Novel Portfolio Selection Models with Parametric Entropy and Sensitivity Analysis under Hesitant Fuzzy Environment

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Research Article

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Posted Date: August 22nd, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1361202/v1

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Novel Portfolio Selection Models with Parametric Entropy and Sensitivity Analysis under Hesitant Fuzzy Environment
Xue Deng¹, Jinyao Zhao¹ *, Qihan Fu²

Abstract: In most risk investments, it is difficult for investors to acquire enough precise data, so they have to rely on uncertain information to make decisions. Taking advantage of hesitant fuzzy sets, whose membership degree is composed of several values in the closed interval between 0 and 1, thus it is easier to depict the uncertainty of events in the practical applications. Based on the above facts, this paper focuses on the portfolio selection problem under hesitant fuzzy environment, using hesitant fuzzy sets’ membership degrees to describe investors’ hesitation and adopting parametric entropy to measure fuzzy sets’ uncertain degree. Firstly, we present a parametric entropy formula satisfying the axiomatic definition of hesitation fuzzy entropy through serious mathematical proof and discussion. Secondly, two portfolio models with parametric hesitant fuzzy entropy are proposed, which adopt score function and entropy to describe return and risk, respectively. In addition, a constraint trisection approach is developed to solve our proposed models according to investors’ different risk preferences. Finally, a case study is conducted to highlight the effectiveness of the proposed models. Sensitivity analysis is carried out to explore the effect of parameter changes on portfolio return, risk and investment ratio, which fully manifest the flexibility of parametric entropy. We come to the conclusion from multiple sets of data that parameter setting can reasonably reflect investors’ information preference. Compared with conventional method, not only parametric entropy introduced as a risk measure for portfolio models, but also the explicit model solving processes listed in this paper contribute to portfolio under hesitant fuzzy environment.

Keywords: Hesitant fuzzy sets; Parametric entropy; Portfolio selection; Risk preferences; Sensitivity analysis

1. Introduction

A reasonable investment behavior of securities and other risky assets requires an effective method to determine the expected return and risk. How to find a balance between the maximization of return and the minimization of risk to carry out a reasonable asset allocation is the key to investment behavior. Modern portfolio selection theory originated from the groundbreaking work of Markowitz (1952) in 1952, who proposed the Mean-Variance model to direct the portfolio selection problem. The method of mathematical statistics was applied to the study of portfolio selection for the first time. Since then, the portfolio selection problem has developed continuously in the research and exploration of experts. Scholars (Sharpe 1963, Lintner 1965, Mossin 1966) developed the Capital Asset Pricing Model (CAPM) which mainly studied the relationship between expected returns and risks in the stock market, and concluded the only reason that an investor would get a higher return is to invest in high risk stocks. Best and Hlouskova (2000) studied the portfolio selection problem of bounded uncorrelated assets. Furthermore, Arbitrage Pricing Theory (APT) proposed by Ross (1976) with a broader applicability was also the theoretical basis of modern finance. There are many ways to measure the risks of portfolio except the variance, such as mean absolute deviation (Konno and Yamazaki 1991), Value at Risk (VaR) (Basak and Shapiro 2001,

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However, most of the above models are based on accurate historical data. In practical applications, such data is difficult to obtain. Thus, when there is uncertainty in the state of events themselves, the qualitative information provided by experts is of greater help to investors' decision-making process. Uncertainty can be divided into uncertainty under random environment and uncertainty under fuzzy environment. A majority of early researches on portfolio are based on stochastic theory, and fuzzy theory is an effective method to analyze qualitative problems. In 1965, Zadeh (1968) proposed the concept of fuzzy sets (FSs), subsequently, many scholars generalized it and derived many other forms of fuzzy sets. Atanassov (1986) introduced intuitionistic fuzzy sets (IFSs) to describe both ‘good’ and ‘bad’ fuzzy information. When it’s difficult to show the preferences of decision makers facing different complex information, hesitant fuzzy sets (HFSs) proposed by Torra (2010) which extended the membership of an element from a single to some possible values in [0,1] are more practical. Dual hesitant fuzzy sets (DHFSs) (Zhu, et al. 2012) and weighted dual hesitant fuzzy sets (WDHFSs) (Zeng, et al. 2021) are then a combination of the HFSs and IFSs. HFSs aim to describe the uncertain information through a set of possible values, therefore, HFSs are more convenient than other fuzzy sets in expressing subjective evaluation information. Xu, et al. (2022) used hesitant fuzzy elements to describe preference information. Liu, et al. (2021) developed a correlation coefficient to measure the strength of the relationship between HFSs, and thus avoid the counter-intuitive decision results. Moreover, three-way decision theory (Liang, et al. 2020, Wang, et al. 2021b, Wang, et al. 2021a) was also applied in decision making problem under hesitant fuzzy environment. Hesitant fuzzy linguistic term sets were also widely used in intelligent decision making (Wei, et al. 2020, Wang, et al. 2022).

Subsequently, many scholars extended the stochastic portfolio selection model to deal with the uncertainty problem described by fuzzy sets. Starting with Carlsson and Fullér (2001), who took the trapezoidal fuzzy number as expected return. Vercher, et al. (2007) proposed two fuzzy portfolio selection models which minimize the risks with a given expected return. Huang (2012) used the mean value of fuzzy numbers to measure returns and the variance to measure risks, then constructed a portfolio selection model based on experts’ evaluation. Li, et al. (2010) defined the skewness of fuzzy variables, which is incorporated into the reference index of portfolio selection. Deng and Pan (2018) compared the multi-objective portfolio models based on intuitionistic fuzzy optimal solution. Zhou and Xu (2018) proposed portfolio selection models for general investors and risk investors under hesitant fuzzy environment. Zhou, et al. (2019) developed the hesitant fuzzy portfolio model based on prospect theory, which considers the investors’ psychology.

Entropy is a method to measure the uncertainty of fuzzy sets proposed by Zadeh (1968). De Luca and Termini (1972) constructed axiomatic criterion of fuzzy entropy. For a long time, entropy is widely used in decision making problem (Narayanamoorthy, et al. 2019, Wei, Rodriguez and Li 2020, Deveci, et al. 2022). Xu and Xia (2012) proposed an axiomatic definition of entropy measure under hesitant fuzzy environment, and provided a series of specific formulas according to the definition. Farhadinia (2013) gave some counter examples and pointed out that entropy formula proposed by Wei, et al. (2016) could not distinguish some hesitant fuzzy elements with obvious differences, thus a series of entropy formulas are proposed based on the distance measure of hesitant fuzzy sets. However, the entropy of Farhadinia (2013) cannot distinguish HFEs from ones with the
same distance as HFE \{0.5\}. Thus, Mei and Li (2019) proposed parametric hesitant fuzzy entropy and introduced parameters to make entropy more flexible.

This paper aims to enrich the research on portfolio selection under hesitant fuzzy environment, the contributions are described as follows. (1) A parametric hesitant fuzzy entropy formula is proved to satisfy the axiomatic definition of entropy measures for HFSs. (2) The portfolio selection models under hesitant fuzzy environment are proposed, where risk is measured by parametric entropy instead of deviation function. (3) We present the specific portfolio selection processes for investors and demonstrate the model solving processes with a numerical example. (4) The sensitivity of parameters is analyzed by exploring their impacts on the returns, risks and investment proportions of the optimal portfolio selection. From what has been discussed above, we suggest that parameters in models represent investors’ preference information of the portfolio selection.

We organize this paper as follows: In Section 2, we review some necessary concepts and operations on the HFS. In Section 3, we introduce a parametric entropy measure of HFS and prove it. In Section 4, two portfolio models based on parametric hesitant fuzzy entropy and score function are constructed. Then, Section 5 uses a numerical example to show the availability of the proposed approaches. Parametric analysis of our proposed models is obtained in Section 6 and Section 7. Finally, we conclude the paper in Section 8.

2. Preliminaries

Hesitant fuzzy sets are proposed to deal with uncertain problems. In this section, we firstly review some fundamental concepts related to hesitant fuzzy sets (HFSs). Moreover, we introduce an axiomatic definition of entropy measure, then two existing portfolio models are referenced. $X = \{x_1, x_2, ..., x_n\}$ denotes the universe of discourses.

2.1 Hesitant fuzzy set

**Definition 1 (Torra 2010)** Let $X$ be a fixed set, a HFS $A$ on $X$ is described as:

\[ A = \\{ (x, h_d(x)) | x \in X \} \]

where $h_d(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$ to $A$. $h_d(x)$ is called a hesitant fuzzy element (HFE).

**Example 1** Let $X = \{x_1, x_2, x_3\}$ be the universe of discourses. Assume that $h_d(x_1) = \{0.3\}$, $h_d(x_2) = \{0.2, 0.6\}$ and $h_d(x_3) = \{0.1, 0.5, 0.9\}$ are HFEs on $x_i \in X (i = 1, 2, 3)$ to set $A$. Then $A = \{ (x_1, \{0.3\}), (x_2, \{0.2, 0.6\}), (x_3, \{0.1, 0.5, 0.9\}) \}$ is a HFS.

