Pairwise Q*-s-(regular and normal) spaces in bitopological spaces

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Pairwise $Q^*$- (regular and normal) spaces in bitopological spaces

P. Padma and Alias B. Khalaf

Abstract: The notion of $\tau_1\tau_2 - Q^*$ - open sets in a bitopological spaces was introduced by K.Kannan and K.Chandrasekhararao. We introduce the notion of pairwise $Q^*$-regular, pairwise $Q^*$-normal, pairwise $s\ Q^*$-normal and obtain some characterizations of pairwise $Q^*$-regularity and pairwise $Q^*$-normality, pairwise $s\ Q^*$-normal.

Keywords: pairwise $Q^*$ - regular ; pairwise $Q^*$ - normal ; pairwise $s\ Q^*$ - normal; pairwise $Q^*$ - normal.

2010 Mathematics Subject Classification: 54E55.

1 Introduction

Separation axioms are properties by which the topology on a space $X$ separates points from points, points from closed sets and closed sets from each other. The various separation axioms give rise to a sequence of successively stronger requirements, which are put upon the topology of a space to separate varying types of subsets. These axioms are also found useful to characterize continuous mappings. In 1963, Levine introduced the concept of semi-open sets. Maheshwari and Prasad have introduced pairwise semi-$T_i$-spaces, $i \in \{0, 1, 2\}$. Using the notion of semi-open sets, Maheshwari, Prasad and Bhamini have defined and studied the notions of pairwise $s$-normal (resp. pairwise irresolutely normal), if for any pair of disjoint $\tau_i$- closed set $A$ and a $\tau_j$- closed set $B$ ($\tau_i$- semi closed set $A$ and a $\tau_j$ - semi-closed set $B$ ), there exists a $\tau_j$ - semi open set $U$ and a $\tau_i$ - semi open set $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$; $i \neq j$, $i, j =1, 2$.

The notion of $Q^*$-open sets in a topological space was introduced by Murugalingam and Lalitha [12, 13]. Mean while J. C. Kelly introduced bitopological space in 1963. There after, several authors studied the above mentioned concepts in bitopological settings. The notion of pairwise semi-$T_0$, pairwise semi-$T_1$, pairwise semi-$T_2$, pairwise $s$-regular and pairwise $s$-normal spaces, $s$-normal (resp. semi normal) spaces were introduced and studied by Maheshwari and Prasad [5, 6, 7, 9, 10, 11]. In this paper, the notion of pairwise $Q^*$-$s$-regular spaces and pairwise $Q^*$-$s$-normal spaces are introduced and their basic properties in bitopological spaces are discussed.

2 Preliminaries

Let $(X, \tau_1, \tau_2)$ or simply $X$ denote a bitopological space. For any subset $A \subseteq X$, $\tau_i - int(A)$ and $\tau_i - cl(A)$ denote the interior and closure of a set $A$ with respect to the topology $\tau_i$, respectively. $A^C$ denotes the complement of $A$ in $X$ unless explicitly stated. We give the following definitions in bitopological spaces.

Definition 2.1 A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise homeomorphism if the induced functions $f : (X, \tau_1) \rightarrow (Y, \sigma_2)$ and $f : (X, \tau_2) \rightarrow (Y, \sigma_1)$ are
homeomorphism.

**Definition 2.2** A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be pairwise semi-homeomorphism if the induced functions \( f : (X, \tau_1) \to (Y, \sigma_1) \) and \( f : (X, \tau_2) \to (Y, \sigma_2) \) are semi-homeomorphism, i.e., the induced function are pre-semi open, irresolute and bijective.

**Lemma 2.3** [12] Let \( X \) be a topological space. Then the family of all \( Q^* \)-open sets in \( X \) with \( \phi \) is a topology. It is denoted by \( \tau_{Q^*} = \sigma^* \).

**Lemma 2.4** [12] Let \( X \) be a topological space. Then the set of all \( Q^* \)-closed sets with \( X \) is a topology. It is denoted by \( \tau_{Q^*} = \mu^* \).

