A Mass Transit Equilibrium Problem with Bottleneck Congestion and Capacity Constraints

Huayan Shang  
Capital University of Economics and Business

Guangbin Tian  
Capital University of Economics and Business

Hongrui Chu  
Capital University of Economics and Business  
chuhongrui@126.com

Tieqiao Tang  
Beihang University

Xin Xing  
Capital University of Economics and Business

Research Article

Keywords: seat capacity, capacity constraints, equilibrium cost, bi-level programming model

Posted Date: January 19th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-1356992/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.  
Read Full License
A Mass Transit Equilibrium Problem with Bottleneck Congestion and Capacity Constraints

Huayan Shang¹, Guangbin Tian¹, Hongrui Chu¹,², Tieqiao Tang², Xin Xing¹

¹School of Management and Engineering, Capital University of Economics and Business, Beijing 100070, China
²School of Transportation Science and Engineering, Beihang University, Beijing 100191, China

Huayan Shang: shanghuayan@126.com
Guangbin Tian: tianguangbin@cueb.edu.cn
Hongrui Chu: chuhongrui@126.com
Tieqiao Tang: tieqiaotang@buaa.edu.cn
Xin Xing: 18811530796@163.com

Abstract

Seat capacity setting greatly affects vehicle capacity. Increasing the number of seats allows more passengers to benefit from comfortable travel conditions but reduces the available space for standing passengers and the total capacity of the vehicle. By reducing the number of seats, the operator increases the capacity of the vehicle but most passengers do not find an available seat. Without changing the size of the bus, the operator chooses the desirable seat capacity when purchasing the bus, which can maximize vehicle capacity and allow passengers to have more opportunities to get a seat. Considering capacity constraints, this paper establishes multiple origins and a single destination equilibrium model to study the travel behaviors during peak-period. A bi-level programming model is considered to meet the equilibrium cost. The result shows that we succeed in finding an optimal seat capacity to make the total travel cost be lowest.

Keywords: seat capacity; capacity constraints; equilibrium cost; bi-level programming model

---

¹ Correspondence author, E-mail: chuhongrui@126.com
1 Introduction

Urban traffic congestion during rush hour is a major problem that affects public transportation sustainability for a long time. In recent years, the service quality of mass transit has gradually declined due to the problems such as travel discomfort and failure to board (de Palma et al. 2015). Challenges are imposed on transit operators to improve the level of service and attraction of mass transit (Bie et al. 2020). The level of service has an important impact on the satisfaction stated by public transport users with the service they receive (de Oña 2020; Li et al. 2020; Li et al. 2021). The main reason for unreliability is the volatility of traffic interval, which further affects waiting times and distributes passengers unevenly across vehicles (Soza-Parra et al. 2019). However, existing studies on alleviating peak-hour in-vehicle crowding in transit networks can be categorized as two aspects: demand management (e.g., peak-fare charging, off-peak discounting, and other forms of rewards) and supply management (e.g., transit schedule or frequency optimization) (An et al. 2020). Starting from the demand side, these subsidies or policies may cause shifts during peak hours; from the supply side, simply increasing the frequency may increase the operating costs. But the control of seat capacity can greatly alleviate traffic pressure, which not only ensures the operator’s cost; but also improves travel satisfaction. As a consequence, two concepts have to be mentioned here, namely seat capacity and vehicle capacity. Vehicle capacity is the sum of seat capacity and maximum standing capacity. Under the premise that the vehicle type is the same and the maximum load of a vehicle is determined, there is an easy way to change seat capacity, that is, by removing or adding the seats. In addition, it was found that reducing the number of seats during peak hours could yield a better result than increasing the bus departure frequency (Hörcher et al. 2018).

It is well known that travelers experience in-vehicle travel time more negatively if the vehicle is crowded, particularly, if it is not possible to get a seat (Cantwell et al. 2009; Kim et al. 2015). Furthermore, waiting and transfer time is typically perceived more negatively than nominal in-vehicle time (Abrantes and Wardman 2011). Jenelius (2020) proposed a methodology for providing personalized, predictive in-vehicle crowding information to public transport travelers via mobile applications or at-stop displays. Three metrics involving seat availability are considered: (1) the probability of getting a seat onboarding, (2) the expected travel time standing, and (3) the excess perceived travel time compared to uncrowded conditions. In addition, Yan et al. (2020) presented a comprehensive review of the crowding effects on public transport vehicles
and seating capacity, including crowding cost, choice probability, passenger crowding, and the size of a seat. Not only do standing passengers experience in-vehicle crowding, but also seated passengers. de Palma et al. (2015) found that the seating capacity had an internal relationship with passenger crowding, by the means of estimating the willingness of passengers to choose a spacious area under different standee densities. Within the maximum allowable total mass, the seating capacity plays a role in adjusting the distribution of the standing passengers in the carriage (Graa et al. 2017).

In practice, reallocating the number of seats is much more economically than purchasing a new vehicle. Inturri et al. (2021) simulated a set of 50 different scenarios, by varying the numbers of vehicles and seat capacity, and considering different demand rates and route choice strategies of the vehicles. Reducing the number of seats increases bus capacity by allowing more standees; thus the number of seats implies a trade-off between comfort and capacity that is allowed for in our model (Tirachini et al. 2014). If the remaining users are denied boarding or cannot get a seat, they have to wait for the next vehicle, incurring an extra waiting time (de Palma et al. 2015). However, it is unclear how the uncertainty due to unknown waiting times affects travelers’ service satisfaction. Therefore, in this article, we consider the additional cost of seated and standing passengers compared with the research of Tian et al. (2007); and further explore its impact on the total travel cost and the equilibrium passenger departure time distribution. Later, a bi-level programming model is set up to solve a mass transit equilibrium problem, that is, the upper-level problem is presented to minimize the total travel cost; the lower-level problem presents the travel distribution under each seat capacity.

The main purpose of this paper is to propose a mass transit equilibrium problem to set an optimal number of seats under the requirement of minimizing the total travel costs. Considering the capacity constrain, passengers need to endure congestion, which incurs crowding costs. In addition, they need to bear the additional costs of having to stand or fail to board. Consequently, we consider in-vehicle passenger crowding effect, schedule delay cost, and the additional cost because of denied boarding or cannot get a seat, and analyze the equilibrium characteristics of commuters during the morning peak hours on multiple origins and a single destination transit system. A bi-level programing model is developed to determine the optimal seat capacity.

The remainder of this paper is organized as follows. Section 2 provides a literature review. In Sect. 3, we present our model, that is multiple origins and a single destination
equilibrium model. Section 4 discusses some equilibrium theorems. Afterward, we introduce the numerical study in Sect. 5. Finally, Section 6 concludes the paper.

2. Literature review

Before traveling, most travelers will weigh various travel modes based on the traffic information they have, and finally, make the desired decision. The bottleneck model was first proposed by Vickrey (1969) and it is based on the deterministic queuing theory in peak period. Small (2015) made a comprehensive interpretation of the bottleneck model. In the bottleneck model, travelers need to weigh the delay cost and crowding cost and then make a decision. At user equilibrium, commuters from the same group have identical individual travel costs, irrespective of the trains used during the rush hour, and this is presented as a series of complementary conditions (Tian et al. 2021). Tian et al. (2007) analyzed the equilibrium properties of commuters’ trip timing during the morning commute on a many-to-one linear corridor transit system with considering in-vehicle passenger crowding effect and schedule delay cost. Yang et al. (2013) investigated the morning commute problem with both bottleneck congestion and parking space constraints. Yang and Tang (2018) provided a framework of the rail transit bottleneck and the user equilibrium with a uniform-fare and the social optimum with service run-dependent fares are determined. Furthermore, Shang et al. (2020) combined the bottleneck model and the activity-based approach to studying commute behavior in the evening rush hours at bus hubs, so as to solve the problem of time allocation between commuters’ activities and trips.

Furthermore, many scholars study travel behavior from the perspective of congestion cost and capacity constraints. Liu et al. (2020) had a study to relate the travel cost variability due to stochastic capacity with commuters’ departure time choice behaviors. de Palma et al. (2017) investigated trip-timing decisions of rail transit users who trade off in-vehicle passenger crowding costs and disutility from traveling early or late. Three fare regimes, namely no fare, an optimal uniform fare, and an optimal time-dependent fare, were studied and compared, together with the determination of the optimal long-run number and capacities of trains. Cao et al. (2018) focused on the vector traffic network equilibrium problem with demands uncertainty and capacity constraints of arcs, in which, the demands are not exactly known and assumed as a discrete set that contains finite scenarios. Due to the discrete vehicle arrivals and the relatively small capacity, it is particularly necessary to study passenger travel behavior during peak hours to accurately describe congestion. Çelebi and Îmre (2020) focused
on the estimation of crowding in public transport and its effect on perceived comfort. They used some specific questions designed to measure the trade-off between travel time, cost and discomfort related to crowding.

In the research field of capacity supply, Tirachini et al. (2014) is the first paper that included the number of seats as a decision variable in a numerical simulation of bus operations and multimodal pricing. They simultaneously optimized seat supply with vehicle size and frequency, and concluded that buses should have as many seats as possible. Different configurations of vehicles regarding the number of seats and space for standees are relevant for the level of crowding and standing externalities in public transport (Whelan and Crockett 2009). Because seated and standing traveling implies significantly different experiences for public transport users. Pefitsi et al. (2021) captured the effective capacity utilization of the train, by considering passengers’ distribution among individual train cars into an agent-based simulation model. Hörcher et al. (2018) investigated with analytical modeling and numerical simulations how the optimal seat supply depends on demand and supply characteristics. They also investigated how decisions on seat provision affect the marginal external cost of traveling, considering the difference between the disutility of standing and seated service usage.

