $F=4.7$ with $P_{(21)}$ and $P_{21}$, respectively, are also observed in Fig. 6 (b), and Fig. 6 (c) is the mapping structures $P_{21}$ and $P_{21}$, that is, the coexistence of two periodic orbits. Fig. 6 (d) shows the coexistence of two asymmetric chaotic attractors with the amplitude value $F=7.5$.

<table>
<thead>
<tr>
<th>Coexisting mapping structures</th>
<th>Ranges of F</th>
<th>Coexisting mapping structures</th>
<th>Ranges of F</th>
</tr>
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<tbody>
<tr>
<td></td>
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</tbody>
</table>
Fig. 5 Switching Boundary Bifurcation Diagram with Parameter $F$:

(a) global bifurcation of $F \in (0, 16)$; (b) local coexistence bifurcation of $F \in (3.80, 7.60)$
Moreover, with varying the spring coefficient $a$, the switching bifurcation for $a \in (0,2.6)$ is depicted in Fig.7, with $b=1, \varepsilon=0.1, F=12, \omega=1, \mu=6$, initial conditions $x_0=0.1, y_0=0.1$, and the step size of 0.001. When $a=1.1$, the system goes through a saddle-node bifurcation, and then goes from a periodic state to a chaotic state. Thereafter, the system is in a chaotic state with multiple periodic windows. The motion of the system does not pass the boundary in the range of $a \in (2.52,2.58)$, and chaotic oscillation occurs in the range of $a \in (2.58,2.59)$. In order to make the coexistence phenomenon in the bifurcation diagram more obvious, the periodic window of the coexistence part in the interval $a \in (1.34,2.00)$ is enlarged in Fig.7 (b). Finally, let the system parameter $a=1,b=1,\varepsilon=0.1,\omega=1,F=12$, the bifurcation diagram of the system with changing the damping coefficient $\mu$ can be obtained in Fig.8. With the increase of parameters, the system gradually changes from the initial periodic state to the chaotic state. Fig.8 (b) shows the local coexistence phase diagram of the system at $(16.5,27.5)$, which details the coexistence mapping structure of the system within the period window.
3.2 Dual-Parameters Mapping Analysis

In order to better analyze the changes of the mapping structure of the system in Eq.(1), the parameter maps are plotted in Fig.9, in which shows the mapping structures of chaotic and periodic orbits with different parameters. Fig.9 (a) is a dual-parameter mapping diagram of the system changing with parameters a and b when the system parameter is $F = 12, \omega = 1, \varepsilon = 0.1, \mu = 6$. Fig.9 (b) is a dual-parameter mapping diagram with the parameter $\mu$ and the amplitude F with $a = 1, b = 1, \varepsilon = 0.1, \omega = 1$. The blue, dark blue, green, yellow, red, and gray in Fig. 9 represent the system mapping structures $P_{21}$, $P_{(21)^2}$, $P_{(21)^3}$, $P_{(21)^4}$, $P_{(21)^{\infty}}$ and $P_{(21)^{\infty}}$, respectively. The mapping structure $P_{(21)^{\infty}}$ indicates the complex motions such as the chaotic state or multi-periodic orbits including the map structure crossing the boundary more than four times, and the mapping structure $P_{(21)^{\infty}}$ means that motion trajectory does not cross the boundary. The colors and corresponding mapping structures in the dual-parameter map are shown in the Table.3. From the gray are in Fig.9 (a), with the decrease of parameter a and the increase of b, the system has experienced an area with no motion trajectory, and the mapping structure gradually changes from $P_{(21)^{\infty}}$ to $P_{21}$. When a=1, the system is in a complex motion state with parameters $b \in (0.50, 0.57) \cup (0.66, 0.73) \cup (0.78, 0.85)$. This is consistent with the corresponding mapping structure in the bifurcation diagrams.
of Fig.3 (b) and Fig.7 (b). In Fig.9 (b), when \(0 < \mu < 70, \ 0 < F < 16\), the system mainly exhibits complex motions, and the mapping structure of the system can be described as \(P_{(21)^*}\). When \(\mu = 6\), the system is in a periodic state when the amplitude \(F \in (5, 5.6) \cup (7,7.5) \cup (11,16)\), and the other regions are in a complex motion state, which corresponds to the switching bifurcation diagram in Fig.5 (a).