**Example 3.2** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{\emptyset, X, \{c\}, \{b\}, \{a, c\}, \{a, b\}\} \), \( SO(X, \tau_1) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}\} \), \( SO(X, \tau_2) = \{\emptyset, X, \{c\}, \{b\}, \{a, c\}, \{a, b\}\} \) and \( \sigma_1 = \{\emptyset, X, \{a, b\}\} \), \( \sigma_2 = \{\emptyset, X, \{b, c\}\} \). Then the space \( X \) is pairwise \( g^* \)-regular but not pairwise \( Q^* \)-regular space.

**Theorem 3.3** For a space \( X \), the following are equivalent:

(a) \((X, \tau_1, \tau_2)\) is pairwise \( Q^* \)-regular.

(b) For each \( x \in X \) and every \( \tau_i - Q^* \)-open set \( U \) containing \( x \), there exists a \( \tau_i \)-semi open set \( H \) such that \( x \in H \subseteq \tau_j - cl(H) \subseteq U \); \( i \neq j \), \( i, j = 1, 2 \).
(c) For every \( \tau_1 - Q^* \)-closed set \( F \), the intersection of all \( \tau_j \)-semi closed, \( \tau_j \)-semi neighborhoods of \( F \) is exactly \( F \); \( i \neq j \), \( i, j = 1, 2 \).

(d) For every set \( A \) and a \( \tau_i - Q^* \)-open set \( B \) such that \( A \cap B = \emptyset \), there exists a \( \tau_i \)-semi open set \( W \) such that \( A \cap W \neq \emptyset \) and \( \tau_j - scl(W) \subseteq B \); \( i \neq j \), \( i, j = 1, 2 \).

(e) For every nonempty set \( A \) and any \( \tau_i - Q^* \)-closed set \( B \) satisfying \( A \cap B = \emptyset \), there exists a \( \tau_i \)-semi open set \( U \) and a \( \tau_j \)-semi open set \( V \) such that \( A \cap U \neq \emptyset \) and \( B \subseteq V \) and \( U \cap V = \emptyset \); \( i \neq j \), \( i, j = 1, 2 \).

**Proof.** (a) \( \rightarrow \) (b) Let \( x \in U \) and \( U \) is \( \tau_i - Q^* \)-open in \( X \). Therefore, \( x \notin X - U \) and \( X - U \) is \( \tau_1 - Q^* \)-closed in \( X \). Since \( X \) is pairwise \( Q^*S \)-regular, there exists a \( \tau_i \)-semi open set \( V \) and a \( \tau_j \)-semi open set \( W \) such that \( x \in V \) and \( X - U \subseteq W \) and \( V \cap W = \emptyset \). Obviously, \( V \subseteq X - W \) and hence \( \tau_j - scl(V) \subseteq X - W \). Hence \( x \in V \subseteq \tau_j - scl(V) \subseteq U \).

(b) \( \rightarrow \) (c) Let \( F \) be a \( \tau_i - Q^* \)-closed subset of \( X \) and \( x \notin F \). Then \( X \) - \( F \) is a \( \tau_i - Q^* \)-open set containing \( x \). Therefore, by (b) there exists a \( \tau_i \)-semi open set \( O \) such that \( x \in O \subseteq scl(O) \subseteq X - F \), which implies that \( F \subseteq (X - \tau_i - scl(O)) \subseteq X - O \). Also \( X - O \) is \( \tau_i \)-semi closed, \( \tau_j \)-semi neighborhood of \( F \) which does not contain \( x \). Hence, the intersection of all \( \tau_i \)-semi closed, \( \tau_j \)-semi neighborhoods of \( F \) is exactly \( F \).