In addition, overcrowding discouraged passengers boarding the train, and this situation may result in fail-to-board, and extra wait/transfer time shall be expected (Qu et al. 2020), which in turn leads to additional costs. Therefore, our paper involves the impact of the additional costs caused by standing in a vehicle and failure to board. If users consider the level of crowding when choosing a public transport line or route, the final demand of each line will be the result of an equilibrium state (Batarce et al. 2016). Thence, this paper proposes a mathematical equilibrium problem to find an optimal number of seats under the requirement of minimizing the total travel costs. Further to the research of Khan and Amin (Khan and Amin 2018), we consider different travel demands and different seat capacities. Based on a full understanding of travel behaviors, a many-to-one transit system is developed to research the equilibrium characteristics during the morning peak hours.

In real life, passengers who cannot get on the bus are divided into two categories: being denied boarding due to capacity constraints and voluntarily giving up riding for comfort reasons. Since the in-vehicle crowding experienced by seated passengers has little to do with the number of onboarding passengers and the comfort level is high for seated passengers, bus seats have attracted more and more attention as an important
factor. In short, the number of seats inside a bus does affect the optimal design of a public transport system if the planner acknowledges that users dislike crowding (Tirachini et al. 2014). A reasonable operation plan is an effective way to solve the contradiction between bus service level and operating cost. So, we propose a mass transit equilibrium problem to find an optimal number of seats while minimizing the total cost. The total travel cost consists of three parts, including in-vehicle passenger crowding cost, schedule delay cost, and the additional cost due to denied boarding or cannot get a seat, and further, the equilibrium characteristics of commuters are analyzed during the morning peak hours.

3 Multiple origins and a single destination equilibrium model

In this section, the multiple origins and a single destination equilibrium model is presented and formulated. Then, an overview of the solution process is given.

3.1 Model formulation

In real life, the living areas of travelers are mostly scattered in various parts of the city, but their workplaces are relatively concentrated, and most travelers’ living areas are far from their workplaces. So, a many-to-one model can be more exactly describe the travel behavior of commuters.

We suppose there is a bus line connecting living area $H$ and workplace $W$. There is a total of $k$ buses with service frequency $h$ during the morning peak-period every day. As shown in Fig. 1, each bus starts at station $H_1$, and passes through stations $H_2, H_3, \ldots, H_s$ to reach working area $W$. During the morning peak-period every day, there are $N_1, N_2, \ldots, N_s$ commuters from their living areas $H_1, H_2, \ldots, H_s$ to their common working area, $W$, and passengers only get off at working area. Among these, subscripts $i$ and $j$ respectively represent station and vehicle, $s$ is the total number of stations, and $k$ is the total number of vehicles.
This paper makes the following assumptions:

(1) All passengers fully understand the bus schedule and ignore the waiting time at stations.

(2) All passengers are homogeneous and they have the same desirable arrival time and the same value of in-vehicle travel time, $\beta$, and bus fare.

(3) Due to capacity constraint, passengers at upstream stops can preferentially choose the vehicles comparing to those at downstream stops.

(4) Passengers board at the bus stop according to the first-in-first-out (FIFO) rules. After getting on the bus, passengers tend to choose available seats instead of standing. If a certain bus has reached the maximum capacity before arriving at the bus stop, the passengers will be rejected and continue to wait for the next bus.

(5) The running time between consecutive stops is constant (including the stopping time at each stop), and is denoted by $T_1,T_2,\ldots,T_s$. And there are no bunching effects.

Unlike a queue on the highway, the vehicles’ arrival and departure can be considered as a continuous process, that is, both supply and demand are continuously generated. In the public transportation system, vehicles arrive at the station under a certain headway, and passengers can only leave after the vehicle arrives, that is, the supply is intermittent. Therefore, in Assumption 1, travelers have a full understanding of the bus schedule, waiting behavior can be avoided, and the discontinuity of bus supply can be overcome. Furthermore, in addition to the cost in the bottleneck model, the crowding cost cannot be ignored in public transportation. This article further considers the crowding cost of commuters in the journey. On the one hand, a strict capacity constraint is set. When the number of in-vehicle passengers reaches the
capacity constraints, no passengers will continue to board the bus; on the other hand, the crowding cost of seated and standing passengers is considered separately, so that the results obtained by the model are more consistent with the reality.

Several of indicators are used to model the congestion and crowding effect, among which load factor and standing passengers per square meter are the two most commonly used notions. Load factor refers to the number of on-board passengers divided by the number of seats in the transit vehicle, and is thus expressed as a percentage. Wardman and Whelan (2011) found load factor ranges from 20% to more than 100%, but when the load factor is less than 60%, the standing passengers will not incur additional costs. In addition, MVA Consulting (2008) reported that standing passengers per square meter can more accurately measure the standing discomfort. The Transit Capacity and Quality of Service Manual (TRB 2013) defined the thresholds for the level of service (LOS) concerning in-transit crowding, shown in Table 1.

**Table 1** The classification of the level of service (LOS) in public transport

<table>
<thead>
<tr>
<th>LOS</th>
<th>Load factor (Passengers/seat)</th>
<th>Standing passenger area (m²/passerger)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.00~0.50</td>
<td>&gt;1.00</td>
<td>No passengers need sit next to another</td>
</tr>
<tr>
<td>B</td>
<td>0.51~0.75</td>
<td>0.76~1.00</td>
<td>Passengers can choose where to sit</td>
</tr>
<tr>
<td>C</td>
<td>0.76~1.00</td>
<td>0.51~0.75</td>
<td>All passengers can sit</td>
</tr>
<tr>
<td>D</td>
<td>1.00~1.25</td>
<td>0.36~0.50</td>
<td>Comfortable standee load for design</td>
</tr>
<tr>
<td>E</td>
<td>1.26~1.50</td>
<td>0.20~0.35</td>
<td>Maximum schedule load</td>
</tr>
<tr>
<td>F</td>
<td>&gt;1.50</td>
<td>&lt;0.20</td>
<td>Crush load</td>
</tr>
</tbody>
</table>

Table 1 shows the general rules on crowding perceived by transit users. According to regulations, it is approved for 0.125m² per standing passenger, and the standing density is up to 8 persons/m². When the vehicle is fully loaded, the load factor of conventional buses in our country is 2.5, which far exceeds Grade F. In-vehicle crowding reduces the attractiveness of public transportation. Some bus companies have to balance service frequency and load factor and adjust service frequency.

Many previous studies focused on reliability indicators such as bus punctuality and running time, but seat availability is an indicator that can intuitively describe the whole congestion. From passengers’ perspective, seat availability can be defined as the
number of remaining or missing seats after passengers have boarded at a particular stop (MVA Consultancy 2008). Seat reliability becomes large as available seats increase. Therefore, this paper considers multiple vehicles on a single line to study the control of seat capacity.

Let $SA_{ji}$ represent seat availability in vehicle $j$ at stop $H_i$, and let $c_{seat}$ represent the number of seats. The value of $SA_{ji}$ is related to the number of passengers and it can reflect the line’s comfort. When $SA_{ji}$ is small, it is difficult for passengers to find empty seats and passengers will feel discomfort. For $SA_{ji} < 0$, the number of passengers exceeds the number of seats, new boarding passengers are forced to stand. When the number of standing passengers exceeded a certain value, passengers would feel overcrowding. If $SA_{ji}$ is large, passengers will have more opportunities to choose empty seats. For $SA_{ji} \geq 0$, all in-vehicle passengers can find empty seats, and the line is exceedingly comfortable. According to Table 1, seat availability can be classified as high, general, low, and poor, which is shown in Table 2.

**Table 2** The classification of seat availability

<table>
<thead>
<tr>
<th>The classification of seat availability</th>
<th>$SA_{ji}$</th>
<th>Passenger feeling</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>$SA_{ji} \geq 0$</td>
<td>No passengers need sit next to another</td>
</tr>
<tr>
<td>General</td>
<td>$-0.25c_{seat} \leq SA_{ji} &lt; 0$</td>
<td>Comfortable standee load for design</td>
</tr>
<tr>
<td>Low</td>
<td>$-0.5c_{seat} \leq SA_{ji} &lt; -0.25c_{seat}$</td>
<td>Maximum schedule load</td>
</tr>
<tr>
<td>Poor</td>
<td>$SA_{ji} &lt; -0.5c_{seat}$</td>
<td>Crush load</td>
</tr>
</tbody>
</table>

Babaei et al. (2014) had shown that headway can significantly affect seat availability, and seat reliability becomes high with low service frequency. However, if a bus company blindly increases headway to improve seat availability, it will not only make its costs uncontrollable; but will also easily lead to bunching effects. Considering that changing the number of seats is an easily accepted solution, this paper builds a
A mathematical equilibrium model to study travel behavior. A reasonable operation plan can be made for the operator to solve the contradiction between bus service level and operating cost.

According to the definition, when a bus leaves a stop, seat availability in vehicle \( j \) at stop \( H_i \) is expressed as:

\[
SA_{ji} = c_{\text{seat}} - \sum_{o=1}^{i} q_{jo}^{\text{seat}}
\]  

(1)

where \( q_{ji} \) is the number of passengers from stop \( H_i \) taking vehicle \( j \), and \( q_{ji}^{\text{seat}} \) (\( q_{ji}^{\text{std}} \)) is the number of seated (standing) passengers from stop \( H_i \) taking vehicle \( j \).