![Parameter maps](image)

(a) (b)

Fig.9 Parameter maps: (a) a-b (\(\mu=6, \omega=1, \varepsilon=0.1,F=12\)); (b) \(\mu-F\) (\(a=1,b=1, \omega=1, \varepsilon=0.1\))

**Table.3** Mapping structure and corresponding color scale

<table>
<thead>
<tr>
<th>Mapping Structure</th>
<th>(P_{21})</th>
<th>(P_{(21)^7})</th>
<th>(P_{(21)^8})</th>
<th>(P_{(21)^4})</th>
<th>(P_{(21)^*})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Color</td>
<td>Blue</td>
<td>Dark blue</td>
<td>Green</td>
<td>Yellow</td>
<td>Red</td>
</tr>
</tbody>
</table>

3.3 Mapping Structural Analysis

In order to understand the switching motion mechanism of the system at the displacement boundary, the symplectic algorithm is used to numerically analyze the Duffing system in this section. With two common mapping structures \(P_{21}\) and \(P_{(21)^*}\), the corresponding phase trajectory diagrams, time series diagrams, and G-function diagrams are drawn, respectively, as shown in Figure 10 and Figure 11, where the switching points though the displacement boundary from \(\Omega_2 \to \Omega_1\) and \(\Omega_1 \to \Omega_2\) are represented by the blue and pink dots, respectively.

First of all, from the bifurcation diagram in Fig. 5 above and the dual-parameter mapping structure diagram in Fig. 9, when the parameter of the system is set to \(a = 1, b = 1, \varepsilon = 0.1, \omega = 1, \mu = 6, F = 4.7\) and the initial value is \((0.3909, -0.9100)\), the
Duffing system works in a periodic motion state with mapping structure $P_{21}$, as shown in Fig.10 (a). Fig.10 (b) shows the time series diagram of the system with respect to the displacement variable $x$, in which the distance between two adjacent blue dots represents one cycle of the system motion, and $x = 0$ is set as the separation line. According to Eq.(14), Eq.(16) and Eq.(17), the G-function time history is illustrated in Fig.10(c), and $G^{(0,2)} = G^{(0,1)} = y$ can be obtained. From the passability conditions of motion, the periodic orbits is always switchable at the displacement boundary, and the G-functions has the same signs. On the one hand, when both $G^{(0,1)}$ and $G^{(0,2)}$ are less than 0, the mapping structure $P_{21}$ does crossing motion against the normal vector by the displacement boundary, and the phase motion trajectory switches from the area $\Omega_1$ on the left side of the separation line to the area $\Omega_2$ on the right side. On the other hand, when both $G^{(0,1)}$ and $G^{(0,2)}$ exceed 0, the mapping structure $P_{21}$ does crossing motion against the normal vector by the displacement, and the phase motion trajectory switches from the area $\Omega_2$ on the right side of the separation line to the area $\Omega_1$ on the left side. The discrete trajectory composed of many red and black dots is shown in Fig. 10(c), where the red dots represent the motion trajectories of the system in area $\Omega_2$, the black dots represent the motion trajectories of the system in area $\Omega_1$, and $T$ represents one motion period of the system. All the switching G-function points during the movement process are plotted in Fig.10 (d). It can be found that the ordinate positions of the blue point and the pink point are the same as those of Fig. 10(a) from $\Omega_2 \rightarrow \Omega_1$ and $\Omega_1 \rightarrow \Omega_2$ at the time displacement boundary. At the same time, it is verified that the mapping structure $P_{21}$ of the system is right at this moment.
In order to compare chaotic motion state with the periodic motion state, the initial value is adjusted to (-2.0723, 0.0712) when the system parameters remain unchanged, the same as in Fig.10. Fig.11 (a) shows the chaotic motion of the system with mapping structure $P_{2\text{I}}$. It can be found from both Fig.11 (b) and Fig.11 (c) that the mapping structure of the system motion still moves through the boundary along the normal vector, but the time of each cycle of the motion is different. The discrete trajectories in Fig.10 (c) are regular and periodic, while the discrete trajectories in Fig.11 (c) are random and chaotic, with no rules to follow. It can also be seen from Fig.11 (d) that there are more switching points at the displacement time boundary, and the curve is formed by discrete solid points, which further proves that the current system is in the mapping structure $P_{2\text{I}}$. 

Fig.10 Periodic motion of $P_{2\text{I}}$: (a) phase plane; (b) time history of state variable $x$; (c) G-function time history; (d) trajectory mapping of G-functions on switching boundaries.
Fig. 11 Periodic motion of $P_e(x)$: (a) phase plane; (b) time history of state variable $x$; (c) $G$-function time history; (d) trajectory mapping of $G$-functions on switching boundaries.