(c) \( \rightarrow \) (d) Let \( A \) be a nonempty subset of \( X \) and \( B \) be a \( \tau_i - Q^* \)-open set such that \( A \cap B \neq \emptyset \). Let \( x \in A \cap B \). Then \( X - B \) is a \( \tau_i - Q^* \)-closed such that \( x \notin X - B \). Therefore, by (c), the intersection of all \( \tau_i \)-semi closed, \( \tau_j \)-semi neighborhood of \( X - B \) is exactly \( X - B \), i.e., there exists a \( \tau_i \)-semi closed set, \( \tau_j \)-semi neighborhood of \( X - B \), say \( V \) such that \( x \notin V \). Thus, there is a \( \tau_i \)-semi open set \( U \) such that \( \tau_j - scl(U) \subseteq X - V \). Take \( W = X - V \). Then \( W \) is a \( \tau_i \)-semi open set containing \( x \) as \( x \notin B \) therefore \( x \notin V \). Hence \( x \in A \) and \( x \in W \) which implies that \( A \cap W \neq \emptyset \). Since \( X - V \subseteq X - U \subseteq B \), therefore, \( \tau_j - scl(X - V) \subseteq X - U \subseteq B \). Hence \( \tau_j - scl(W) \subseteq B \).

(d) \( \rightarrow \) (e) Let \( A \cap B = \emptyset \), where \( A \) is nonempty and \( B \) is a \( \tau_i - Q^* \)-closed set, then \( A \cap X - B \neq \emptyset \), where \( X - B \) is a \( \tau_i - Q^* \)-open set. Therefore by (d), there exists a \( \tau_i \)-semi open set \( G \) such that \( A \cap G \neq \emptyset \), and \( \tau_j - scl(G) \subseteq X - B \). Now, put \( M = X - \tau_i - scl(G) \). Then \( B \subseteq M \) and \( M \) and \( G \) are \( \tau_j \)-semi open sets such that \( G \cap M = \emptyset \).

(e) \( \rightarrow \) (a) Let \( F \) be a \( \tau_i - Q^* \)-closed subset of \( X \) and \( x \notin F \). Then \( \{ x \} \) and \( F \) are disjoint. Therefore by (e), there exists a \( \tau_i \)-semi open set \( U \) and a \( \tau_j \)-semi open set \( V \) such that \( \{ x \} \cap U \neq \emptyset \), \( F \subseteq M \) and \( U \cap V = \emptyset \); i.e., \( x \in U \). Hence \( X \) is pairwise \( Q^*S \)-regular.

**Definition 3.4** A space \( X \) is said to be \( bi-Q^* \)-symmetric if every singleton \( \{ x \} \) is \( \tau_i - Q^* \)-closed, \( i = 1, 2 \).

**Remark 3.5** Every \( bi - Q^* \)-symmetric is \( bi \)-symmetric but the converse need not be true in general. The following example supports our claim.

**Example 3.6** Let \( X = \{ a, b \} \), \( \tau_1 = \tau_2 = \{ \emptyset, X, \{ a \}, \{ b \} \} \). Then \( X \) is \( bi \)-symmetric but not \( bi - Q^* \)-symmetric.

**Theorem 3.7** Every pairwise \( Q^*S \)-regular, \( bi - Q^* \)-symmetric space is pairwise semi-\( T_2 \).

**Proof.** Let \( X \) be a pairwise \( Q^*S \)-regular and \( bi - Q^* \)-symmetric space. Let \( x, y \) be any two distinct points of \( X \). Since \( X \) is \( bi - Q^* \)-symmetric implies \( \{ x \} \) is \( \tau_i - Q^* \)-
closed for \(i = 1, 2\). Also \(y \notin \{x\}\). Since \(X\) is pairwise \(Q^*\) - regular, there exists a \(\tau_i\) - semi open set \(U\) and \(\tau_j\) - semi open set \(V\) such that \(\{x\} \in V, y \in U\) and \(U \cap V = \emptyset\); \(i \neq j, i, j = 1,2\). Hence \(X\) is pairwise semi-\(T_2\).

**Example 3.8** In Example 3.2, the space \(X\) is pairwise \(Q^*\)-normal but not bi-\(Q^*\)-symmetric and pairwise \(Q^*\)-regular.