Obviously, \( SA_{ji} \in \left[c_{\text{seat}} - \sum_{o=1}^{i} q_{jo}^{\text{seat}}, c_{\text{seat}}\right] \).

Assuming that vehicles \( 1, 2, \ldots, m, m+1, \ldots, k \) are dispatched from the depot, if and only if passengers take vehicle \( m+1 \), they can arrive at workplace \( W \) on time. Let \( \theta(j) \) denote the early/late arrival penalty for passengers taking vehicle \( j \) to the workplace \( W \). It is consistent with Tian et al. (2007), and we see the expression in Appendix A.1.

In addition to the early/late penalties, in-vehicle passengers have to bear the crowding cost. Assuming that passengers with seated and standing passengers experience different crowding level, the crowding cost of standing passengers is related to the number of standing passengers in a vehicle. The crowding penalty on stop \( H_i \) taking vehicle \( j \) can be written as:

\[
\varphi(q_{ji}) = \begin{cases} 
\varphi_0, & q_{ji} \leq SA_{ji} \\
\varphi_{\text{std}} \left(q_{ji}^{\text{std}}\right) + \varphi_0, & q_{ji} > SA_{ji}
\end{cases}
\]  

(2)

where \( q_{ji} \) represents the number of passengers from stop \( H_i \) taking vehicle \( j \), and \( q_{ji}^{\text{std}} \) is the number of standing passengers from stop \( H_i \) taking vehicle \( j \). \( \varphi_0 \) is a constant and represents the crowding cost of seated passengers.
In order to simplify the calculation, it is assumed that the crowding cost has a linear relationship with the number of standing passengers. So, we have \( \varphi^{\text{std}}(q_{ji}^{\text{std}}) = \eta_3 q_{ji}^{\text{std}} \), where \( \eta_3 \) denotes the crowding penalty coefficient of standing passengers. \( \varphi^{\text{std}}(q_{ji}^{\text{std}}) \) is a monotonically increasing function of the number of standing passengers in a vehicle, and \( \varphi^{\text{std}}(q_{ji}^{\text{std}}) \geq 0 \), \( \varphi^{\text{std}}(0) = 0 \). The crowding level experienced by seated passengers is always less than that of standing passengers, so \( 0 < \varphi_0 < \varphi^{\text{std}}(1) \). This definition is consistent with the actual background, passengers always prefer to choose available seats, and the crowding cost of standing passengers increases as the number of standing passengers in a vehicle increases. Table 3 shows the passenger travel information matrix of the line.

**Table 3** Passenger travel condition table for a many-to-one route

<table>
<thead>
<tr>
<th>( q_{ji} )</th>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>...</th>
<th>( H_i )</th>
<th>...</th>
<th>( H_j )</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_{i1}^{\text{seat}} )</td>
<td>( q_{i2}^{\text{seat}} )</td>
<td>...</td>
<td>( q_{i1}^{\text{seat}} )</td>
<td>...</td>
<td>( q_{i1}^{\text{seat}} )</td>
<td>( \sum_j q_{i1}^{\text{seat}} )</td>
<td></td>
</tr>
<tr>
<td>Vehicle 1</td>
<td>( q_{i1}^{\text{std}} )</td>
<td>( q_{i2}^{\text{std}} )</td>
<td>...</td>
<td>( q_{i1}^{\text{std}} )</td>
<td>...</td>
<td>( q_{i1}^{\text{std}} )</td>
<td>( \sum_j q_{i1}^{\text{std}} )</td>
</tr>
<tr>
<td>( q_{21}^{\text{seat}} )</td>
<td>( q_{22}^{\text{seat}} )</td>
<td>...</td>
<td>( q_{21}^{\text{seat}} )</td>
<td>...</td>
<td>( q_{21}^{\text{seat}} )</td>
<td>( \sum_j q_{21}^{\text{seat}} )</td>
<td></td>
</tr>
<tr>
<td>Vehicle 2</td>
<td>( q_{21}^{\text{std}} )</td>
<td>( q_{22}^{\text{std}} )</td>
<td>...</td>
<td>( q_{21}^{\text{std}} )</td>
<td>...</td>
<td>( q_{21}^{\text{std}} )</td>
<td>( \sum_j q_{21}^{\text{std}} )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>( q_{ji}^{\text{seat}} )</td>
<td>( q_{ji}^{\text{seat}} )</td>
<td>...</td>
<td>( q_{ji}^{\text{seat}} )</td>
<td>...</td>
<td>( q_{ji}^{\text{seat}} )</td>
<td>( \sum_j q_{ji}^{\text{seat}} )</td>
<td></td>
</tr>
<tr>
<td>Vehicle ( j )</td>
<td>( q_{ji}^{\text{std}} )</td>
<td>( q_{ji}^{\text{std}} )</td>
<td>...</td>
<td>( q_{ji}^{\text{std}} )</td>
<td>...</td>
<td>( q_{ji}^{\text{std}} )</td>
<td>( \sum_j q_{ji}^{\text{std}} )</td>
</tr>
</tbody>
</table>
Let $\beta$ denote the value of in-vehicle travel time. Combining with Assumption 2 and Assumption 5, it can be known that bus fare $p$ and the in-vehicle time cost departing from the same station have no effect on the passenger equilibrium, so we assume $p + \beta T_i = 0$. The travel cost $C_{ji}$ from stop $H_i$ taking vehicle $j$ is expressed as follows:

$$
C_{ji} = \begin{cases}
\phi_0 \sum_{a=i}^{s} T_i + \theta(j) + e_{ji}^{\text{seat}}, & q_{ji} \leq SA_{ji} \\
\sum_{a=i}^{s} \phi^{\text{std}} \left( \sum_{a=0}^{d} q^{\text{std}}_{jo} \right) T_u + \theta(j) + e_{ji}^{\text{std}}, & q_{ji} > SA_{ji}
\end{cases}
$$

(3)

where $\theta(j)$ is the early/late penalty in Eq. (A1). $e_{ji}^{\text{seat}}$ and $e_{ji}^{\text{std}}$ respectively represent the additional cost of seated passengers and standing passengers. The former used to describe the risk of passengers arriving at the station early in order to obtain available seats, and the latter represents denied boarding because the vehicle reach the maximum capacity. When the bus arrives at the stop $H_i$, there are no available seats or standing space, passengers need to pay additional costs to obtain this scarce resource.

In order to better illustrate the cost relationship between two successive stations, we use 3 stations and 6 vehicles as an example to derive in Appendix A.2 and the final general result is

$$
\begin{align*}
C_{ji(i-1)}^{\text{seat}} - C_{ji}^{\text{seat}} &= T_{i-1} \phi_0 + e_{ji(i-1)}^{\text{seat}} - e_{ji}^{\text{seat}}, & q_{ji} \leq SA_{ji} \\
C_{ji(i-1)}^{\text{std}} - C_{ji}^{\text{std}} &= \phi^{\text{std}} \left( \sum_{a=i}^{d} q_{jo}^{\text{std}} \right) T_{i-1} + e_{ji(i-1)}^{\text{std}} - e_{ji}^{\text{std}}, & q_{ji} > SA_{ji}
\end{align*}
$$

(4)
Suppose that all passengers attempt to minimize their individual travel cost. In an equilibrium state, all seated (standing) passengers boarding at the same station have the same travel cost, and no passengers will unilaterally change the travel strategy to make the travel cost less than the equilibrium cost. The relationship corresponding to the equilibrium state is:

\[
\begin{align*}
1. \, e_{ji}^{\text{seat}} &= 0, \, e_{ji}^{\text{std}} \geq 0 \text{ if } q_{ji}^{\text{seat}} = 0, \, q_{ji}^{\text{std}} = 0, \, \text{ non-existent} \\
2. \, e_{ji}^{\text{seat}} &\geq 0, \, e_{ji}^{\text{std}} = 0 \text{ if } q_{ji}^{\text{seat}} = 0, \, q_{ji}^{\text{std}} = 0, \, \text{ non-existent} \\
3. \, e_{ji}^{\text{seat}} &\geq 0, \, e_{ji}^{\text{std}} \geq 0 \text{ if } q_{ji}^{\text{seat}} = 0, \, q_{ji}^{\text{std}} = 0, \, \text{ non-existent} \\
4. \, e_{ji}^{\text{seat}} &\geq 0, \, e_{ji}^{\text{std}} = 0 \text{ if } q_{ji}^{\text{seat}} = 0, \, q_{ji}^{\text{std}} > 0, \, \text{ non-existent} \\
5. \, e_{ji}^{\text{seat}} = 0, \, e_{ji}^{\text{std}} \geq 0 \text{ if } q_{ji}^{\text{seat}} = 0, \, q_{ji}^{\text{std}} > 0, \, \text{ non-existent} \\
6. \, e_{ji}^{\text{seat}} = 0, \, e_{ji}^{\text{std}} \geq 0 \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} = 0, \, \text{ non-existent} \\
7. \, e_{ji}^{\text{seat}} = 0, \, e_{ji}^{\text{std}} \geq 0 \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} = 0, \, \text{ non-existent} \\
8. \, e_{ji}^{\text{seat}} = 0, \, e_{ji}^{\text{std}} = 0 \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} > 0, \, \text{ non-existent} \\
9. \, e_{ji}^{\text{seat}} = 0, \, e_{ji}^{\text{std}} \geq 0 \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} > 0, \, \text{ non-existent} \\
10. \, e_{ji}^{\text{seat}} \geq e_{ji}^{\text{seat}}, \, e_{ji}^{\text{std}} \geq e_{ji}^{\text{std}} \text{ if } q_{ji}^{\text{seat}} = 0, \, q_{ji}^{\text{std}} = 0 \\
11. \, e_{ji}^{\text{seat}} \geq e_{ji}^{\text{seat}}, \, e_{ji}^{\text{std}} \geq e_{ji}^{\text{std}} \text{ if } q_{ji}^{\text{seat}} = 0, \, q_{ji}^{\text{std}} = 0, \, \text{ and } \sum_{o=1}^{i-1} q_{io}^{\text{seat}} = \sum_{o=1}^{i-1} q_{io}^{\text{std}} \leq C_{ji}^{\text{std}} \\
12. \, e_{ji}^{\text{seat}} \geq e_{ji}^{\text{seat}}, \, e_{ji}^{\text{std}} \geq e_{ji}^{\text{std}} \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} = 0, \, \text{ and } \sum_{o=1}^{i-1} q_{io}^{\text{seat}} = \sum_{o=1}^{i-1} q_{io}^{\text{std}} \leq C_{ji}^{\text{std}} \\
13. \, e_{ji}^{\text{seat}} = e_{ji}^{\text{seat}}, \, e_{ji}^{\text{std}} \geq e_{ji}^{\text{std}} \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} = 0, \, \text{ and } \sum_{o=1}^{i-1} q_{io}^{\text{seat}} = \sum_{o=1}^{i-1} q_{io}^{\text{std}} \leq C_{ji}^{\text{std}} \\
14. \, e_{ji}^{\text{seat}} = e_{ji}^{\text{seat}}, \, e_{ji}^{\text{std}} \geq e_{ji}^{\text{std}} \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} = 0, \, \text{ and } \sum_{o=1}^{i-1} q_{io}^{\text{seat}} = \sum_{o=1}^{i-1} q_{io}^{\text{std}} \leq C_{ji}^{\text{std}} \\
15. \, e_{ji}^{\text{seat}} = e_{ji}^{\text{seat}}, \, e_{ji}^{\text{std}} \geq e_{ji}^{\text{std}} \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} = 0, \, \text{ and } \sum_{o=1}^{i-1} q_{io}^{\text{seat}} = \sum_{o=1}^{i-1} q_{io}^{\text{std}} \leq C_{ji}^{\text{std}} \\
16. \, e_{ji}^{\text{seat}} = e_{ji}^{\text{seat}}, \, e_{ji}^{\text{std}} \geq e_{ji}^{\text{std}} \text{ if } q_{ji}^{\text{seat}} > 0, \, q_{ji}^{\text{std}} = 0, \, \text{ and } \sum_{o=1}^{i-1} q_{io}^{\text{seat}} = \sum_{o=1}^{i-1} q_{io}^{\text{std}} \leq C_{ji}^{\text{std}}
\end{align*}
\]
the smallest and equal to the (seated or standing) equilibrium cost. If there are no passengers taking vehicle \( j \), the total travel cost is greater than or equal to the equilibrium cost.

In order to meet the equilibrium cost under each seat capacity, we establish a bi-level programming model, that is, the upper-level problem is presented to minimize the total travel cost; the lower-level problem presents the equilibrium analysis under each seat capacity, and each passenger chooses their own travel vehicles. The feedback of the upper-level and lower-level problems forms the optimal travel strategy.

Since the watershed site of each vehicle is different, \( \sum_{j=1}^{v} q_{ji}^{\text{seat}} = c_{\text{seat}} \) is reached at the watershed site \( H_{\nu} \) of vehicle \( j \). Let \( c_{\text{seat}} \) denote the optimization variable of the upper-level objective function, so the upper-level objective function can be obtained,

\[
\text{Min} TC = \sum_{j=1}^{k} \left( c_{\text{seat}} \sum_{i=1}^{v} C_{i}^{\text{seat}} + \left( \sum_{i=1}^{s} N_{i} - c_{\text{seat}} \right) \left( \sum_{j=i+1}^{s} C_{i}^{\text{std}} \right) \right)
\]

(6)

The equilibrium passenger travel distribution in the lower level \( q = \{ q_{ji} | j \in \{1, \ldots, k\}, i = 1, \ldots, s \} \) can be given by solving the following minimization problem:

\[
\text{Min} C(q) = \sum_{i=1}^{s} \sum_{j=1}^{k} q_{ji}^{\text{seat}} \left( \sum_{b=1}^{b_i} T_{bj} \right) + \sum_{j=1}^{k} \sum_{i=1}^{s} q_{ji}^{\text{std}} \left( \sum_{b=1}^{b_i} T_{bj} \right) + \sum_{j=1}^{k} \left( q_{ji}^{\text{seat}} + q_{ji}^{\text{std}} \right) \theta(j)
\]

(7)

subject to

\[
\sum_{j=1}^{k} q_{ji}^{\text{seat}} = N_{i}^{\text{seat}}, \quad j \in \{1, \ldots, k\}, i = 2, \ldots, s,
\]

(8)

\[
\sum_{j=1}^{k} q_{ji}^{\text{std}} = N_{i}^{\text{std}}, \quad j \in \{1, \ldots, k\}, i = 2, \ldots, s,
\]

(9)

\[
\sum_{i=1}^{s} q_{ji}^{\text{seat}} \leq c_{\text{seat}} + \gamma M, \quad j \in \{1, \ldots, k\},
\]

(10)

\[
\sum_{i=1}^{s} q_{ji}^{\text{std}} \leq c_{\text{std}} + (1 - \gamma) M, \quad j \in \{1, \ldots, k\},
\]

(11)
\[ q_{ji}^{\text{seat}}, q_{ji}^{\text{std}} \geq 0, \ j \in \{1, \ldots, k\}, i = 1, 2, \ldots, s, \]  
\hspace{1cm} (12)

where \( y = \begin{cases} 0, & q_{ji}^{\text{seat}} \geq 0 \\ 1, & q_{ji}^{\text{std}} \geq 0 \end{cases} \), \( M \) is an infinite positive number, and

\[ f(x) = \int_0^1 [\varphi_j^{\text{std}}(w) + \varphi_0] \, dw. \]  

The three terms in the objective function are expressed as the sum of the crowding cost of all seated passengers and standing passengers and the early/late penalties suffered by all passengers. Constraint (8) is the conservation condition of the number of seated passengers at station \( H_i \), constraint (9) is the conservation condition of the number of standing passengers at station \( H_i \), and constraints (10)-(11) respectively limit the seat capacity and standing space at each station. The number of seated and standing passengers do not exceed the established seating capacity and standing capacity, which can ensure seat reliability. Constraint (12) is a non-negative constraint on the number of passengers. Since Eq. (7) is composed of \( s \) minimization problem, the feasible region is a convex set, and the equilibrium solution that satisfies the user equilibrium state is unique.

The first-order optimality conditions for the problem (7)-(12) can be expressed as:

\[ q_{ji}^{\text{seat}} \left[ \varphi_0 T_b + \theta(j) + e_{ji}^{\text{seat}} - \lambda_j^{\text{seat}} \right] + yM = 0 \]  
\hspace{1cm} (13)

\[ \varphi_0 T_b + \theta(j) + e_{ji}^{\text{seat}} - \lambda_j^{\text{seat}} + yM \geq 0 \]  
\hspace{1cm} (14)

\[ q_{ji}^{\text{std}} \left[ \varphi_{i}^{\text{std}} \left( \sum_{j=1}^{k} q_{ji}^{\text{std}} \right) T_i + \theta(j) + e_{ji}^{\text{std}} - \lambda_i^{\text{std}} \right] + (1-y)M = 0 \]  
\hspace{1cm} (15)

\[ \varphi_{i}^{\text{std}} \left( \sum_{j=1}^{k} q_{ji}^{\text{std}} \right) T_i + \theta(j) + e_{ji}^{\text{std}} - \lambda_i^{\text{std}} + (1-y)M \geq 0 \]  
\hspace{1cm} (16)

\[ \sum_{j=1}^{k} q_{ji}^{\text{seat}} = N_i^{\text{seat}}, \quad \sum_{j=1}^{k} q_{ji}^{\text{std}} = N_i^{\text{std}} \]  
\hspace{1cm} (17)

\[ e_{ji}^{\text{seat}} \left( c_{\text{seat}} - \sum_{j=1}^{k} q_{ji}^{\text{seat}} \right) + yM = 0 \]  
\hspace{1cm} (18)

\[ e_{ji}^{\text{std}} \left( c_{\text{std}} - \sum_{j=1}^{k} q_{ji}^{\text{std}} \right) + (1-y)M = 0 \]  
\hspace{1cm} (19)
\[ q^\text{seat}_{ji}, q^\text{std}_{ji} \geq 0 \]  \hspace{1cm} (20)

\[ e^\text{seat}_{ji} + yM \geq 0 , \quad e^\text{std}_{ji} + (1 - y)M \geq 0 \]  \hspace{1cm} (21)

Similarly, where  
\[ y = \begin{cases} 0, & q^\text{seat}_{ji} \geq 0 \\ 1, & q^\text{std}_{ji} \geq 0 \end{cases} \]

\[ M \]  is an infinite positive number, and \( \lambda^\text{seat}_i \), \( \lambda^\text{std}_i \), \( e^\text{seat}_{ji} \) and \( e^\text{std}_{ji} \) are the multipliers associated with constraints (8)-(11). In the equilibrium state, the equilibrium cost of all seated passengers is constant \( \lambda^\text{seat}_i \), and the equilibrium cost of all standing passengers is constant \( \lambda^\text{std}_i \). Eqs. (13)-(16) indicate that if there are seated (standing) passengers taking vehicle \( j \) at stop \( H_i \), the travel cost is equal to a constant \( \lambda^\text{seat}_i \) (\( \lambda^\text{std}_i \)). If no passengers take vehicle \( j \), the passenger travel cost is not less than constant \( \lambda^\text{seat}_i \) (\( \lambda^\text{std}_i \)). Eqs. (17)-(19) show that at stop \( H_i \), when vehicle \( j \) has available seats, there has \( e^\text{seat}_{ji} = 0 \); when no available seats, passengers need to pay additional costs if they want to obtain available seats, that is \( e^\text{seat}_{ji} > 0 \). When vehicle \( j \) has surplus standing capacity, there is \( e^\text{std}_{ji} = 0 \); when vehicle \( j \) is fully loaded, the passengers will also need to pay extra cost if they want to board the bus, that is \( e^\text{std}_{ji} > 0 \). And Eq. (21) can be obtained. Therefore, the first-order optimality conditions for the problem (7)-(12) is equivalent to the dynamic user equilibrium condition (5).