4. FPGA implementation

From the perspective of hardware design, FPGA has the characteristics of high efficiency, low cost and strong stability, which is one of the effective methods to digitally implement analog circuits. With the system in Eq. (1), In order to realize the Duffing system with absolute function and external excitation on FPGA, the discrete model of the Rayleigh-Duffing-like system is implemented by the second-order Runge-Kutta method, and described as

$$
\begin{align*}
\dot{x}_{i+1} &= x_i + T*(K_{11} + K_{12})/2 \\
\dot{y}_{i+1} &= y_i + T*(K_{21} + K_{22})/2
\end{align*}
$$

(27)

with

$$
\begin{align*}
K_{11} &= y_i \\
K_{21} &= ax_i - bx_i|x_i| + \varepsilon[\mu(1-y_i^2)y_i + F \sin(\omega t)]
\end{align*}
$$

(28)

and

$$
\begin{align*}
K_{12} &= y_i + T*K_{21} \\
K_{22} &= a(x_i + T*K_{11}) - b(x_i + T*K_{11})(x_i + T*K_{11}) \\
&+ \varepsilon[\mu(1-(y_i + T*K_{21})^2)(y_i + T*K_{21}) + F \sin(\omega(t+T))] \\
\end{align*}
$$

(29)

where $T$ is the iterative step.

From Eqs. (27)-(29), the entire circuit consists of four modules, namely Module_Top, Module_Intermediate, Module_Iteration, Module_Conversion, where
Module_Top is the top-level module, and the other three modules are sub-modules. Module_Top is designed to invoke the remaining sub-modules in order to adjust the calculation steps of the system. Module_Intermediate is used to calculate each recursive parameter $K_{11}$, $K_{21}$ and $K_{12}$, $K_{22}$ in the discrete equation. Module_Iteration is to find the median of each item to obtain new values $x_{i+1}$ and $y_{i+1}$. Module_Conversion is mainly composed of the high-speed Digital/Analog converter modules, which is convenient for subsequent D/A chips can perform digital-to-analog conversion.

In the simulation environment of Xilinx-ISE, by using Verilog language to write, compile and burn the system program to the FPGA development board, and connect both ends of the development board to the digital oscilloscope and PC through JTAG, D/A converter and double bus connectors. The observed waveforms are shown in Fig 12, which show the velocity-displacement time series and x-y phase plots of CH1 and CH2 channels in Fig.12 (a) and (b), respectively, where the blue line represents the x time series and the yellow line represents the y time series. By selecting the system parameters with the same as in Fig.4, the phase diagrams displayed with the mapping structures $P_{21}$, $P_{(21)^*}$, $P_{(21)^*}$, $P_{21}$, respectively, in Fig.13 (a)-(d) are in consistent with those shown in Fig. 4, which verifies the consistency between the numerical simulation results and the circuit simulation results.

![Fig.12](image-url) Chaos attractor and sequence diagram based on FPGA: (a) the time series of x and y (blue: x; yellow: y) (b) x-y phase diagram
5. Conclusion

In this paper, from the theory of flow switching, a Rayleigh-Duffing-like system with absolute function and external excitation is studied and the judgment conditions for the switching motion of system on the boundary are deduced. The concepts of $G$ function and mapping structure are applied to analyze the discontinuous dynamic characteristics of the system in the vector field. The motion trajectory of the system is represented by the mapping structure, and the bifurcation characteristics and coexistence phenomenon of Rayleigh-Duffing-like are analyzed by bifurcation diagram and phase diagram respectively, and the multistability phenomenon of the system on the boundary is also studied in detail. Moreover, through the dual-parameter maps, it is observed that the system exhibits more complex dynamic behaviors when the two parameters change simultaneously. In addition, the design and experiment of the corresponding hardware circuit are completed with the help of FPGA, and the correctness of the theoretical analysis is verified.

Acknowledgements: This work is supported by National Natural Science Foundation of China under Grant No. 61971228, 61871230, the Natural Science Foundations of Jiangsu Higher Education Institutions of China under Grant No.19KJB520042, and the Postgraduate Research and Practice Innovation Program of Jiangsu of China under Grant No. KYCX21_1390 and Grant No. KYSX21_564.

Conflict of Interest: Compliance with ethical standards. The authors declare that they have no conflict of interest.

Availability of data and materials
The datasets supporting the conclusions of this article are included within the article and its additional files.

Reference