**Theorem 3.9** Let \(f: X \rightarrow Y\) be a pairwise homeomorphism. Then \(X\) is pairwise \(Q^*\) - regular if and only if \(Y\) is pairwise \(Q^*\)-regular.

**Proof.** Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a pairwise homeomorphism. Let \(X\) is pairwise \(Q^*\) - regularity. Let \(F\) be a \(\sigma_i - Q^*\) - closed subset in \(Y\) such that \(y \notin F\). Then \(x \notin f^{-1}(F)\), where \(y = f(x)\) and \(f^{-1}(F)\) is a \(\tau_i - Q^*\) - closed since \(f\) is pairwise homeomorphism. By pairwise \(Q^*\) - regularity of \(X\), there exists a \(\tau_i\) - semi open set \(U\) and a \(\tau_j\) - semi open set \(V\) such that \(x \in U, f^{-1}(F) \subseteq V\) and \(U \cap V = \emptyset\). Hence \(y \in f(U), F \subseteq f(V)\) and \(f(U) \cap f(V) = \emptyset\). Since \(f\) is pairwise homeomorphism implies \(f\) is pairwise semi homeomorphism implies \(f\) is pairwise pre semi open. Therefore, \(f(U)\) and \(f(V)\) are \(\sigma_i\) - semi open and \(\sigma_j\) - semi open sets respectively. Hence \(Y\) is pairwise \(Q^*\) - regular.

Conversely, Let \(Y\) be pairwise \(Q^*\)-regular and let \(G\) be any \(\tau_i - Q^*\) - closed set in \(X\) such that \(x \notin G\). Then \(y \notin f(G)\) a \(\sigma_i - Q^*\) - closed set in \(Y\) since \(f\) is pairwise homeomorphism. By pairwise \(Q^*\) - regularity of \(Y\), there exists a \(\sigma_i\) - semi open set \(U\) and a \(\sigma_j\) - semi open set \(V\) in \(Y\) such that \(y \in U\) and \(f(G) \subseteq V\). Hence \(x \in f^{-1}(U)\) and \(G \subseteq f^{-1}(V)\) with \(f^{-1}(U) \cap f^{-1}(V) = \emptyset\). Since \(f\) is pairwise homeomorphism implies \(f\) is pairwise semi homeomorphism implies \(f\) is pairwise irresolute. Therefore, \(f^{-1}(U)\) and \(f^{-1}(V)\) are \(\tau_i\) - semi open and \(\tau_j\) - semi open sets respectively in \(X\). Hence, \(X\) is pairwise \(Q^*\) - regular.

4 Pairwise \(Q^*\)-normal spaces

In this section, we introduce the concept of pairwise \(Q^*\)-normal spaces and we establish some properties of this concept.

**Definition 4.1** A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise \(Q^*\) - normal if for every pair of disjoint \(\tau_i - Q^*\) - closed set \(A\) and \(\tau_j - Q^*\) - closed set \(B\), there exists a \(\tau_j\) - semi open set \(U\) and a \(\tau_i\) - semi open set \(V\) such that \(A \subseteq U, B \subseteq V\) and \(U \cap V = \emptyset\); \(i \neq j, i, j = 1, 2\).

**Example 4.2** In Example 3.2, shows that the space \((X, \tau_1, \tau_2)\) is pairwise \(Q^*\)-normal but not pairwise \(Q^*\)-normal.

**Definition 4.3** A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise \(s^*\) - \(Q^*\) - normal if for every pair of disjoint \(\tau_i\) - semi closed set \(A\) and \(\tau_j\) - semi closed set \(B\) in \(X\), there exists disjoint \(\tau_j - Q^*\) - open set \(U\) and a \(\tau_i - Q^*\) - open set \(V\) such that \(A \subseteq U, B \subseteq V\) and \(U \cap V = \emptyset\); \(i \neq j, i, j = 1, 2\).