### 3.2 Solution

An overview of the solution procedure is outlined as follows:

**Step 1** All commuters are assigned to the \( k \) buses according to their departure time. And there are two cases of standing or having a seat in a vehicle, namely \( q^\text{seat}_{ji} \) and \( q^\text{std}_{ji} \). We can get the passenger distribution table shown in Table 3.

**Step 2** Find the watershed site \( H_{ji} \) of vehicle \( j \), the passenger load before the watershed site is \( q^\text{seat}_{ji} \); and the load after the watershed site is \( q^\text{std}_{ji} \). Then the load
before the watershed site $H_{\nu}$ is $c_{\text{seat}}$, that is $\sum_{i=1}^{s} q_{ji}^{\text{seat}}$; the load after the watershed site $H_{\nu}$ is $\sum_{i=1}^{s} N_{i} - c_{\text{seat}}$, that is $\sum_{i=s+1}^{s} q_{ji}^{\text{std}}$.

**Step 3** Get the travel distribution of all passengers under different seat capacity.

Get the passenger travel distribution for each vehicle at each station. For example, when seat capacity is 10, what is the travel distribution?

**Step 4** Calculate the total travel cost under different seat capacity.

We can get the crowding penalty by Eq. (2), $\phi(q_{j})$. Based on the assumption of early/late penalty, only vehicle $m+1$ can arrive at workplace $W$ on time, than $\theta(j)$ can be obtained. Additional costs can be given based on equilibrium state (5). Finally, we can determine the total travel cost, $C_{ji}$, by Eq. (3).

The seat capacity corresponding to the smallest total travel cost is the optimal number of seats that should be set under this travel distribution.

### 4 Equilibrium theorem

In order to more comprehensively analyze and understand the characteristics of passenger travel, this section is to further study the theorem of the equilibrium state.

Let $S_{i} = \{ j \mid q_{ji}^{\text{std}} > 0, \forall j \in \{1, \ldots, k\} \}$ denote the set of vehicles where passengers need to stand boarding the bus at stop $H_{i}$, and $P_{i} = \{ j \mid q_{ji}^{\text{seat}} > 0, \forall j \in \{1, \ldots, k\} \}$ denote the set of vehicles where passengers can get available seats boarding the bus at stop $H_{i}$. And, let $M = \left\{ j \mid \sum_{o=1}^{t} q_{jo}^{\text{std}} = c_{\text{std}}, \forall j \in \{1, \ldots, k\} \right\}$ denote the set of fully loaded vehicles, $N = \left\{ j \mid \sum_{o=1}^{t} q_{jo}^{\text{std}} < c_{\text{std}}, \forall j \in \{1, \ldots, k\} \right\}$ denote the set of no fully loaded vehicles.

**Theorem 1.** At equilibrium, at station $H_{i}$, when $j \in \{1, \ldots, m\}$ and $q_{ji}^{\text{std}} > 0$, if $c_{ji}^{\text{std}} \neq 0$, then there must be $c_{(j+1)i}^{\text{std}} \neq 0$; when $j \in \{m+1, \ldots, k\}$ and $q_{ji}^{\text{std}} > 0$, if $c_{ji}^{\text{std}} \neq 0$, then there must be $c_{(j+1)i}^{\text{std}} \neq 0$. 


\( e^{std}_{(j+1)i} \neq 0 \), then there must be \( e^{std}_{ji} \neq 0 \).

At the same time, when \( j \in \{m+2, \ldots, k\} \) and \( q^{std}_{ji} > 0 \), if \( e^{std}_{ji} \neq 0 \), then there must be \( e^{std}_{(j+1)i} \neq 0 \).

**Proof.** We use contradiction to proof the theorem. Suppose that there exists \( j \) when \( j \in \{1, \ldots, m\} \) and \( q^{std}_{ji} > 0 \), so that \( e^{std}_{(j+1)i} = 0 \). According to Eq. (3), we know

\[
C^{std}_{ji} = \sum_{a=1}^{d} \varphi^{std}_{ai} \left( \sum_{a=1}^{d} q^{std}_{j\mu} \right) T_a + \eta h(m+1-j) + e^{std}_{ji}
\]

and

\[
C^{std}_{(j+1)i} = \sum_{a=1}^{d} \varphi^{std}_{ai} \left( \sum_{a=1}^{d} q^{std}_{(j+1)i\mu} \right) T_a + \eta h(m-j),
\]

then there is \( C^{std}_{(j+1)i} - C^{std}_{ji} \neq 0 \). According to the equilibrium states, it is always true that \( C^{std}_{(j+1)i} = C^{std}_{ji} = C^{std}_i \). This contradicts with \( C^{std}_{j+1} - C^j \neq 0 \) got above. So, the hypothesis is not true and the original proposition is true.

Similarly, we know if \( j \in \{m+1, \ldots, k\} \) and \( q^{std}_{ji} > 0 \), there does not exist \( j \) such that when \( e^{std}_{(j+1)i} \neq 0 \), then \( e^{std}_{ji} = 0 \); if \( j \in \{m+2, \ldots, k\} \) and \( q^{std}_{ji} > 0 \), there does not exist \( j \) such that when \( e^{std}_{(j+1)i} \neq 0 \), then \( e^{std}_{ji} = 0 \).

Theorem 1 states that at equilibrium, for station \( H_i \), when \( j \in \{1, \ldots, m\} \), if vehicle \( j \) passengers boarding is fully loaded, then vehicle \( j+1 \) must be fully loaded. When \( j \in \{m+1, \ldots, k\} \), if vehicle \( j+1 \) passengers boarding is fully loaded, then vehicle \( j \) must be fully loaded. When \( j \in \{m+2, \ldots, k\} \), if vehicle \( j \) passengers boarding is fully loaded, then vehicle \( j-1 \) must be fully loaded.

**Theorem 2.** At equilibrium, for station \( H_i \), when \( j \in \{1, \ldots, m\} \), if \( e^{cat}_{(j+1)i} = 0 \), then there must be \( q^{cat}_{ji} = 0 \). When \( j \in \{m+1, \ldots, k\} \), if \( q^{cat}_{ji} = 0 \), then there must be \( q^{cat}_{(j+1)i} = 0 \).
Proof. We use contradiction to prove the theorem. Suppose that there exists \( j \) when \( j \in \{1, \ldots, m\} \), if \( e_{(j+1)i}^{\text{seat}} = 0 \), then \( q_{ji}^{\text{seat}} \neq 0 \).

According to the equilibrium condition (5), we know \( C_{ji}^{\text{seat}} = C_j^{\text{seat}} \). From Eq. (3), we have \( C_{(j+1)i}^{\text{seat}} = \phi_j T_i + \eta_i h (m-j) \) and \( C_{ji}^{\text{seat}} = \phi_j T_i + \eta_i h (m+1-j) + e_{ji}^{\text{seat}} \). So, there is \( C_{(j+1)i}^{\text{seat}} - C_{ji}^{\text{seat}} = -\eta_i h - e_{ji}^{\text{seat}} \neq 0 \). But it is \( C_{(j+1)i}^{\text{seat}} = C_{ji}^{\text{seat}} \) according to the equivalence condition. The above two types are contradictory. So, when \( j \in \{1, \ldots, m\} \), if \( e_{(j+1)i}^{\text{seat}} = 0 \), then there must be \( q_{ji}^{\text{seat}} = 0 \).

Similarly, when \( j \in \{m+1, \ldots, k\} \), there does not exist \( j \) such that if \( q_{ji}^{\text{seat}} = 0 \), then \( q_{(j+1)i}^{\text{seat}} \neq 0 \).

Theorem 2 states that at equilibrium, for station \( H \), when \( j \in \{1, \ldots, m\} \), if passengers were able to get available seats when taking vehicle \( j+1 \), no passengers would take vehicle \( j \); when \( j+1 \in \{m+1, \ldots, k\} \), if no passengers took vehicle \( j \), then there are no passengers taking vehicle \( j+1 \).

Theorem 3. For any \( m \in M^i \) and any \( n \in N^i \), then \( \theta(m) < \theta(n) \).