**Definition 2** Let $h = \{\gamma_1, ..., \gamma_l\}$ be a HFE, $l$ is the number of the elements in $h$, which is also called the length of $h$.

Then the score function (Xia and Xu 2010) of $h$ is

\[ s(h) = \frac{1}{l} \sum_{j=1}^{l} \gamma^j \]

the deviation function (Zhou and Xu 2015) of $h$ is
\[
\eta(h) = \frac{\sum_{i=1}^{l} |h'_i - s(h)_i|}{l} .
\]  \hspace{1cm} (2)

2.2 Operations of HFSs

**Definition 3** (Xia and Xu 2010) Let \( h, h_1, h_2 \) be three HFEs, several operations on HFEs can be represented as follows:

(i) \( h^C = \bigcup_{\gamma \in h} \{1 - \gamma\} \);

(ii) \( h_1 \bigcup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\} \);

(iii) \( h \bigcap h_2 = \bigcap_{\gamma_1 \in h, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\} \);

(iv) \( h^\theta = \bigcup_{\gamma \in h} \{\gamma^\theta\} \), \( \theta > 0 \);

(v) \( \partial h = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^\theta\} \);

(vi) \( h_1 \bigoplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\} \);

(vii) \( h_1 \bigotimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\} \).

According to Definition 2, we can obtain a formula which will be used frequently:

\[
\bigoplus_{i=1}^{n} \omega_h = \bigcup_{\gamma \in h} \left\{1 - \prod_{i=1}^{n} (1 - \gamma_i)^{\omega}\right\} .
\]  \hspace{1cm} (3)

**Example 2** Let \( h_3(x_i) = \{0.3\} \), \( h_4(x_i) = \{0.2, 0.6\} \), and \( h_5(x_i) = \{0.1, 0.5, 0.9\} \) be three HFEs, then based on Formula (3),

\[
\bigoplus_{i=1}^{n} \omega_h = \bigcup_{\gamma \in h} \left\{1 - \prod_{i=1}^{n} (1 - \gamma_i)^{\omega}\right\}
= \left\{(1 - 0.7^n \cdot 0.8^n \cdot 0.9^n), (1 - 0.7^n \cdot 0.8^n \cdot 0.5^n), (1 - 0.7^n \cdot 0.8^n \cdot 0.1^n), (1 - 0.7^n \cdot 0.4^n \cdot 0.9^n), (1 - 0.7^n \cdot 0.4^n \cdot 0.5^n), (1 - 0.7^n \cdot 0.4^n \cdot 0.1^n)\right\} .
\]  \hspace{1cm} (4)

Entropy is used to measure uncertain degree of fuzzy sets. Based on the axiomatic definition of fuzzy entropy, Xu and Xia (2012) proposed the axiomatic definition of entropy measure for HFEs.

2.3 Entropy measure of HFE

**Definition 4** (Xu and Xia 2012) Real-valued function \( E : H \rightarrow [0,1] \) is called the entropy measure of HFE \( h \), if it satisfies the following conditions:

(1) \( E(h) = 0 \) if and only if \( h = \{0\} \) or \( h = \{1\} \);

(2) \( E(h) = 1 \) if and only if \( h' + h'' = 1 \), for all \( i = 1, 2, \ldots, l \);
(3) \( E(h_1) \leq E(h_2) \) if \( h'_1 \leq h'_2 \) for \( h'_1 + h''_{i+1} \leq 1 \), or \( h'_1 \geq h'_2 \) for \( h'_1 + h''_{i+1} \geq 1 \) (\( h_1 \) and \( h_2 \) are HFEs, they can get the same length \( l \) by adding the same element in the shorter one):

(4) \( E(h) = E(h^c) \).

Four entropy formulas (Xu and Xia 2012) are given to satisfy Definition 4 which can be used to calculate entropy of HFEs:

\[
E_1(h) = \frac{1}{l(\sqrt{2} - 1)} \sum_{i=1}^{l} \left\{ \sin \left( \frac{h'_i + h''_{i+1}}{4} \right) + \sin \left( \frac{2 - h'_i - h''_{i+1}}{4} \right) - 1 \right\},
\]

(5)

\[
E_2(h) = \frac{1}{l(\sqrt{2} - 1)} \sum_{i=1}^{l} \left\{ \cos \left( \frac{h'_i + h''_{i+1}}{4} \right) + \cos \left( \frac{2 - h'_i - h''_{i+1}}{4} \right) - 1 \right\},
\]

(6)

\[
E_3(h) = -\frac{1}{l \ln(2)} \left\{ \sum_{i=1}^{l} \frac{h'_i + h''_{i+1}}{2} \ln \frac{h'_i + h''_{i+1}}{2} + \frac{2 - h'_i - h''_{i+1}}{2} \ln \frac{2 - h'_i - h''_{i+1}}{2} \right\},
\]

(7)

\[
E_4(h) = \frac{1}{l \left( 2^{\frac{n}{l}} - 1 \right)} \sum_{i=1}^{l} \left\{ \left( \frac{h'_i + h''_{i+1}}{2} \right)^n + \left( \frac{2 - h'_i - h''_{i+1}}{2} \right)^n \right\} - 1 \right\},
\]

(8)

2.4 Some existing portfolio models under hesitant fuzzy environment

In traditional Mean-Variance model proposed by Markowitz (1952), risks are measured by variance while returns are measured by mean. However, when it’s difficult to obtain statistics data, we need new measures to depict returns and risks. Zhou and Xu (2018) constructed portfolio selection models under the hesitant fuzzy environment with score function \( s(h) \) and deviation function \( d(h) \) describing returns and risks separately, which are shown as follow. In addition, \( S \) is the min-score degree and \( D \) is the max-deviation degree.

\[
\begin{align*}
\text{Model 1:} & \quad \min d\left[ \bigoplus_{i=1}^{n} w_i \check{h}_i \right] \\
\text{s.t.} & \quad s\left[ \bigoplus_{i=1}^{n} w_i \check{h}_i \right] \geq S \\
& \quad \bigoplus_{i=1}^{n} w_i \check{h}_i = \bigcup_{x_1, x_2, \ldots, x_n \in \mathbb{X}} \left\{ 1 - \prod_{i=1}^{n} (1 - \gamma_i)^{y_i} \right\} \\
& \quad d(h) = \frac{1}{\# h} \sum_{y \in h} |\gamma - s(h)| \\
& \quad s(h) = \frac{1}{\# h} \sum_{y \in h} \gamma \\
& \quad \sum_{i=1}^{n} w_i \leq 1, l_i \leq w_i \leq u_i, i = 1, \ldots, n.
\end{align*}
\]

Similarly, taking maximum return as the objective function, Model 2 is proposed.
In this paper, we apply the score function as before, but the entropy will be substituted for deviation function.

3. Definition and proof of parametric hesitant fuzzy entropy

Mei and Li (2019) proposed the improved axiomatic definition of hesitant fuzzy entropy to address the shortcoming of existing definitions such as Xu and Xia (2012) and constructed a parametric hesitant fuzzy entropy formula. A series of hesitant fuzzy entropy values are available through the transformation of parameter values, which can avoid the counterintuitive situation under certain conditions.

3.1 Improved axiomatic definition of hesitation fuzzy entropy

Definition 5 (Mei and Li 2019) Let $X$ be a fixed set. Assume that $A_i = \{ (x_i, h_i(x_i)) \mid x_i \in X \}$ and $B_i = \{ (x_i, h_b(x_i)) \mid x_i \in X \}$ are two HFSs on $X$ with $h_i(x_i) = \{ \gamma_i^1, \gamma_i^2, \ldots, \gamma_i^l \}$ and $h_b(x_i) = \{ \sigma_i^1, \sigma_i^2, \ldots, \sigma_i^n \}$.

Real-valued function $E: HFS(X) \rightarrow [0,1]$ is called the entropy measure of HFS $A$, if it satisfies the following conditions:

(i) $E(A) = 0$ if and only if $A$ is a crisp set, which means $\gamma_i^1 = 0$ or $\gamma_i^l = 1$ for any $i = 1, 2, \ldots, n$ and $l = 1, \ldots, l_i$;

(ii) $E(A) = 1$ if and only if $A$ is the fuzziest set, which means $\gamma_i^1 = \frac{1}{2}$ for any $i = 1, 2, \ldots, n$ and $l = 1, \ldots, l_i$;

(iii) $E(A) = E(A^c)$, $A^c$ is the complementary set of $A$;

(iv) $E(A) \geq E(B)$ if $\frac{1}{2} \geq \gamma_i^l \geq \sigma_i^l$ or $\frac{1}{2} \leq \gamma_i^l \leq \sigma_i^l$ for $i = 1, 2, \ldots, n$ and $l = 1, \ldots, l_i$. The length of $h_i(x_i)$ and $h_b(x_i)$ is $l_i$, which means $h_i(x_i) = \{ \gamma_i^1, \gamma_i^2, \ldots, \gamma_i^l \}$, $h_b(x_i) = \{ \sigma_i^1, \sigma_i^2, \ldots, \sigma_i^n \}$.
3.2 A parametric hesitant fuzzy entropy formula and its proof

**Theorem 1 (Mei and Li 2019)** Let $X$ be a non-empty finite set and $A = \{(x_i, h_i(x_i)) | x_i \in X\}$ be a HFS on $X$. Assume that $h_i(x_i) = \{\gamma_i^1, \gamma_i^2, ..., \gamma_i^l\}$ and $l_i$ represent the length of $h_i$ with

$$E_\alpha^\beta(A) = \frac{2 - \beta}{n(2 - \beta - \alpha)} \sum_{i=1}^{n} l_i \left[ \frac{1}{l_i} \sum_{j=1}^{l_i} \left( \gamma_i^j \frac{\alpha}{2 - \beta} + (1 - \gamma_i^j) \frac{\alpha}{2 - \beta} \right) \right].$$

Then $E_\alpha^\beta(A)$ is the parametric hesitant fuzzy entropy of HFS $A$, where $\alpha > 0$, $\beta \in [0,1], \alpha + \beta \neq 2$.