**Definition 4.4** \([4]\) A space \(X\) is said to be pairwise \(s^*\) - normal if for any two disjoint \(\tau_i\) - semi closed set \(A\) and \(\tau_j\) - semi closed set \(B\), there exists a disjoint \(\tau_j\) - semi open set \(U\) and \(\tau_i\) - semi open set \(V\) such that \(A \subseteq U, B \subseteq V\) and \(U \cap V = \emptyset\); \(i \neq j, i, j = 1, 2\).
Definition 4.5 [19] A space X is said to be pairwise gs - normal if for any two disjoint \( \tau_i - g - \) closed set A and \( \tau_j - g - \) closed set B, there exists a disjoint \( \tau_i - \) semi open set U and \( \tau_j - \) semi open set V such that \( A \subseteq U, B \subseteq V \) and \( U \cap V = \phi; \ i \neq j, i, j = 1, 2. \)

Definition 4.6 [4] A space X is said to be pairwise s - normal if for any two disjoint \( \tau_i - \) closed set A and \( \tau_j - \) closed set B, there exists a disjoint \( \tau_i - \) semi open set U and \( \tau_j - \) semi open set V such that \( A \subseteq U, B \subseteq V \) and \( U \cap V = \phi; \ i \neq j, i, j = 1, 2. \)

Theorem 4.7 For a space X, the following are equivalent:

(a) \((X, \tau_1, \tau_2)\) is pairwise Q* s - normal.

(b) For each \( \tau_i - Q^*\) - closed set F and a \( \tau_j - Q^*\) - open set K containing F, there exists a \( \tau_j - \) semi open set U such that \( F \subseteq U \subseteq \tau_i - scl(U) \subseteq K.\)

(c) For every \( \tau_i - Q^*\) - closed set A and a \( \tau_j - Q^*\) - closed set B disjoint from A, there exists a \( \tau_i - \) semi open set U containing A such that \( \tau_j - scl(U) \cap B = \phi.\)

Proof. \((a) \implies (b)\) Let X be pairwise Q* s - normal and let K be a \( \tau_j - Q^*\) - open set containing a \( \tau_i - Q^*\) - closed set F. Then F and X - K are disjoint \( \tau_i - Q^*\) - closed and \( \tau_j - Q^*\) - closed sets respectively. So by (a), there exists a \( \tau_j - \) semi open set U and a \( \tau_i - \) semi open set V such that \( F \subseteq U \subseteq X - K \subseteq V \) and \( U \cap V = \phi.\) Thus \( U \subseteq X - V,\) which implies that \( \tau_i - scl(U) \subseteq X - V.\) Hence, \( F \subseteq U \subseteq \tau_i - scl(U) \subseteq X - V.\)

(b) \implies (c) Let A and B be respectively \( \tau_i - Q^*\) - closed and \( \tau_j - Q^*\) - closed subsets of X such that \( A \cap B = \phi,\) which implies \( A \subseteq X - B,\) a \( \tau_j - Q^*\) - open set. So by (b), there exists a \( \tau_j - \) semi open set U such that \( A \subseteq U \subseteq \tau_i - scl(U) \subseteq X - B.\) Hence, \( \tau_i - scl(U) \cap B = \phi.\)

(c) \implies (a) Let A be a \( \tau_i - Q^*\) - closed set and B be a \( \tau_j - Q^*\) - closed set disjoint from A. Then, by (c), there is a \( \tau_j - \) semi open set U such that \( A \subseteq U \) and \( \tau_i - scl(U) \cap B = \phi.\) Now \( \tau_i - scl(U)\) is semi closed. Hence, \( B \subseteq X - \tau_i - scl(U),\) let \( V = X - \tau_i - scl(U).\) Then V is a \( \tau_i - \) semi open set such that \( B \subseteq V \) and \( U \cap V = \phi.\) Hence, X is pairwise Q* s - normal.

\(\square\)

Theorem 4.8 For a space X, the following are equivalent:

(a) \((X, \tau_1, \tau_2)\) is pairwise s' Q* - normal.