Proof. According to the definition of sets \( M \) and \( N \), we get \( e_{mi}^{\text{std}} \geq 0 \) and \( e_{mi}^{\text{std}} = 0 \). According to Eq. (4), we can obtain \( C_i^{\text{std}} = C_{mi}^{\text{std}} = \sum_{a=1}^{\overline{a}} \phi^a \left( \sum_{m=1}^{\overline{m}} q_{mo}^{\text{std}} \right) T_a + \theta(m) + e_{mi}^{\text{std}} \) and

\[
C_i^{\text{std}} = C_{mi}^{\text{std}} = \sum_{a=1}^{\overline{a}} \phi^a \left( \sum_{m=1}^{\overline{m}} q_{mo}^{\text{std}} \right) T_a + \theta(n) + e_{mi}^{\text{std}}.
\]

From the above two formulas, there has

\[
\sum_{a=1}^{\overline{a}} \phi^a \left( \sum_{m=1}^{\overline{m}} q_{mo}^{\text{std}} \right) T_a + \theta(m) \leq \sum_{a=1}^{\overline{a}} \phi^a \left( \sum_{m=1}^{\overline{m}} q_{mo}^{\text{std}} \right) T_a + \theta(n).
\]

Combining

\[
\sum_{a=1}^{\overline{a}} \phi^a \left( \sum_{m=1}^{\overline{m}} q_{mo}^{\text{std}} \right) T_a = \sum_{a=1}^{\overline{a}} \phi^a \left( c_{std} \right) T_a
\]
and
\[ \sum_{a=i}^{i} \phi_{std} \left( \sum_{\alpha=1}^{\mu} q_{\alpha \mu}^{std} \right) T_{a} - \sum_{a=i}^{i} \phi_{std} (c_{std}) T_{a}, \quad \theta(m) < \theta(n) \] can be obtained, the proposition is proved.

Theorem 3 shows that if the bus leaving station \( H_{i} \) is fully loaded, the full load will be concentrated on the lower early/late penalty vehicles, and passengers will also pay extra cost for being able to take the full-loaded vehicles with small early/late penalty. If station \( H_{i} \) is the terminal station, the early/late penalty for arriving at the terminal station with a full load is smaller.

**Theorem 4.** At equilibrium, for any \( i \in \{1, 2, \ldots, s-1\} \), if \( S_{i} \neq \emptyset \), then \( P_{i+1} = \emptyset \).

It is consistent with Tian et al. (2007), and we see the detailed proof in Appendix A.3. Theorem 4 states that at equilibrium, if the passengers boarding vehicle \( j \) from station \( H_{i} \) need to stand up, then there has no seat for passengers from station \( H_{i+1} \).

**Lemma 1.** There exists \( \nu \in \{1, 2, \ldots, s\} \), \( \forall \omega \neq \nu \), \( S_{i} = \emptyset \); \( \forall \tau > \nu \), \( P_{i} = \emptyset \).

Lemma 1 shows that for any vehicles, there exists station \( H_{\nu} \) in station \( H_{1}, H_{2}, \ldots, H_{s} \), and passengers at the upstream of station \( H_{\nu} \) can get available seats, and passengers at the downstream of station \( H_{\nu} \) will be forced to stand. \( \nu \) denotes the watershed site.

**Theorem 5.** At equilibrium, for station \( H_{i} \) \( (i > \nu) \) and any \( j \in S_{i} \), if \( q_{j \nu}^{std} > 0 \), then \( \sum_{\alpha=1}^{i-1} q_{j \alpha}^{std} > 0 \).

It is similar to Tian et al. (2007), but our main consideration is the travel of standing passengers. And we see the detailed proof in Appendix A.3. Theorem 5 shows that for any vehicle, if there are standing passengers on the bus at any stop that is not the originating station, then someone must have taken the vehicle and stood before the station. (Different vehicles have different watershed sites.)

**5 Numerical study**
In order to ensure seat reliability, this article requires that seat availability should reach at least general reliability, that is \[ SA_j \geq -0.5c_{\text{seat}} \cdot \] Taking Beijing’s most common Northern Bus BFC6809GBEV as an example and excluding the driver’s seat area, the usable area is 22.4m², the maximum standing capacity is 64 (seat capacity is 0) and the maximum number of seats is 42 (the standing capacity is 1). According to the requirement of seat availability, the average standing area in a vehicle is at least 0.35m²/person. So let \( c = \pi c_{\text{seat}} + c_{\text{std}} \), where \( c \) represents the maximum standing capacity (\( c = 64 \)), \( c_{\text{std}} \) represents the number of standing passengers who can be accommodated after the bus has adjusted the seat capacity, and \( \pi \) represents the ratio of the average standing passenger area to the seating area and \( \pi = 1.5 \). If the operator intends to purchase the North Bus BFC6809GBEV model, it can request the North Bus Company to design the seat capacity. Through calculation, we can get the different seat capacity in Table 4. The parameter values needed for model are shown in Table 5.

**Table 4** The distribution with different seat capacity

<table>
<thead>
<tr>
<th>Seat capacity</th>
<th>Standing capacity</th>
<th>Total</th>
<th>Seat capacity</th>
<th>Standing capacity</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{\text{seat}} )</td>
<td>( c_{\text{std}} )</td>
<td>( c_{\text{seat}} + c_{\text{std}} )</td>
<td>( c_{\text{seat}} )</td>
<td>( c_{\text{std}} )</td>
<td>( c_{\text{seat}} + c_{\text{std}} )</td>
</tr>
<tr>
<td>0</td>
<td>64</td>
<td>64</td>
<td>22</td>
<td>31</td>
<td>53</td>
</tr>
<tr>
<td>2</td>
<td>61</td>
<td>63</td>
<td>24</td>
<td>28</td>
<td>52</td>
</tr>
<tr>
<td>4</td>
<td>58</td>
<td>62</td>
<td>26</td>
<td>25</td>
<td>51</td>
</tr>
<tr>
<td>6</td>
<td>55</td>
<td>61</td>
<td>28</td>
<td>22</td>
<td>50</td>
</tr>
<tr>
<td>8</td>
<td>52</td>
<td>60</td>
<td>30</td>
<td>19</td>
<td>49</td>
</tr>
<tr>
<td>10</td>
<td>49</td>
<td>59</td>
<td>32</td>
<td>16</td>
<td>48</td>
</tr>
<tr>
<td>12</td>
<td>46</td>
<td>58</td>
<td>34</td>
<td>13</td>
<td>47</td>
</tr>
<tr>
<td>14</td>
<td>43</td>
<td>57</td>
<td>36</td>
<td>10</td>
<td>46</td>
</tr>
<tr>
<td>16</td>
<td>40</td>
<td>56</td>
<td>38</td>
<td>7</td>
<td>45</td>
</tr>
<tr>
<td>18</td>
<td>37</td>
<td>55</td>
<td>40</td>
<td>4</td>
<td>44</td>
</tr>
</tbody>
</table>
Table 5 Input values of calculation example parameters

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_1 )</td>
<td>5Y/h</td>
<td>( \varphi_0 )</td>
<td>0.5Y</td>
</tr>
<tr>
<td>( \eta_2 )</td>
<td>12Y/h</td>
<td>( h )</td>
<td>0.167h</td>
</tr>
<tr>
<td>( \eta_3 )</td>
<td>0.025Y</td>
<td>( k )</td>
<td>6</td>
</tr>
<tr>
<td>( s )</td>
<td>3</td>
<td>( T_1=T_2=T_3 )</td>
<td>0.25h</td>
</tr>
<tr>
<td>( m )</td>
<td>3</td>
<td>( N_1=N_2=N_3 )</td>
<td>80</td>
</tr>
</tbody>
</table>

Table 5 assumes a mass transit line with 3 boarding stops (\( s = 3 \)) and 6 vehicles (\( k = 6 \)), which departs at 10-minute intervals (\( h = 0.167h \)) during peak hours. \( m = 3 \) means that vehicle 4 will arrive on time. Let us take the bus line of the above 3 stops as an example, and we use the expression in Section 3 to get.

Table 6 Passenger travel distribution table of 3 boarding stops

<table>
<thead>
<tr>
<th>Vehicle 1</th>
<th>Station 1</th>
<th>Station 2</th>
<th>Station 3</th>
<th>Vehicle 4</th>
<th>Station 1</th>
<th>Station 2</th>
<th>Station 3</th>
<th>Vehicle 5</th>
<th>Station 1</th>
<th>Station 2</th>
<th>Station 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( q_{11}^{\text{seat}} )</td>
<td>( q_{12}^{\text{seat}} )</td>
<td>( q_{13}^{\text{seat}} )</td>
<td>4</td>
<td>( q_{41}^{\text{seat}} )</td>
<td>( q_{42}^{\text{seat}} )</td>
<td>( q_{43}^{\text{seat}} )</td>
<td>5</td>
<td>( q_{51}^{\text{seat}} )</td>
<td>( q_{52}^{\text{seat}} )</td>
<td>( q_{53}^{\text{seat}} )</td>
</tr>
<tr>
<td></td>
<td>( q_{11}^{\text{std}} )</td>
<td>( q_{12}^{\text{std}} )</td>
<td>( q_{13}^{\text{std}} )</td>
<td></td>
<td>( q_{41}^{\text{std}} )</td>
<td>( q_{42}^{\text{std}} )</td>
<td>( q_{43}^{\text{std}} )</td>
<td></td>
<td>( q_{51}^{\text{std}} )</td>
<td>( q_{52}^{\text{std}} )</td>
<td>( q_{53}^{\text{std}} )</td>
</tr>
</tbody>
</table>