Below we show the detailed mathematic proof and related discussion.

**Proof:** We need to prove $E_\alpha^\beta(A)$ in Theorem 1 satisfies four conditions in Definition 5.

(i) If $A$ is a crisp set, $\gamma_i^1 = 0$ or $\gamma_i^1 = 1$ for any $i = 1, 2, ..., n$ and $\lambda = 1, ..., l_i$, substitute it into Formula (11), we have

$$E_\alpha^\beta(A) = \frac{2 - \beta}{n(2 - \beta - \alpha)} \sum_{i=1}^{n} l_i \left[ \frac{1}{l_i} \sum_{j=1}^{l_i} \left( \gamma_i^j \frac{\alpha}{2 - \beta} + (1 - \gamma_i^j) \frac{\alpha}{2 - \beta} \right) \right] = 0. \quad (12)$$

If $E(A) = 0$, derive $E_\alpha^\beta(A)$ with respect to $\gamma_i^1$, the outcome is

$$\frac{\partial E_\alpha^\beta(A)}{\partial \gamma_i^1} = \frac{2 - \beta}{n l_i (2 - \beta - \alpha)} \left[ \frac{\alpha}{2 - \beta} \left( \gamma_i^1 \frac{\alpha}{2 - \beta - 1} - (1 - \gamma_i^1) \frac{\alpha}{2 - \beta - 1} \right) \right]. \quad (13)$$

Due to the fact that $\alpha + \beta \neq 2$, this problem will be discussed in two cases:

**Case 1:**
When $\alpha + \beta > 2$, which means $\frac{\alpha}{2 - \beta} > 1$, we have $\frac{2 - \beta}{2 - \beta - \alpha} < 0$. This shows that

$$\frac{\partial E_\alpha^\beta(A)}{\partial \gamma_i^1} > 0 \text{ for } \gamma_i^1 \in \left(0, \frac{1}{2}\right), \quad \frac{\partial E_\alpha^\beta(A)}{\partial \gamma_i^1} < 0 \text{ for } \gamma_i^1 \in \left(\frac{1}{2}, 1\right).$$

**Case 2:**
When $\alpha + \beta < 2$, which means $0 < -\frac{\alpha}{2 - \beta} < 1$, we have $\frac{2 - \beta}{2 - \beta - \alpha} > 0$. This shows that

$$\frac{\partial E_\alpha^\beta(A)}{\partial \gamma_i^1} > 0 \text{ for } \gamma_i^1 \in \left(0, \frac{1}{2}\right), \quad \frac{\partial E_\alpha^\beta(A)}{\partial \gamma_i^1} < 0 \text{ for } \gamma_i^1 \in \left(\frac{1}{2}, 1\right).$$

From what has been discussed above, $E_\alpha^\beta(A)$ is monotonically increasing for $\gamma_i^1 \in \left(0, \frac{1}{2}\right)$ and
monotonically decreasing for \( \gamma_i \in \left[ \frac{1}{2}, 1 \right] \).

(ii) If \( A \) is the fuzziest set, \( \gamma_i = \frac{1}{2} \) for any \( i = 1, 2, \ldots, n \) and \( \lambda = 1, 2, \ldots, l \).

Substitute it into Formula (11), we have

\[
E^0_\alpha(A) = \frac{2 - \beta}{n(2 - \beta - \alpha)} \sum_{i=1}^{l} \left[ \frac{1}{l} \sum_{i=1}^{l} \left( (\gamma_i^\alpha)^{\alpha \beta} + (1 - \gamma_i^\alpha)^{\alpha \beta} \right) \right] = 1. \tag{14}
\]

On the other hand, we get \( \frac{\partial E^0_\alpha(A)}{\partial \gamma_i^\alpha} = 0 \) for \( \gamma_i = \frac{1}{2} \) from the proof above. According to the sufficient condition of the existent extreme value, we need to estimate the symbol of

\[
\frac{\partial^2 E^0_\alpha(A)}{\partial \gamma_i^\alpha \partial \gamma_i^\alpha}
\]

when \( \gamma_i = \frac{1}{2} \), for \( \lambda = 1, 2, \ldots, l, k = 1, 2, \ldots, l \).

Derive Formula (13) with respect to \( \gamma_i^\alpha \), the outcome is

\[
\frac{\partial^2 E^0_\alpha(A)}{(\partial \gamma_i^\alpha)^2} = \frac{2 - \beta}{nl(2 - \alpha - \beta)} (Q_1 + Q_2), \tag{15}
\]

\[
Q_1 = -\frac{\alpha}{2 - \beta} \left( \frac{\alpha}{2 - \beta - 1} \right) \left( (\gamma_i^\alpha)^{\alpha \beta - 1} + (1 - \gamma_i^\alpha)^{\alpha \beta - 1} \right) \right) \ln 2,
\]

\[
Q_2 = -\frac{1}{l} \left( \frac{\alpha}{2 - \beta} \right)^2 \left( (\gamma_i^\alpha)^{\alpha \beta - 1} - (1 - \gamma_i^\alpha)^{\alpha \beta - 1} \right)^2 \right) \ln 2. \tag{16}
\]

We can easily get \( Q_2 = 0 \) when \( \gamma_i = \frac{1}{2} \), then we consider \( Q_1 \) when \( \gamma_i = \frac{1}{2} \).

Case 1:

When \( \alpha + \beta > 2 \), which means \( \frac{\alpha}{2 - \beta} > 1 \), we have \( Q_1 > 0 \) and \( \frac{2 - \beta}{2 - \beta - \alpha} < 0 \). Therefore,

\[
\frac{\partial^2 E^0_\alpha(A)}{(\partial \gamma_i^\alpha)^2} < 0.
\]

Case 2:

When \( \alpha + \beta < 2 \), which means \( 0 < \frac{\alpha}{2 - \beta} < 1 \), we have \( Q_1 < 0 \) and \( \frac{2 - \beta}{2 - \beta - \alpha} > 0 \). Therefore,

\[
\frac{\partial^2 E^0_\alpha(A)}{(\partial \gamma_i^\alpha)^2} < 0.
\]
From what has been discussed above, \( \frac{\partial^2 E_\alpha^\beta (A)}{\partial \gamma_i^\alpha \partial \gamma_i^\alpha} < 0 \) is always satisfied when \( \gamma_i^\alpha = \frac{1}{2} \) for \( \alpha > 0, \beta \in [0,1], \alpha + \beta \neq 2 \). Derive Formula (13) with respect to \( \gamma_i^\alpha \), the outcome is

\[
\frac{\partial^3 E_\alpha^\beta (A)}{\partial \gamma_i^\alpha \partial \gamma_i^\alpha} = \frac{2 - \beta}{nl_i^2 (2 - \alpha - \beta)} \left[ \left( \gamma_i^\alpha \right)^{2 - \beta} - \left( 1 - \gamma_i^\alpha \right)^{2 - \beta} \right] \ln 2 \left( \frac{1}{l_i} \sum_{k=1}^{l_i} \left( \left( \gamma_i^\alpha \right)^{2 - \beta} + \left( 1 - \gamma_i^\alpha \right)^{2 - \beta} \right) \right). \tag{18}
\]

We finally receive \( \frac{\partial^3 E_\alpha^\beta (A)}{\partial \gamma_i^\alpha \partial \gamma_i^\alpha} = 0 \) when \( \gamma_i^\alpha = \gamma_i^\beta = \frac{1}{2} \).