(b) For each \( \tau_i - \) semi closed set F and a \( \tau_j - \) semi open set K containing F, there exists a \( \tau_j - Q^*\) - open set U such that \( F \subseteq U \subseteq \mu^*_i - cl(U) \subseteq K.\)

(c) For every \( \tau_i - \) semi closed set A and a \( \tau_j - \) semi closed set B disjoint from A, there exists a \( \tau_j - Q^*\) - open set U containing A such that \( \mu^*_j - cl(U) \cap B = \phi.\)

Proof. \((a) \implies (b)\) Let X be pairwise s' Q* - normal and let K be a \( \tau_i - \) semi open set containing a \( \tau_j - \) semi closed set F. Then F and X - K are disjoint \( \tau_j - \) semi closed set and \( \tau_i - \) semi closed sets respectively. So by (a), there exists a \( \tau_j - Q^*\) - open set U and a \( \tau_i - Q^*\) - open set V such that \( F \subseteq U \subseteq X - K \subseteq V \) and \( U \cap V = \phi.\) Thus \( U \subseteq X - V,\) which implies that \( \mu^*_i - cl(U) \subseteq X - V.\) Hence, \( F \subseteq U \subseteq \mu^*_i - cl(U) \subseteq X - V.\)

(b) \implies (c) Let A and B be respectively \( \tau_i - \) semi closed set and \( \tau_j - \) semi closed subsets of X such that \( A \cap B = \phi,\) which implies \( A \subseteq X - B,\) a \( \tau_j - Q^*\) - open set. So by (b), there exists a \( \tau_j - Q^*\) - open set U such that \( A \subseteq U \subseteq \mu^*_i - cl(U) \subseteq X - B.\) Hence, \( \mu^*_j - cl(U) \cap B = \phi.\) (c) \implies (a) Let A be a \( \tau_i - \) semi closed and B be a \( \tau_j - \) semi closed set disjoint from A. Then, by (c), there is a \( \tau_j - Q^*\) - open set U such that \( A \subseteq U \) and \( \mu^*_i - cl(U) \cap B = \phi.\) Now \( \mu^*_i - cl(U)\) is \( Q^*\) - closed. Hence, \( B \subseteq X - \mu^*_i - cl(U),\) let \( V = X - \mu^*_i - cl(U).\) Then V is a \( \tau_j - Q^*\) - open set such that \( B \subseteq V \) and \( U \cap V = \phi.\) Hence, X is pairwise s' Q* - normal. \(\square\)
**Theorem 4.9** Every pairwise $Q^*$-normal and $Q^*$-symmetric space $X$ is $Q^*$-regular.

**Proof.** Let $F$ be a $\tau_j - Q^*$ closed subset of $X$ with $x \notin F$. Since $X$ is bi-$Q^*$-symmetric so $\{x\}$ is $\tau_i - Q^*$ closed; $i \neq j$ and $i, j = 1, 2$. So $\{x\}$ and $F$ are disjoint $\tau_i - Q^*$ closed and $\tau_j - Q^*$ closed sets respectively in $X$. Since $X$ is pairwise $Q^*$-normal, there exist disjoint $\tau_j$ - semi open set $U$ and $\tau_i$ - semi open set $V$ such that $\{x\} \subseteq U, F \subseteq V$. Hence $X$ is pairwise $Q^*$-regular.

**Theorem 4.10** Every pairwise $Q^*$-normal and bi-$Q^*$-symmetric space $X$ is pairwise $Q^*$-regular.

**Proof.** Let $F$ be a $\tau_j - Q^*$ closed subset of $X$ with $x \notin F$. Since $X$ is bi-$Q^*$-symmetric so $\{x\}$ is $\tau_i - Q^*$ closed. So $\{x\}$ and $F$ are disjoint $\tau_i - Q^*$ closed and $\tau_j - Q^*$ closed sets respectively in $X$. Since $X$ is pairwise $Q^*$-normal, there exists a disjoint $\tau_j$ - open set $U$ and $\tau_j - Q^*$ open set $V$ such that $\{x\} \subseteq U, F \subseteq V$. Hence $X$ is pairwise $Q^*$-regular.