| Vehicle 2 | \( q_{21}^{\text{seat}} \) | \( q_{22}^{\text{seat}} \) | \( q_{23}^{\text{seat}} \) | 4         | \( q_{41}^{\text{seat}} \) | \( q_{42}^{\text{seat}} \) | \( q_{43}^{\text{seat}} \) | 5         | \( q_{51}^{\text{seat}} \) | \( q_{52}^{\text{seat}} \) | \( q_{53}^{\text{seat}} \) |
|           | \( q_{21}^{\text{std}} \)  | \( q_{22}^{\text{std}} \)  | \( q_{23}^{\text{std}} \)  |           | \( q_{41}^{\text{std}} \) | \( q_{42}^{\text{std}} \) | \( q_{43}^{\text{std}} \) |           | \( q_{51}^{\text{std}} \) | \( q_{52}^{\text{std}} \) | \( q_{53}^{\text{std}} \) |

| Vehicle 3 | \( q_{31}^{\text{seat}} \) | \( q_{32}^{\text{seat}} \) | \( q_{33}^{\text{seat}} \) | 4         | \( q_{41}^{\text{seat}} \) | \( q_{42}^{\text{seat}} \) | \( q_{43}^{\text{seat}} \) | 5         | \( q_{51}^{\text{seat}} \) | \( q_{52}^{\text{seat}} \) | \( q_{53}^{\text{seat}} \) |
|           | \( q_{31}^{\text{std}} \)  | \( q_{32}^{\text{std}} \)  | \( q_{33}^{\text{std}} \)  |           | \( q_{41}^{\text{std}} \) | \( q_{42}^{\text{std}} \) | \( q_{43}^{\text{std}} \) |           | \( q_{51}^{\text{std}} \) | \( q_{52}^{\text{std}} \) | \( q_{53}^{\text{std}} \) |

| Vehicle   | \( q_{61}^{\text{seat}} \) | \( q_{62}^{\text{seat}} \) | \( q_{63}^{\text{seat}} \) |              | \( q_{61}^{\text{std}} \) | \( q_{62}^{\text{std}} \) | \( q_{63}^{\text{std}} \) |              | \( q_{61}^{\text{std}} \) | \( q_{62}^{\text{std}} \) | \( q_{63}^{\text{std}} \) |
Then, we get the crowding cost for all seated passengers, $A$,

$$A = \sum_{i=1}^{k} \sum_{j=1}^{4} \left[ \phi_i q_{ji} \left( \sum_{b=1}^{s} T_b \right) \right]$$

$$= \phi_0 q_{11} (T_1 + T_2 + T_3) + \phi_0 q_{21} (T_1 + T_2 + T_3) + \phi_0 q_{31} (T_1 + T_2 + T_3) + \phi_0 q_{41} (T_1 + T_2 + T_3) + \phi_0 q_{51} (T_1 + T_2 + T_3) + \phi_0 q_{61} (T_1 + T_2 + T_3) + \phi_0 q_{12} (T_2 + T_3) + \phi_0 q_{22} (T_2 + T_3) + \phi_0 q_{32} (T_2 + T_3) + \phi_0 q_{42} (T_2 + T_3) + \phi_0 q_{52} (T_2 + T_3) + \phi_0 q_{62} (T_2 + T_3) + \phi_0 q_{13} T_3 + \phi_0 q_{23} T_3 + \phi_0 q_{33} T_3 + \phi_0 q_{43} T_3 + \phi_0 q_{53} T_3 + \phi_0 q_{63} T_3$$

The early/late arrival penalty for all passengers, $B$,

$$B = \sum_{j=1}^{k} \sum_{i=1}^{4} \left[ (q_{ji}^{\text{seat}} + q_{ji}^{\text{std}}) \right] \theta(j)$$

$$= \theta(1) \left[ (q_{11}^{\text{seat}} + q_{11}^{\text{std}}) + (q_{12}^{\text{seat}} + q_{12}^{\text{std}}) + (q_{13}^{\text{seat}} + q_{13}^{\text{std}}) + (q_{14}^{\text{seat}} + q_{14}^{\text{std}}) + (q_{15}^{\text{seat}} + q_{15}^{\text{std}}) + (q_{16}^{\text{seat}} + q_{16}^{\text{std}}) \right]$$

$$+ \theta(2) \left[ (q_{21}^{\text{seat}} + q_{21}^{\text{std}}) + (q_{22}^{\text{seat}} + q_{22}^{\text{std}}) + (q_{23}^{\text{seat}} + q_{23}^{\text{std}}) + (q_{24}^{\text{seat}} + q_{24}^{\text{std}}) + (q_{25}^{\text{seat}} + q_{25}^{\text{std}}) + (q_{26}^{\text{seat}} + q_{26}^{\text{std}}) \right]$$

$$+ \theta(3) \left[ (q_{31}^{\text{seat}} + q_{31}^{\text{std}}) + (q_{32}^{\text{seat}} + q_{32}^{\text{std}}) + (q_{33}^{\text{seat}} + q_{33}^{\text{std}}) + (q_{34}^{\text{seat}} + q_{34}^{\text{std}}) + (q_{35}^{\text{seat}} + q_{35}^{\text{std}}) + (q_{36}^{\text{seat}} + q_{36}^{\text{std}}) \right]$$

$$+ \theta(5) \left[ (q_{51}^{\text{seat}} + q_{51}^{\text{std}}) + (q_{52}^{\text{seat}} + q_{52}^{\text{std}}) + (q_{53}^{\text{seat}} + q_{53}^{\text{std}}) + (q_{54}^{\text{seat}} + q_{54}^{\text{std}}) + (q_{55}^{\text{seat}} + q_{55}^{\text{std}}) + (q_{56}^{\text{seat}} + q_{56}^{\text{std}}) \right]$$

$$+ \theta(6) \left[ (q_{61}^{\text{seat}} + q_{61}^{\text{std}}) + (q_{62}^{\text{seat}} + q_{62}^{\text{std}}) + (q_{63}^{\text{seat}} + q_{63}^{\text{std}}) + (q_{64}^{\text{seat}} + q_{64}^{\text{std}}) + (q_{65}^{\text{seat}} + q_{65}^{\text{std}}) + (q_{66}^{\text{seat}} + q_{66}^{\text{std}}) \right]$$

We know that vehicle 4 will arrive on time from Table 5. So, there have

$$\theta(1) = 3\eta h, \quad \theta(2) = 2\eta h, \quad \theta(3) = \eta h, \quad \theta(4) = 0, \quad \theta(5) = \eta_2 h, \quad \theta(6) = 2\eta_2 h$$

The crowding cost for all standing passengers, $C$, 

$$C = \sum_{i=1}^{k} \sum_{j=1}^{4} \left[ (q_{ji}^{\text{std}}) \right]$$

$$= \phi_0 q_{11}^{\text{std}} (T_1 + T_2 + T_3) + \phi_0 q_{21}^{\text{std}} (T_1 + T_2 + T_3) + \phi_0 q_{31}^{\text{std}} (T_1 + T_2 + T_3) + \phi_0 q_{41}^{\text{std}} (T_1 + T_2 + T_3) + \phi_0 q_{51}^{\text{std}} (T_1 + T_2 + T_3) + \phi_0 q_{61}^{\text{std}} (T_1 + T_2 + T_3) + \phi_0 q_{12}^{\text{std}} (T_2 + T_3) + \phi_0 q_{22}^{\text{std}} (T_2 + T_3) + \phi_0 q_{32}^{\text{std}} (T_2 + T_3) + \phi_0 q_{42}^{\text{std}} (T_2 + T_3) + \phi_0 q_{52}^{\text{std}} (T_2 + T_3) + \phi_0 q_{62}^{\text{std}} (T_2 + T_3) + \phi_0 q_{13}^{\text{std}} T_3 + \phi_0 q_{23}^{\text{std}} T_3 + \phi_0 q_{33}^{\text{std}} T_3 + \phi_0 q_{43}^{\text{std}} T_3 + \phi_0 q_{53}^{\text{std}} T_3 + \phi_0 q_{63}^{\text{std}} T_3$$
\[ C = \sum_{i=1}^{n} \left[ \sum_{j=1}^{m} f \left( \sum_{i=1}^{n} q_{ij} \right) \right] T_i \]

\[ = f \left( q_{i1} \right) T_i + f \left( q_{i2} \right) T_i + f \left( q_{i3} \right) T_i + f \left( q_{i4} \right) T_i + f \left( q_{i5} \right) T_i + f \left( q_{i6} \right) T_i \]

\[ + f \left( q_{i1} + q_{i2} \right) T_i + f \left( q_{i3} + q_{i4} \right) T_i + f \left( q_{i5} + q_{i6} \right) T_i \]

\[ + f \left( q_{i1} + q_{i2} + q_{i3} \right) T_i + f \left( q_{i4} + q_{i5} + q_{i6} \right) T_i + f \left( q_{i1} + q_{i2} + q_{i5} \right) T_i \]

\[ + f \left( q_{i1} + q_{i2} + q_{i3} \right) T_i + f \left( q_{i4} + q_{i5} + q_{i6} \right) T_i + f \left( q_{i1} + q_{i2} + q_{i3} \right) T_i \]

\[ = T_1 \int_{0}^{\xi} \left[ \lambda \left( w \right) + \varphi \right] dw + T_2 \int_{0}^{\xi} \left[ \lambda \left( w \right) + \varphi \right] dw + T_3 \int_{0}^{\xi} \left[ \lambda \left( w \right) + \varphi \right] dw \]

\[ + T_4 \int_{0}^{\xi} \left[ \lambda \left( w \right) + \varphi \right] dw + T_5 \int_{0}^{\xi} \left[ \lambda \left( w \right) + \varphi \right] dw + T_6 \int_{0}^{\xi} \left[ \lambda \left( w \right) + \varphi \right] dw \]

According to Table 4 and substituting the values in Table 5 into the established model, we obtain the passenger travel schedule distribution of different seat capacity, which are shown in Fig. 2(a) and (b).
Fig. 2 Passenger travel distribution. (a) Passenger travel distribution when seat capacity is 6; (b) Passenger travel distribution when seat capacity is 10

In Figure 2, when passengers at Station 1 stand up to travel, the passengers at downstream Stations 2 and 3 will also be forced to stand. Vehicles 2, 3, 4, and 5 of Station 2 are saturated vehicles, and the number of passengers in each vehicle is the same and greater than that in the peak vehicles. If a passenger takes a certain vehicle at Station 3, there must be someone already taking that vehicle at Station 2, and the vehicles at Station 3 are the saturated vehicles of Station 2. (In the peak-period riding behavior research, Tian et al. (2005) considered capacity constraints, and peak-period and saturated-period are defined. Vehicles reach the upper limit of capacity during the saturated-period.) In addition, it also shows that the desirable vehicle has the largest number of passengers. Since the early penalty is less than the late penalty, the number of early passengers is significantly greater than the number of late passengers. Compared with the desirable vehicle, the travel distribution is different on the two symmetrical vehicles.