The Hessian Matrix \( \frac{\partial^3 E_\alpha^\beta (A)}{\partial \gamma_i^\alpha \partial \gamma_i^\alpha} \) of \( E_\alpha^\beta (A) \) is

\[
H \left( E_\alpha^\beta (A) \right) = \begin{pmatrix}
E_\alpha^\beta (A)_{11} & 0 & \cdots & 0 \\
0 & E_\alpha^\beta (A)_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E_\alpha^\beta (A)_{l_i}
\end{pmatrix}.
\]

Because \( \frac{\partial^2 E_\alpha^\beta (A)}{\partial \gamma_i^\alpha \partial \gamma_i^\alpha} \) is always less than 0 when \( \gamma_i^\alpha = \frac{1}{2} \), the eigenvalues of \( H \left( E_\alpha^\beta (A) \right) \) is less than 0. Thus \( H \left( E_\alpha^\beta (A) \right) \) is a negative definite matrix, \( E_\alpha^\beta (A) \) reaches its maximum value at \( \gamma_i^\alpha = \frac{1}{2} \).

(iii) According to the definition of the complementary set of HFS, \( A^C = \left\{ \left( x_i, h_{x_i} (x_i) \right) \left| x_i \in X \right. \right\} \),

\[
h_{x_i} (x_i) = \left\{ 1 - \gamma_i^\alpha, 1 - \gamma_i^\alpha, \ldots, 1 - \gamma_i^\alpha \right\}.
\]

Substitute it into (11), we have \( E(A) = E \left( A^C \right) \).

(iv) It can be seen from the proof of (i) that \( E_\alpha^\beta (A) \) is monotonically increasing for \( \gamma_i^\alpha \in \left( 0, \frac{1}{2} \right) \)

and monotonically decreasing for \( \gamma_i^\alpha \in \left( \frac{1}{2}, 1 \right) \). Hence, \( E(A) \geq E(B) \) for \( \frac{1}{2} \geq \gamma_i^\alpha \geq \sigma_i^\alpha \) or

\[
\frac{1}{2} \leq \gamma_i^\alpha \leq \sigma_i^\alpha.
\]

Formula (11) satisfies the conditions in Definition 4, thus \( E_\alpha^\beta (A) \) is an entropy measure for HFS.

**Example 3.** Now we use the data from Example 1 to calculate the relationship between \( E_\alpha^\beta (A) \)
and the two parameters. Figure 1 shows that the entropy values under different values of $\beta$ when $\alpha = 0.1$. We can find that $E^\beta_{\alpha}(A)$ decreases monotonically with increasing parameter $\beta$. Similarly, Figure 2 shows that the entropy values under different values of $\alpha$ when $\beta = 0.1$. We can find that $E^\alpha_{\beta}(A)$ decreases monotonically with increasing parameter $\alpha$.

![Figure 1](image1.png)

**Figure 1** $E^\beta_{\alpha}(A)$ under different values of $\beta$ when $\alpha = 0.1$ in Formula (11)

![Figure 2](image2.png)

**Figure 2** $E^\alpha_{\beta}(A)$ under different values of $\alpha$ when $\beta = 0.1$ in Formula (11)

4. **Model construction and solution process under hesitant fuzzy environment**

In this section, we firstly propose two portfolio models with parametric hesitant fuzzy entropy Formula (11) measuring risks, and give a detailed explanation of the symbols in the models. Then, the solving processes of the models are further illustrated.

4.1 **Portfolio model construction with parametric hesitant fuzzy entropy**

This subsection introduces portfolio models taking the score function and fuzzy entropy to measure the return and risk values under hesitant fuzzy environment.
Assume that an investor chooses from \( n \) stocks \( \{x_1, x_2, \ldots, x_n\} \) by \( m \) criteria \( \{y_1, y_2, \ldots, y_m\} \).

Hesitant fuzzy matrix \( H = [h_{ij}]_{n \times m} \) is composed of hesitant fuzzy set \( h_{ij}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m) \), which can be transformed into a collective column vector \( \overline{H} = [\overline{h}_{i}]_{n \times 1} \). In order to obtain the optimal investment ratios and the optimal portfolio, the following portfolio models based on parametric hesitant fuzzy entropy are proposed.

In the following models, we continue to adopt the constraint trisection approach to expected returns and acceptable risks, which is shown by Zhou and Xu (2018) for risk investors under the hesitant fuzzy environment.

4.1.1 Minimizing parametric entropy portfolio model with expected returns

Firstly, Model 3 seeks a minimum risk level with an expected return \( S \), which changes with the risk preferences of investors.

\[
\begin{align*}
\text{Model 3:} & \quad \min \ E_\alpha^\beta \left[ \bigoplus_{i=1}^n w_i \overline{h}_i \right] \\
& \text{s.t. } \frac{s}{\bigoplus_{i=1}^n w_i \overline{h}_i} \geq \overline{S}, \\
& \quad \bigoplus_{i=1}^n w_i \overline{h}_i = \bigcup_{y_1 \in \tilde{y}_1, y_2 \in \tilde{y}_2, \ldots, y_m \in \tilde{y}_m} \left[ 1 - \prod_{j=1}^m (1 - y_j)^\alpha \right], \\
& \quad E_\alpha^\beta(h) = \frac{2 - \beta}{2 - \beta - \alpha} \left[ \frac{1}{\#h} \sum_{y \in \tilde{h}} (y)^\alpha \frac{\#h}{\gamma} + (1 - y)^\alpha \frac{\#h}{\gamma} \right], \\
& \quad s(h) = \frac{1}{\#h} \sum_{y \in \tilde{h}} \gamma, \\
& \quad \sum_{i=1}^w w_i \leq 1, l_i \leq w_i \leq u_i, i = 1, \ldots, n.
\end{align*}
\]

Where \( s(h) \) is the score function of HFE \( h \), \( \overline{h}_i \) is the aggregated HFE based on \( \overline{h}_i = \bigoplus_{j=1}^n h_{ij} \), \( h_{ij} \) is the hesitant fuzzy information of the alternative \( x_i \) with respect to the criterion \( y_j \), \( W = \{w_1, w_2, \ldots, w_n\} \) and \( w_i \) denote the optimal investment ratios of this fund on these stocks, \( l_i \) and \( u_i \) denote the upper limit and lower limit of \( i \)-th stock, \( \#h \) denotes the length of HFE \( h \), \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \).

In Model 3, the min-score degree \( \overline{S} \) can be used to represent investors with different risk preference. Assume that the range of the min-score degree \( \overline{S} \) is \( [S_{\min}, S_{\max}] \), then we have

- **Case 1:** For a risk seeker, the min-score degree \( \overline{S} \) is set as \( \overline{S}_1 = S_{\max} - \frac{1}{3}(S_{\max} - S_{\min}) \);
- **Case 2:** For a risk neutral, the min-score degree \( \overline{S} \) is set as \( \overline{S}_2 = S_{\max} - \frac{2}{3}(S_{\max} - S_{\min}) \);
- **Case 3:** For a risk averter, the min-score degree \( \overline{S} \) is set as \( \overline{S}_3 = S_{\min} \).
4.1.2 Maximizing score function portfolio model with acceptable risks

Similarly, Model 4 seeks the maximum return level with a given acceptable risk \( \overline{E} \), which also changes with the risk preferences of investors.

\[
\begin{align*}
\text{Model 4:} & \quad \max \ s \left( \Theta_{i=1}^n w_i \overline{h}_i \right) \\
& \quad \text{s.t.} \quad E_{\alpha}^\beta \left[ \Theta_{i=1}^n w_i \overline{h}_i \right] \leq \overline{E} \\
& \quad \Theta_{i=1}^n w_i \overline{h}_i = \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_n} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i)^{\alpha_i} \right\} \\
& \quad E_{\alpha}^\beta (h) = \frac{2 - \beta}{2 - \beta - \alpha} \ln \left[ \frac{1}{h} \sum_{j=1}^m \left( \gamma_j^{\alpha_j} + (1 - \gamma_j)^{\alpha_j} \right) \right] \\
& \quad s(h) = \frac{1}{h} \sum_{j=1}^m \gamma_j \\
& \quad \sum_{i=1}^n w_i \leq 1, l_i \leq w_i \leq u_i, i = 1, \ldots, n.
\end{align*}
\]

In Model 4, the max-entropy degree \( \overline{E} \) can be used to represent investors with different risk preferences. Assume that the range of the max-entropy degree \( \overline{E} \) is \( [E_{\min}, E_{\max}] \), then we obtain:

**Case 1:** For a risk seeker, the max-entropy degree \( \overline{E} \) of is set as \( \overline{E}_1 = E_{\max} \);

**Case 2:** For a risk neutral, the max-entropy degree \( \overline{E} \) of is set as \( \overline{E}_2 = E_{\max} + \frac{2}{3} (E_{\max} - E_{\min}) \);

**Case 3:** For a risk averter, the max-entropy degree \( \overline{E} \) of is set as \( \overline{E}_3 = E_{\min} + \frac{1}{3} (E_{\max} - E_{\min}) \).

4.2 Portfolio solution processes under hesitant fuzzy environment

Assume that an investor wants to put a fund on \( n \) new stocks \( \{x_1, x_2, \ldots, x_n\} \) according to \( m \) criteria \( \{y_1, y_2, \ldots, y_m\} \). The qualitative data he collected is presented by a hesitant fuzzy matrix \( H = [h_{ij}]_{n \times m} \). Generally, portfolio selection process can be divided into two categories: Process I is for investors who seek fewer risks when achieving expected returns, and Process II is for investors who consider the maximum returns primarily at a low risk level.