**Example 4.11** Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{b, c\}\}$ and $\tau_2 = \{\phi, X, \{a, c\}\}$. Therefore, the space $X$ is pairwise $Q^*$-normal but not bi-$Q^*$-symmetric and pairwise $Q^*$-regular.

**Theorem 4.12** Let $f : X \rightarrow Y$ is a pairwise homeomorphism. Then $X$ is pairwise $Q^*$-normal if and only if $Y$ is pairwise $Q^*$-normal.

**Proof.** Let $Y$ be pairwise $Q^*$-normal. Let $A$ and $B$ be two disjoint $\tau_i - Q^*$ closed set and $\tau_j - Q^*$ closed sets in $X$. Then $f(A)$ and $f(B)$ are $\sigma_i - Q^*$ closed set and $\sigma_j - Q^*$ closed sets in $Y$. Since $Y$ is pairwise $Q^*$-normal, there exist disjoint $\sigma_i$ - semi open set $U$ and $\sigma_j$ - semi open set $V$ in $Y$ such that $f(A) \subseteq U, f(B) \subseteq V$. Hence, $A \subseteq f^{-1}(U), B \subseteq f^{-1}(V)$, and $f^{-1}(U) \cap f^{-1}(V) = \phi$ as $U \cap V = \phi$. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are $\tau_i$ - semi open and $\tau_j$ - semi open sets; since $f$ is pairwise irresolute. Hence $X$ is pairwise $Q^*$-normal. Conversely, Let $X$ is pairwise $Q^*$-normal. Let $A$ and $B$ be two disjoint $\sigma_i - Q^*$ closed set and $\sigma_j - Q^*$ closed sets in $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are $\tau_i - Q^*$ closed and $\tau_j - Q^*$ closed sets in $X$. Since $X$ is pairwise $Q^*$-normal, there exists a disjoint $\tau_j$ - semi open set $U$ and $\tau_j$ - semi open set $V$ in $X$ such that $f^{-1}(A) \subseteq U, f^{-1}(B) \subseteq V$. Hence $A \subseteq f(U), B \subseteq f(V)$, and $f(U) \cap f(V) = \phi$ as $U \cap V = \phi$. Since $f$ is pairwise homeomorphism implies $f$ is pairwise semi homeomorphism implies $f$ is pairwise pre - semi open. Therefore, $f(U)$ and $f(V)$ are $\sigma_i$ - semi open and $\sigma_i$ - semi open in $Y$ respectively. Hence, $Y$ is pairwise $Q^*$-normal.

5 Comparison

**Remark 5.1** We summarize the relationship between various special types of normal spaces in the following diagram. None of the implications is reversible.
Theorem 5.2 Every pairwise $Q^*S$-normal space is pairwise $S$-normal.

Proof. Let $X$ be a pairwise $Q^*s$-normal space. To show that $X$ is pairwise $S$-normal. Let $A$ be $\tau_i - Q^*$-closed and $B$ be $\tau_j - Q^*$-closed. Since $X$ is pairwise $Q^*s$-normal, there exists a disjoint $\tau_j$-semi open set $U$ and $\tau_i$-semi open set $V$ such that $A \subseteq U$ and $B \subseteq V$. Since every $Q^*$-closed set is closed we have $A$ is $\tau_i$-closed and $B$ is $\tau_j$-closed. Hence $X$ is pairwise $s$-normal. □

Remark 5.3 Converse of the above theorem need not be true in general.

Example 5.4 In Example 3.2, $X$ is pairwise $S$-normal but not pairwise $Q^*s$-normal. Here $\{ b, c \}$ $\tau_i$-closed but not $\tau_i - Q^*$-closed.

Theorem 5.5 Every pairwise $Q^*S$-normal space is pairwise semi-normal.