4.2.1 The concrete solution process for Model 3

According to the discussion above, Model 3 is suitable for **Process I**.

**Step 1:** Transform the hesitant fuzzy matrix \( H = [h_{ij}]_{n \times m} \) into a collective column vector \( \overline{H} = [\overline{h}_j]_{1 \times m} \) by aggregating all the values on one line \( (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m) \).

**Step 2:** Construct the parametric hesitant fuzzy entropy-based portfolio Model 3, where the investor chooses to take the minimum risk with the accepted return.

**Step 3:** Calculate the value range of the score function, written as min-score degree \( \overline{S} \in [S_{\min}, S_{\max}] \).
Determine the value of the lower limit \( i_l \) and upper limit \( i_u \) of \( i \)-th stock’s investment ratios, respectively.

**Step4:** Determine the expected return \( \overline{S} \). In Model 3, we have three cases

**Case 1:** If there is a risk seeker, set \( \overline{S}_1 = S_{\max} - \frac{1}{3}(S_{\max} - S_{\min}) \);

**Case 2:** If there is a risk neutral, set \( \overline{S}_2 = S_{\max} - \frac{2}{3}(S_{\max} - S_{\min}) \);

**Case 3:** If there is a risk averter, set \( \overline{S}_3 = S_{\min} \).

**Step5:** Solve the portfolio model, then obtain the optimal investment ratios \( w_i \) \( (i = 1, 2, ..., n) \).

### 4.2.2 The concrete solution process for Model 4

Similarly, the solution process of Model 4 is as follows by **Process II**.

**Step1:** Transform the hesitant fuzzy matrix \( H = [h_{ij}]_{n \times m} \) into a collective column vector \( \overline{H} = [\overline{h}_i]_{n \times 1} \) by aggregating all the values on one line \( (i = 1, 2, ..., n; j = 1, 2, ..., m) \).

**Step2:** Construct the parametric hesitant fuzzy entropy-based portfolio Model 5, where the investor chooses to take the maximum return with the acceptable risk.

**Step3:** Calculate the value range of the score function, written as max-entropy degree \( \overline{E} \in [E_{\min}, E_{\max}] \). Determine the value of the lower limit \( i_l \) and upper limit \( i_u \) of \( i \)-th stock’s investment ratios, respectively.

**Step4:** Determine the accepted risk \( \overline{E} \). In Model 4, we have three cases

**Case1:** If there is a risk seeker, set \( \overline{E}_1 = E_{\max} \);

**Case 2:** If there is a risk neutral, set \( \overline{E}_2 = E_{\max} + \frac{2}{3}(E_{\max} - E_{\min}) \);

**Case 3:** If there is a risk averter, set \( \overline{E}_3 = E_{\min} + \frac{1}{3}(E_{\max} - E_{\min}) \).

**Step5:** Solve the portfolio model, then obtain the optimal investment ratios \( w_i \) \( (i = 1, 2, ..., n) \).

### 5. Case study

In this section, we demonstrate the validity of the models proposed in Section 4 through empirical analysis. Meanwhile, we compare the results with existing models proposed by Zhou and Xu (2018) to present the models in this paper are canonical.

### 5.1 Example and calculations

In this subsection, we use the same data as paper (Zhou and Xu 2018). Assume that the investor constructs the hesitant fuzzy matrix \( H = [h_{ij}]_{4 \times 3} \) of \( x_i \) \( (i = 1, 2, 3, 4) \) with respect to the criteria
\( y_j (j = 1, 2, 3) \), which is shown in Table 1.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>{0.45}</td>
<td>{0.35, 0.95}</td>
<td>{0.15}</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>{0.35, 0.65, 0.90}</td>
<td>{0.10}</td>
<td>{0.55}</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>{0.75}</td>
<td>{0.15}</td>
<td>{0.35}</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>{0.10, 0.70}</td>
<td>{0.30}</td>
<td>{0.65}</td>
</tr>
</tbody>
</table>

5.1.1 Solution Process I for Model 3 in Case Study

We demonstrate the portfolio selection process for risk investors, and the concrete Process I is as follows for Model 3.

**Step1:** Transform the hesitant fuzzy matrix \( H = \left[ h_{ij} \right]_{4 \times 3} \) into a collective column vector, then we get \( \tilde{H} = \left[ \tilde{h}_i \right]_{4 \times 1} = \{0.6961, 0.9766, 0.7368, 0.8583, 0.9595, 0.8619, 0.7795, 0.9265\}^T \).

**Step2:** Construct the parametric hesitant fuzzy entropy-based portfolio model, where the investor chooses to take the minimum risk with the accepted return.

\[
\begin{align*}
\text{Model 3:} & \quad E_\alpha^\beta \left( h \right) = \frac{2 - \beta}{2 - \beta - \alpha} \log \left[ \frac{1}{\#h} \sum_{r \in h} \left( \left( \gamma^\frac{\alpha}{\beta} \right)^{\frac{\beta}{\beta - \alpha}} + \left( 1 - \gamma^\frac{\alpha}{\beta} \right)^{\frac{\beta}{\beta - \alpha}} \right) \right] \\
\text{s.t.} & \quad s \left[ \bigoplus_{i=1}^4 w_i \tilde{h}_i \right] \geq \tilde{S} \\
\Delta_{i=1}^4 w_i \tilde{h}_i & = \bigcup_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \left\{ 1 - \prod_{i=1}^4 (1 - \gamma_i)^n \right\} \\
\sum_{i=1}^4 w_i & \leq 1, 0.05 \leq w_i \leq 1, i = 1, \ldots, 4.
\end{align*}
\]

During the solution process, we set \( \alpha = 0.8, \beta = 0.5 \).

**Step3:** Calculate the value range of the score function, written as min-score degree \( \tilde{S} \in [0.8608, 0.8771] \).

**Step4:** Set the expected return \( \bar{S} \). In Model 3,

- **Case 1:** If there is a risk seeker, set \( \bar{S}_1 = S_{\max} - \frac{1}{3} (S_{\max} - S_{\min}) = 0.8716 \);
- **Case 2:** If there is a risk neutral, set \( \bar{S}_2 = S_{\max} - \frac{2}{3} (S_{\max} - S_{\min}) = 0.8662 \);
- **Case 3:** If there is a risk averter, set \( \bar{S}_3 = S_{\min} = 0.8608 \).
Step 5: Solve the portfolio model, then obtain the optimal investment ratios \( w_i(i = 1, 2, 3, 4) \), which is shown in Table 2.

For a risk seeker, he can allocate the fund in the following ratios: \( w_1 = 0.5527 \), \( w_2 = 0.3473 \), \( w_3 = 0.0500 \), \( w_4 = 0.0500 \). In other words, if he plans to invest 10,000 dollars in these four stocks, he would place 5527, 3473, 500 and 500 dollars in \( x_1 \), \( x_2 \), \( x_3 \) and \( x_4 \) respectively.

For the risk neutral, the ratios of the fund are \( w_1 = 0.6669 \), \( w_2 = 0.2331 \), \( w_3 = 0.0500 \), \( w_4 = 0.0500 \).

Likewise, for the risk averter, we obtain \( w_1 = 0.7464 \), \( w_2 = 0.1536 \), \( w_3 = 0.0500 \), \( w_4 = 0.0500 \).

5.1.2 Solution Process II for Model 4 in case study

Below we state the results of Model 4 with the same parameter values as Model 3. Since Step 1 is the same as Process I, we perform Process II from Step 2. The concrete Process II is as follows for Model 4.

Step 1: The same as Process I.

Step 2: Construct the parametric hesitant fuzzy entropy-based portfolio model, where the investor chooses to take the maximum returns with the acceptable risk.

\[
\begin{align*}
\text{Model 4:} & \quad \begin{cases}
\max & \sum_{i=1}^{4} w_i h_i \\
\text{s.t.} & \quad E\left[ \sum_{i=1}^{4} w_i h_i \right] \leq \bar{E} \\
& \quad \Theta^{4} \sum_{i=1}^{4} w_i h_i = \bigcup_{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{4} \in \Xi} \left\{ 1 - \prod_{i=1}^{4} (1 - \gamma_{i})^{w_{i}} \right\} \\
& \quad E_{\alpha}^{\beta}(h) = \frac{2-\beta}{2-\beta-\alpha} lb \left[ \frac{1}{\# h} \sum_{\gamma \in \# h} \left( \gamma^{\alpha} + (1 - \gamma)^{\beta} \right) \right] \\
& \quad s(h) = \frac{1}{\# h} \sum_{\gamma} \gamma \\
& \quad \sum_{i=1}^{4} w_i \leq 1, 0.05 \leq w_i \leq 1, i = 1, \ldots, 4.
\end{cases}
\end{align*}
\]

During the solution process, we set \( \alpha = 0.8 \), \( \beta = 0.5 \).

Step 3: Calculate the value range of the entropy function, written as max-entropy degree

\( \bar{E} \in [0.6648, 0.7333] \).

Step 4: Set the acceptable risk \( \bar{E} \). In Model 2,
**Case 1:** if there is a risk seeker, set $E_1 = E_{\text{max}} = 0.7333$;

**Case 2:** if there is a risk neutral, set $E_2 = E_{\text{min}} + \frac{2}{3}(E_{\text{max}} - E_{\text{min}}) = 0.7104$;

**Case 3:** if there is a risk averter, set $E_3 = E_{\text{min}} + \frac{1}{3}(E_{\text{max}} - E_{\text{min}}) = 0.6876$.

**Step 5:** Solve the portfolio model, then obtain the optimal investment ratios $w_i (i = 1, 2, 3, 4)$, we can obtain that:

**Case 1:** For a risk seeker, he can allocate the fund in the following ratios: $w_1 = 0.3164$, $w_2 = 0.3263$, $w_3 = 0.0855$, $w_4 = 0.2717$. In other words, if he plans to invest 10,000 dollars in these four stocks, he would place 1757, 1821, 4877 and 1545 dollars in $x_1$, $x_2$, $x_3$ and $x_4$ respectively;

**Case 2:** For the risk neutral, the ratios of the fund are $w_1 = 0.3164$, $w_2 = 0.3263$, $w_3 = 0.0855$, $w_4 = 0.2717$;

**Case 3:** For the risk averter, we obtain $w_1 = 0.3182$, $w_2 = 0.3282$, $w_3 = 0.0804$, $w_4 = 0.2732$.

### 5.2 Discussion of results

According to the calculations in Section 5.1, the optimal investment ratios are different for risk preferences. To illustrate the effectiveness of the model proposed in Chapter 4, we solve the Model 1 and Model 2 based on the data in Table 1, and present the results in Table 2 and Table 3. These models all apply score function to measure returns.

| Table 2 | Comparison of portfolio Model 1 and Model 3 |
|-----------------|-----------------|-----------------|-----------------|
| **Ratios** | **Risk-seeking** | **Risk-neutral** | **Risk-averse** |
| | Model 3 | Model 1 | Model 3 | Model 1 | Model 3 | Model 1 |
| $w_1$ | 0.5527 | 0.1170 | **0.6669** | 0.0500 | **0.7464** | 0.0500 |
| $w_2$ | **0.3473** | 0.1522 | **0.2331** | 0.0500 | **0.1536** | 0.0500 |
| $w_3$ | 0.0500 | 0.6531 | 0.0500 | 0.8500 | 0.0500 | 0.8500 |
| $w_4$ | 0.0500 | 0.0777 | 0.0500 | 0.0500 | 0.0500 | 0.0500 |
| Returns | 0.8716 | 0.8717 | 0.8662 | 0.8666 | 0.8608 | 0.8666 |

From Table 2, we can conclude that:

1. In the existing portfolio selection model with deviation function measuring risks, the optimal
investment ratios are more dispersed for risk seekers, yet concentrated in $x_3$ for the other two kinds of investors. As we have discussed in Section 5.1, when it turns to measure risks by entropy, the investors mainly allocate funds to $x_1$ and $x_2$.

(2) The returns of the optimal portfolios are similar for the risk seeker and risk neutral under two risk measures. By contrast, returns change more significantly with risk appetite in entropy-based portfolio models.

(3) As the expected returns $\bar{S}$ rise, investors tend to place more funds in $x_2$. The reason for this phenomenon is that the entropy of $x_1$ and $x_2$ is less than $x_3$ and $x_4$, but the returns of $x_2$ are more than $x_1$. When it comes to pursue higher returns, more funds should be placed in $x_2$.

| Table 3 Comparison of portfolio Model 2 and Model 4 |
|-----------------|-----------------|-----------------|-----------------|
| Ratios          | Risk-seeking    | Risk-neutral    | Risk-averse     |
| Model 4         | Model 2         | Model 4         | Model 2         | Model 4         | Model 2         |
| $w_1$           | 0.3164          | 0.3164          | **0.3164**      | 0.3182          | 0.1589          |
| $w_2$           | 0.3264          | 0.3264          | **0.3264**      | 0.3282          | 0.1975          |
| $w_3$           | 0.0085          | 0.0085          | **0.0085**      | 0.0804          | 0.5075          |
| $w_4$           | 0.2717          | 0.2717          | **0.2717**      | 0.2732          | 0.1371          |
| Returns         | 0.8771          | 0.8771          | 0.8771          | 0.8771          | 0.8739          |

From Table 3, we can extract:

(1) The optimal investment ratios of Model 4 are the same for risk seeking and the risk neutral investors, while the optimal investment ratios of Model 2 change with the risk preferences. With the known information, Model 2 has greater diversity of results.

(2) The returns of the optimal portfolios are similar for the risk seeker and the risk neutral under two risk measures; however, Model 4 has higher returns for risk-averse investors.

(3) For risk-averse investors, Model 2 tends to place the most funds in $x_3$, but Model 4 shows the opposite results, which is a significant difference between the two models. This fully illustrates the variability of two risk measures, and inspires us to use multiple measures to mitigate risks in practical problems.

The discussion above suggests that the proposed portfolio selection model is effective, and the optimal portfolios change more significantly with the expected returns.

6. Sensitivity analysis of our proposed Model 3

In this section, we focus on the impacts of parameters on the returns, risks and investment proportions of our proposed Model 3 for risk neutral investors. Although the parameters $\alpha$ and $\beta$ only appear in the entropy formula, the influence of parameter changes on investment will
eventually be reflected in many aspects. To begin with, we need to clarify the entropy Formula (11) decreases with these two parameters, which has been pointed out by Example 3 in Chapter 3.

According to Formula (11), we should follow the value range of the parameters, to be specific, \( \alpha > 0, \beta \in [0,1], \alpha + \beta \neq 2 \). We change one parameter within a reasonable range while the other one is fixed. To make the results more visible, the returns (see Figure 3) and risks (see Figure 5) of the optimal portfolio selection for risk neutral investors under different values of \( \beta \) under a fixed \( \alpha \) are shown in Table 4. The returns (see Figure 4) and risks (see Figure 6) of the optimal portfolio selection for risk neutral investors under different values of \( \alpha \) when \( \beta = 0.1 \) are shown in Table 5.

The result is discussed as follows.

**Table 4** Returns and risks for risk neutral investors under different values of \( \beta \) when \( \alpha = 2 \) in Model 3

<table>
<thead>
<tr>
<th>Parameter value</th>
<th>Returns</th>
<th>Risks</th>
<th>Parameter value</th>
<th>Returns</th>
<th>Risks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta = 0.1 )</td>
<td>0.8723</td>
<td>0.4894</td>
<td>( \beta = 0.6 )</td>
<td>0.8741</td>
<td>0.4098</td>
</tr>
<tr>
<td>( \beta = 0.2 )</td>
<td>0.8727</td>
<td>0.4848</td>
<td>( \beta = 0.7 )</td>
<td>0.8748</td>
<td>0.3919</td>
</tr>
<tr>
<td>( \beta = 0.3 )</td>
<td>0.8730</td>
<td>0.4596</td>
<td>( \beta = 0.8 )</td>
<td>0.8753</td>
<td>0.3732</td>
</tr>
<tr>
<td>( \beta = 0.4 )</td>
<td>0.8733</td>
<td>0.4437</td>
<td>( \beta = 0.9 )</td>
<td>0.8756</td>
<td>0.3540</td>
</tr>
<tr>
<td>( \beta = 0.5 )</td>
<td>0.8735</td>
<td>0.4271</td>
<td>( \beta = 1.0 )</td>
<td>0.8759</td>
<td>0.3344</td>
</tr>
</tbody>
</table>

**Table 5** Returns and risks for risk neutral investors under different values of \( \alpha \) when \( \beta = 0.1 \) in Model 3

<table>
<thead>
<tr>
<th>Parameter value</th>
<th>Returns</th>
<th>Risks</th>
<th>Parameter value</th>
<th>Returns</th>
<th>Risks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0.5 )</td>
<td>0.8608</td>
<td>0.8091</td>
<td>( \alpha = 5.5 )</td>
<td>0.8760</td>
<td>0.2728</td>
</tr>
<tr>
<td>( \alpha = 1.5 )</td>
<td>0.8687</td>
<td>0.5674</td>
<td>( \alpha = 6.5 )</td>
<td>0.8755</td>
<td>0.2518</td>
</tr>
<tr>
<td>( \alpha = 2.5 )</td>
<td>0.8735</td>
<td>0.4305</td>
<td>( \alpha = 7.5 )</td>
<td>0.8743</td>
<td>0.2364</td>
</tr>
<tr>
<td>( \alpha = 3.5 )</td>
<td>0.8757</td>
<td>0.3512</td>
<td>( \alpha = 8.5 )</td>
<td>0.8734</td>
<td>0.2242</td>
</tr>
<tr>
<td>( \alpha = 4.5 )</td>
<td>0.8761</td>
<td>0.3034</td>
<td>( \alpha = 9.5 )</td>
<td>0.8728</td>
<td>0.2142</td>
</tr>
</tbody>
</table>