Proof. Let $X$ be a pairwise $Q^*s$-normal space. To show that $X$ is pairwise semi-normal. Let $A$ and $B$ be a two disjoint $\tau_i$-semi closed set $A$ and $\tau_j$-semi closed set $B$ in $X$. Since $X$ is $Q^*s$-normal, there exists a disjoint $\tau_j$-semi open set $U$ and $\tau_i$-semi open set $V$ such that $A \subseteq U$ and $B \subseteq V$. Since every $Q^*$-open set is semi-open, there exists a disjoint $\tau_j$-semi open set $U$ and $\tau_i$-semi open set $V$ such that $A \subseteq U$ and $B \subseteq V$. Hence $X$ is semi-normal. □

Remark 5.6 But the converse of the above theorem need not be true in general. i.e) every pairwise semi-normal space is not pairwise $Q^*s$-normal.

Example 5.7 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, c\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{a, c\}\}$. Then the space $X$ is pairwise semi-normal but not pairwise $Q^*s$-normal.

Theorem 5.8 Every pairwise $S^*Q^*$-normal space is pairwise semi-normal.

Proof. Let $X$ be a pairwise $S^*Q^*$-normal space. To show that $X$ is pairwise semi-normal. Let $A$ and $B$ be a two disjoint $\tau_i$-semi closed set $A$ and $\tau_j$-semi closed set $B$ in $X$. Since $X$ is $S^*Q^*$-normal, there exists a disjoint $\tau_j$-open set $U$ and $\tau_i$-open set $V$ such that $A \subseteq U$ and $B \subseteq V$. Since every $Q^*$-open set is semi-open, there exists a disjoint $\tau_j$-semi open set $U$ and $\tau_i$-semi open set $V$ such that $A \subseteq U$ and $B \subseteq V$. Hence $X$ is semi-normal. □

Remark 5.9 But the converse of the above theorem need not be true in general. i.e) every pairwise semi-normal space is not pairwise $S^*Q^*$-normal.
Example 5.10 In example 5.2, the space X is pairwise semi normal but not pairwise $S^*Q^*$ - normal.

Theorem 5.11 Every pairwise $Q^*$ - normal space is pairwise $Q^*s$ - normal.

Proof. Let X be a pairwise $Q^*$ - normal space. To show that X is pairwise $Q^*s$ - normal. Let A and B be two disjoint $\tau_i - Q^*$ closed set A and $\tau_j - Q^*$ closed set B in X. Since X is pairwise $Q^*$ - normal, there exists a disjoint $\tau_i - Q^*$ open set U and $\tau_j - Q^*$ open set V such that $A \subseteq U$ and $B \subseteq V$. Since every $Q^*$ - open set is semi - open, there exists a disjoint $\tau_i$ - semi open set U and $\tau_j$ - semi open set V such that $A \subseteq U$ and $B \subseteq V$. Hence X is pairwise $Q^*s$ - normal. □

Remark 5.12 But the converse of the above theorem need not be true in general. i.e) every pairwise $Q^*s$ - normal space is not pairwise $Q^*$ - normal.

Theorem 5.13 Every pairwise $Q^*s$ - normal space is pairwise $gs$ - normal.

Proof. Let X be a pairwise $Q^*s$ - normal space. To show that X is pairwise $gs$ - normal. Let A and B be two disjoint $\tau_i - Q^*$ closed set A and $\tau_j - Q^*$ closed set B in X. Since X is pairwise $Q^*s$ - normal, there exists a disjoint $\tau_i$ - semi open set U and $\tau_j$ - semi open set V such that $A \subseteq U$ and $B \subseteq V$. Since every $Q^*$ - closed set is $g$ - closed we have A and B are $\tau_i$ - $g$ closed and $\tau_j$ - $g$ closed sets. Hence X is pairwise $gs$ - normal. □

Remark 5.14 But the converse of the above theorem need not be true in general. i.e) every pairwise $gs$ - normal space is not pairwise $Q^*s$ - normal.

Example 5.15 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then the space X is pairwise $gs$ normal but not pairwise $Q^*s$ - normal.

References
Pairwise $Q^*$-(regular and normal) spaces in bitopological spaces


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