### 6.1 Impacts of Parameters \( \alpha, \beta \) on Returns

From Table 4, we can obtain when the value of \( \alpha \) is fixed, the higher the value of \( \beta \), the higher the return of the optimal portfolio selection. From Table 5, we can also be aware that returns show a growing trend with an increasing value of parameter \( \alpha \). Here we can view the value of parameters as investor’s preference information of the portfolio selection. Due to the fact that the entropy Formula (11) decreases with \( \alpha \) and \( \beta \), a higher parameter value means the investor is subjectively more confident with the stocks he has chosen, which would bring fewer risks. Thus, the investor can get higher returns.
Figure 3 Returns for risk neutral investors under different values of $\beta$ when $\alpha = 2$ in Model 3

Figure 4 Returns for risk neutral investors under different values of $\alpha$ when $\beta = 0.1$ in Model 3

6.2 Impacts of parameters $\alpha$, $\beta$ on risks

From Table 4, we can tell when the value of $\alpha$ is fixed, the higher the value of $\beta$, the lower the risk of the optimal portfolio selection. For the fixed $\beta$, the risk of the optimal portfolio selection decreases rapidly which is shown in Table 5. The result is reasonable because the entropy always decreases with the parameters $\alpha$ and $\beta$. When exploring the impact of $\alpha$ on risks, the risk of the optimal portfolio selection changes a lot because the parameter ranges are wider.
Figure 5 Risks for risk neutral investors under different values of $\beta$ when $\alpha = 2$ in Model 3

Figure 6 Risks for risk neutral investors under different values of $\alpha$ when $\beta = 0.1$ in Model 3

The change of returns and risks discussed above enlightens us that an appropriate parameter value is important in the portfolio selection with information preference.

6.3 Impacts of parameter $\alpha$, $\beta$ on investment proportions

6.3.1 Impacts of parameter $\alpha$ on investment proportions

Now we compare the investment proportions for risk neutral investors under different values of $\alpha$ when $\beta = 0.1$ (see Figure 7), the specific data is shown in Table 6. The result is discussed as follows.

<table>
<thead>
<tr>
<th>Parameter value</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>Parameter value</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.5$</td>
<td>0.6669</td>
<td>0.2331</td>
<td>0.0500</td>
<td>0.0500</td>
<td>$\alpha = 5.5$</td>
<td>0.4150</td>
<td>0.3753</td>
<td>0.0500</td>
<td>0.1587</td>
</tr>
<tr>
<td>$\alpha = 1.5$</td>
<td>0.6226</td>
<td>0.2774</td>
<td>0.0500</td>
<td>0.0500</td>
<td>$\alpha = 6.5$</td>
<td>0.4339</td>
<td>0.3860</td>
<td>0.0500</td>
<td>0.1301</td>
</tr>
<tr>
<td>$\alpha = 2.5$</td>
<td>0.4894</td>
<td>0.4106</td>
<td>0.0500</td>
<td>0.0500</td>
<td>$\alpha = 7.5$</td>
<td>0.4650</td>
<td>0.4085</td>
<td>0.0500</td>
<td>0.0765</td>
</tr>
<tr>
<td>$\alpha = 3.5$</td>
<td>0.4260</td>
<td>0.3809</td>
<td>0.0500</td>
<td>0.1431</td>
<td>$\alpha = 8.5$</td>
<td>0.4928</td>
<td>0.4072</td>
<td>0.0500</td>
<td>0.0500</td>
</tr>
<tr>
<td>$\alpha = 4.5$</td>
<td>0.4096</td>
<td>0.3727</td>
<td>0.0500</td>
<td>0.1677</td>
<td>$\alpha = 9.5$</td>
<td>0.5178</td>
<td>0.3822</td>
<td>0.0500</td>
<td>0.0500</td>
</tr>
</tbody>
</table>

(1) We can notice that $w_1$ and $w_2$ are always higher than $w_3$ and $w_4$, which is due to the fact that the risks of $w_1$ and $w_2$ are lower. In the portfolio Process I, investors pursue lower risks over
With the increase of the parameter $\alpha$, $w_2$ becomes higher. It’s reasonable because the value of score function of $x_2$ is higher than $x_1$, as we have discussed in Section 6.1.2, the returns of the optimal portfolio selection tend to increase with the parameter $\alpha$, the increase in $w_2$ contributes to the growth in returns.

### Figure 7
Investment proportions for risk neutral investors under different values of $\alpha$ when $\beta = 0.1$ in Model 3

#### 6.3.2 Impacts of parameter $\beta$ on investment proportions

Now we compare the investment proportions for risk neutral investors under different values of $\beta$ when $\alpha = 2$ (see Figure 8), the specific data is shown in Table 7. The result is discussed as follows.

<table>
<thead>
<tr>
<th>Parameter value</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>$w_3$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0.1$</td>
<td>0.5323</td>
<td>0.3677</td>
<td>0.0500</td>
<td>0.0500</td>
<td>0.4717</td>
<td>0.4079</td>
<td>0.0500</td>
<td>0.0704</td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>0.5202</td>
<td>0.3798</td>
<td>0.0500</td>
<td>0.0500</td>
<td>0.4548</td>
<td>0.3970</td>
<td>0.0500</td>
<td>0.0982</td>
</tr>
<tr>
<td>$\beta = 0.3$</td>
<td>0.5087</td>
<td>0.3913</td>
<td>0.0500</td>
<td>0.0500</td>
<td>0.4400</td>
<td>0.3884</td>
<td>0.0500</td>
<td>0.1216</td>
</tr>
<tr>
<td>$\beta = 0.4$</td>
<td>0.4977</td>
<td>0.4023</td>
<td>0.0500</td>
<td>0.0500</td>
<td>0.4276</td>
<td>0.3817</td>
<td>0.0500</td>
<td>0.1407</td>
</tr>
<tr>
<td>$\beta = 0.5$</td>
<td>0.4873</td>
<td>0.4127</td>
<td>0.0500</td>
<td>0.0500</td>
<td>0.4178</td>
<td>0.3767</td>
<td>0.0500</td>
<td>0.1554</td>
</tr>
</tbody>
</table>

(1) $w_1$ and $w_2$ are always higher than $w_3$ and $w_4$, which is reasonable because the risks of $w_1$ and $w_2$ are lower. In the portfolio solving Process 1, investors pursue lower risks over higher returns.

(2) With the increase of the parameter $\beta$, $w_2$ becomes higher. It’s reasonable because the value of score function of $x_2$ is higher than $x_1$, as we have discussed in Section 6.1.2, the returns of the optimal portfolio selection tend to increase with the parameter $\beta$, the increase in $w_2$ contributes to the growth in returns.
7. Sensitivity analysis of our proposed Model 4

In this section, we focus on the impacts of parameters on the returns, risks and investment proportions of our proposed Model 5 for risk-averse investors, which take the maximum returns as the objective function. As emphasized earlier, the entropy Formula (11) decreases with these two parameters.

Like what has been done in Chapter 6, we change one parameter within a reasonable range while the other one is fixed, specifically, the parameter values $\alpha = 0.2$ and $\beta = 0.1$ separately. To make the results more visible, the returns (see Figure 9) and risks (see Figure 11) of the optimal portfolio selection for risk-averse investors under different values of $\beta$ with a fixed $\alpha = 0.2$ are shown in Table 8. The returns (see Figure 10) and risks (see Figure 12) of the optimal portfolio selection for risk-averse investors under different values of $\alpha$ when $\beta = 0.1$ are shown in Table 9. The result is discussed as follows.

<table>
<thead>
<tr>
<th>Table 8</th>
<th>Returns and risks for risk-averse investors under different values of $\beta$ when $\alpha = 0.2$ in Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter value</td>
<td>Returns</td>
</tr>
<tr>
<td>$\beta = 0.1$</td>
<td>0.8744</td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>0.8745</td>
</tr>
<tr>
<td>$\beta = 0.3$</td>
<td>0.8746</td>
</tr>
<tr>
<td>$\beta = 0.4$</td>
<td>0.8747</td>
</tr>
<tr>
<td>$\beta = 0.5$</td>
<td>0.8748</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 9</th>
<th>Returns and risks for risk-averse investors under different values of $\alpha$ when $\beta = 0.1$ in Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter value</td>
<td>Returns</td>
</tr>
<tr>
<td>$\alpha = 0.2$</td>
<td>0.8745</td>
</tr>
<tr>
<td>$\alpha = 0.4$</td>
<td>0.8754</td>
</tr>
<tr>
<td>$\alpha = 0.6$</td>
<td>0.8763</td>
</tr>
<tr>
<td>$\alpha = 0.8$</td>
<td>0.8769</td>
</tr>
<tr>
<td>$\alpha = 1.0$</td>
<td>0.8771</td>
</tr>
</tbody>
</table>

7.1 Impacts of the parameters $\alpha$, $\beta$ on returns

From Figure 9, we can obtain when the value of $\alpha$ is fixed, the higher the value of $\beta$, the higher the return of the optimal portfolio selection, which is consistent with the results of Model 1. From Figure 10, we can also be aware that returns show a growing trend with an increasing value of...
parameter $\alpha$. Here we can view the value of parameters as investor’s preference information of the portfolio selection. Due to the fact that the entropy Formula (11) decreases with $\alpha$ and $\beta$, a higher parameter value means the investor is subjectively more confident with the stocks he has chosen, which would bring fewer risks. Thus, the investor can get higher returns.

**Figure 9** Returns for risk-averse investors under different values of $\beta$ when $\alpha = 0.2$ in Model 4

**Figure 10** Returns for risk-averse investors under different values of $\alpha$ when $\beta = 0.1$ in Model 4

### 7.2 Impacts of the parameters $\alpha, \beta$ on risks

From Table 4, we can tell when the value of $\alpha$ is fixed, the higher the value of $\beta$, the lower the risk of the optimal portfolio selection. For the fixed $\beta$, the risk of the optimal portfolio selection decreases rapidly which is shown in Table 5. The result is reasonable because the entropy always decreases with the parameters $\alpha$ and $\beta$. When exploring the impact of $\alpha$ on risks, the risk of the optimal portfolio selection changes a lot because the parameter ranges are wider.
7.3 Impacts of the parameter $\alpha$, $\beta$ on investment proportions

7.3.1 Impacts of the parameter $\alpha$ on investment proportions

Now we compare the investment proportions for risk-averse investors under different values of $\alpha$ when $\beta = 0.1$ (see Figure 13), the specific data is shown in Table 10. The result is discussed as follows.

Table 10 Investment proportions for risk-averse investors under different values of $\alpha$ when $\beta = 0.1$ in Model 4

<table>
<thead>
<tr>
<th>Parameter value</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>Parameter value</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.2$</td>
<td>0.4767</td>
<td>0.3611</td>
<td>0.0500</td>
<td>0.1122</td>
<td>$\alpha = 1.2$</td>
<td>0.3164</td>
<td>0.3264</td>
<td>0.0085</td>
<td>0.2717</td>
</tr>
<tr>
<td>$\alpha = 0.4$</td>
<td>0.4434</td>
<td>0.3574</td>
<td>0.0500</td>
<td>0.1492</td>
<td>$\alpha = 1.4$</td>
<td>0.3164</td>
<td>0.3264</td>
<td>0.0085</td>
<td>0.2717</td>
</tr>
<tr>
<td>$\alpha = 0.6$</td>
<td>0.4042</td>
<td>0.3516</td>
<td>0.0500</td>
<td>0.1942</td>
<td>$\alpha = 1.6$</td>
<td>0.3164</td>
<td>0.3264</td>
<td>0.0085</td>
<td>0.2717</td>
</tr>
<tr>
<td>$\alpha = 0.8$</td>
<td>0.3601</td>
<td>0.3444</td>
<td>0.0500</td>
<td>0.2455</td>
<td>$\alpha = 1.8$</td>
<td>0.3164</td>
<td>0.3264</td>
<td>0.0085</td>
<td>0.2717</td>
</tr>
<tr>
<td>$\alpha = 1.0$</td>
<td>0.3164</td>
<td>0.3264</td>
<td>0.0085</td>
<td>0.2717</td>
<td>$\alpha = 2.0$</td>
<td>0.3164</td>
<td>0.3264</td>
<td>0.0085</td>
<td>0.2717</td>
</tr>
</tbody>
</table>

(1) $w_1$ and $w_2$ are always higher than $w_3$ and $w_4$, which is a tradeoff between returns and risks. $x_1$ and $x_2$ offer higher returns with lower risks, relatively.

(2) With the increase of the parameter $\alpha$, $w_1$ becomes higher while $w_3$ and $w_4$ decline. It's reasonable because the descending order of score function values is $x_1, x_4, x_2, x_3$. Further, the increase of the parameter $\alpha$ reduces the risk difference between these four stocks, the
optimal portfolio selection tends to concentrate on stocks with higher returns.

(3) When the parameter value $\alpha$ is large enough, the investment proportion remains unchanged. This indicates the portfolio achieves an optimal solution under the known information, which has fewer risks than the acceptable risk.

![Figure 13: Investment proportions for risk-averse investors under different values of $\alpha$ when $\beta = 0.1$ in Model 4](image)

### 7.3.2 Impacts of the Parameter $\beta$ on Investment Proportions

Now we compare the investment proportions for risk neutral investors under different values of $\beta$ when $\alpha = 0.2$ (see Figure 14), the specific data is shown in Table 11. The result is discussed as follows.

**Table 11: Investment proportions for risk-averse investors under different values of $\beta$ when $\alpha = 0.2$ in Model 4**

<table>
<thead>
<tr>
<th>Parameter value $\beta$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>Parameter value $\beta$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0.1$</td>
<td>0.4767</td>
<td>0.3611</td>
<td>0.0500</td>
<td>0.1122</td>
<td>$\beta = 0.6$</td>
<td>0.4655</td>
<td>0.3600</td>
<td>0.0500</td>
<td>0.1245</td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>0.4750</td>
<td>0.3609</td>
<td>0.0500</td>
<td>0.1141</td>
<td>$\beta = 0.7$</td>
<td>0.4621</td>
<td>0.3596</td>
<td>0.0500</td>
<td>0.1283</td>
</tr>
<tr>
<td>$\beta = 0.3$</td>
<td>0.4731</td>
<td>0.3608</td>
<td>0.0500</td>
<td>0.1161</td>
<td>$\beta = 0.8$</td>
<td>0.4580</td>
<td>0.3592</td>
<td>0.0500</td>
<td>0.1328</td>
</tr>
<tr>
<td>$\beta = 0.4$</td>
<td>0.4709</td>
<td>0.3605</td>
<td>0.0500</td>
<td>0.1186</td>
<td>$\beta = 0.9$</td>
<td>0.4531</td>
<td>0.3586</td>
<td>0.0500</td>
<td>0.1384</td>
</tr>
<tr>
<td>$\beta = 0.5$</td>
<td>0.4684</td>
<td>0.3603</td>
<td>0.0500</td>
<td>0.1213</td>
<td>$\beta = 1.0$</td>
<td>0.4470</td>
<td>0.3578</td>
<td>0.0500</td>
<td>0.1452</td>
</tr>
</tbody>
</table>

(1) Note the fact that $w_1$ and $w_2$ are always higher than $w_3$ and $w_4$, which is a tradeoff between returns and risks. $x_1$ and $x_2$ offer relatively higher returns with lower risks.

(2) With the increase of the parameter $\alpha$, $w_4$ becomes higher slightly while $w_1$ and $w_2$ decline. Due to the fact that the change of parameter value $\beta$ results in the less change of risks, comparatively, the change in investment proportions is not dramatic.
Figure 14 Investment proportions for risk-averse investors under different values of $\beta$ when $\alpha = 0.2$ in Model 4

8. Conclusions

There have been many approaches to the decision-making problem under fuzzy environment, but when facing the portfolio selection where precise historical data is not available, the qualitative model proposed in this paper is effective. The main contributions of this paper are as follows. As a further application of parametric entropy proposed by Mei and Li (2019), we take entropy as the objective function, which provides a new way to measure portfolio risks under hesitant fuzzy environment. To make this risk measure reasonable, we show the rigorous mathematic proof and related discussion of parametric entropy. Corresponding to models of Zhou and Xu (2018), we propose two novel portfolio selection models from two perspectives with parametric entropy measuring risks, where portfolio return is measured by score function as before. On the one hand, we advance a minimizing parametric entropy portfolio model with expected return. On the other hand, we put forward a maximizing score function portfolio model with acceptable risk. Then, we provide a detailed description of the proposed models’ solving processes for risk investors and demonstrate processes with a numerical example. In the end, we conduct an intensive analysis of the results, including the comparison of proposed models and existing ones, and exploration of their impacts on the returns, risks and investment proportions of the optimal portfolio selection. Through full and accurate data obtained, we can draw the conclusion that the introduced parameters allow investors more flexibility in portfolio and provide a more comprehensive picture of preference information under uncertain environment.

Admittedly, research on portfolio selection under hesitant fuzzy environment is still in its infancy. In further studies on portfolio, we are committed to dividing the value range of parameters to adapt to the investors’ risk appetites. In the meantime, other risk measures should be considered to adopt in portfolio selection process. We will continue to work on finding a more practical and impeccable portfolio selection model under hesitant fuzzy environment.

Acknowledgements: This research was supported by the “National Social Science Foundation Projects of China, No. 21BTJ069”. The authors are highly grateful to the referees and editor in-chief for their very helpful advices and comments.

Author Contributions

Xue Deng was responsible for the overall understanding of the structure of the article, the main idea
of the article, the conclusion analysis of the article and the revision of the full text.

Jinyao Zhao performed the data analyses and wrote the manuscript.

Qihan Fu revised it critically for important intellectual content.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest

We declare that we have no conflict of interest.

Informed consent

Informed consent was obtained from all individual participants included in the study.

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