It can be seen from Fig. 3(a) that there exists watershed station 2, and passengers traveling at upstream station 1 can get available seats, and passengers traveling at downstream station 3 will be forced to stand up. In Fig. 3(b), due to the larger seat capacity, Station 3 is the watershed station, and passengers traveling at upstream stations 1 and 2 can get available seats. (At the watershed site, there are both standing passengers and seating passengers.) Furthermore, Fig. 3 also shows that even if the
passengers traveling at Station 3 are nearest to the destination and the remaining capacity is small, the passengers are still concentrated on the early/late vehicles with lower cost. When the number of available seats for vehicles 2, 3, 4, and 5 is 0 and vehicles 1 and 6 have available seats, passengers will choose the early vehicles with relatively lower cost. Although vehicle 6 has empty seats, the late arrival penalty is too high and there are no passengers. Comparing Figs. 2-3, we find that when seat capacity is small, the watershed site is close to the departure station.

**Fig. 3** Passenger travel distribution. (a) Passenger travel distribution when seat capacity is 26; (b) Passenger travel distribution when seat capacity is 40

Fig. 4 shows that the closer to the desirable vehicle, the full load will always be reached first. When passengers can take a bus close to the ideal vehicle, no passengers will take on a bus farther away from the ideal vehicle. The early/late penalties for vehicles 1 and 6 reach saturated state at the latest.

When seat capacity is small and standing capacity is large, passengers tend to choose either vehicle 3 or 4, which makes these vehicles fully load. The travel comfort is high as seat capacity increases. At this time, vehicle carrying ability is gradually reduced and the number of early or delayed passengers will increase. The number of passengers choosing each vehicle is gradually evened, and the corresponding vehicles are fully loaded. This phenomenon means that when the travel comfort is high, passengers will change the departure time to avoid being denied, which is consistent with the actual situation. When seat capacity is sufficient but the maximum capacity is
small, passengers feel less discomfort due to in-vehicle congestion. They only need to be measured between the early or late arrival time penalty, so passenger travel is dispersed and each vehicle is gradually equilibrium state. In this process, it can also be clearly seen that the saturated-period has continuous characteristics.

![Bar chart showing passenger travel distribution for different models and different vehicles](image)

**Fig. 4** Passenger travel distribution for different models and different vehicles

Figs. 2-4 show the travel situation that the demand is 80 people, \((N_1, N_2, N_3) = (80, 80, 80)\), for all three pick-up stops. In addition, we randomly calculated the travel costs of 10 groups of travel demands, such as \((N_1, N_2, N_3) = (60, 70, 70)\), as shown in Table 7 and Fig. 5. The total cost is greater with more travelers, but seat capacity has a turning point where it has the lowest total cost. When the travel demand is different, the turning point is different. The best model can be selected according to passenger’ demand so that the total cost under the equilibrium state is the lowest. For example, when the number of travelers is 255, it is best to design seat capacity to 32.
Table 7 The total travel cost with each seat capacity in the case of different travel demand

<table>
<thead>
<tr>
<th>Number of travelers at each station</th>
<th>60-70-70</th>
<th>70-70-70</th>
<th>70-75-75</th>
<th>75-75-80</th>
<th>75-80-80</th>
<th>80-80-80</th>
<th>80-80-85</th>
<th>80-85-85</th>
<th>85-85-85</th>
</tr>
</thead>
<tbody>
<tr>
<td>Travel demand 200</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>518.36</td>
<td>565.39</td>
<td>599.83</td>
<td>624.38</td>
<td>639.35</td>
<td>660.07</td>
<td>685.42</td>
<td>700.96</td>
<td>722.35</td>
</tr>
<tr>
<td>2</td>
<td>494.92</td>
<td>540.24</td>
<td>574.33</td>
<td>598.03</td>
<td>612.90</td>
<td>633.36</td>
<td>657.87</td>
<td>673.31</td>
<td>694.45</td>
</tr>
<tr>
<td>4</td>
<td>473.36</td>
<td>517.28</td>
<td>551.02</td>
<td>573.86</td>
<td>588.64</td>
<td>608.85</td>
<td>632.50</td>
<td>647.84</td>
<td>668.73</td>
</tr>
<tr>
<td>6</td>
<td>453.23</td>
<td>496.22</td>
<td>529.87</td>
<td>551.89</td>
<td>566.57</td>
<td>586.53</td>
<td>609.32</td>
<td>624.57</td>
<td>645.21</td>
</tr>
<tr>
<td>8</td>
<td>434.69</td>
<td>476.32</td>
<td>509.93</td>
<td>531.76</td>
<td>546.51</td>
<td>566.34</td>
<td>588.33</td>
<td>603.48</td>
<td>623.86</td>
</tr>
<tr>
<td>10</td>
<td>417.95</td>
<td>458.00</td>
<td>491.40</td>
<td>512.59</td>
<td>527.38</td>
<td>547.24</td>
<td>569.13</td>
<td>584.36</td>
<td>604.64</td>
</tr>
<tr>
<td>12</td>
<td>403.06</td>
<td>441.43</td>
<td>474.57</td>
<td>494.98</td>
<td>509.73</td>
<td>529.40</td>
<td>550.73</td>
<td>566.18</td>
<td>586.71</td>
</tr>
<tr>
<td>14</td>
<td>389.32</td>
<td>426.72</td>
<td>459.52</td>
<td>479.16</td>
<td>493.84</td>
<td>513.33</td>
<td>533.92</td>
<td>549.46</td>
<td>569.97</td>
</tr>
<tr>
<td>16</td>
<td>376.58</td>
<td>413.19</td>
<td>445.57</td>
<td>464.84</td>
<td>479.47</td>
<td>498.64</td>
<td>518.89</td>
<td>534.48</td>
<td>554.80</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
<td>26</td>
<td>28</td>
<td>30</td>
<td>32</td>
<td>34</td>
</tr>
<tr>
<td>---</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td></td>
<td>365.02</td>
<td>354.72</td>
<td>345.70</td>
<td>338.01</td>
<td>331.99</td>
<td>327.64</td>
<td>323.82</td>
<td>320.52</td>
<td>317.89</td>
</tr>
<tr>
<td></td>
<td>400.64</td>
<td>389.21</td>
<td>379.05</td>
<td>370.16</td>
<td>362.56</td>
<td>356.48</td>
<td>352.62</td>
<td>349.32</td>
<td>346.69</td>
</tr>
<tr>
<td></td>
<td>432.58</td>
<td>420.64</td>
<td>409.83</td>
<td>400.32</td>
<td>392.12</td>
<td>385.24</td>
<td>380.34</td>
<td>376.85</td>
<td>374.06</td>
</tr>
<tr>
<td></td>
<td>451.46</td>
<td>439.13</td>
<td>418.07</td>
<td>409.41</td>
<td>392.53</td>
<td>402.07</td>
<td>396.05</td>
<td>392.53</td>
<td>390.05</td>
</tr>
<tr>
<td></td>
<td>466.10</td>
<td>453.88</td>
<td>433.05</td>
<td>424.42</td>
<td>411.24</td>
<td>417.10</td>
<td>409.41</td>
<td>408.00</td>
<td>405.76</td>
</tr>
<tr>
<td></td>
<td>485.01</td>
<td>472.50</td>
<td>451.05</td>
<td>442.11</td>
<td>428.54</td>
<td>417.10</td>
<td>411.24</td>
<td>408.00</td>
<td>405.76</td>
</tr>
<tr>
<td></td>
<td>505.01</td>
<td>492.21</td>
<td>470.13</td>
<td>461.11</td>
<td>467.48</td>
<td>471.00</td>
<td>461.11</td>
<td>462.71</td>
<td>447.38</td>
</tr>
<tr>
<td></td>
<td>520.65</td>
<td>507.89</td>
<td>486.18</td>
<td>477.48</td>
<td>497.03</td>
<td>509.70</td>
<td>507.89</td>
<td>491.00</td>
<td>464.37</td>
</tr>
<tr>
<td></td>
<td>540.69</td>
<td>527.66</td>
<td>516.01</td>
<td>517.90</td>
<td>511.62</td>
<td>511.70</td>
<td>507.58</td>
<td>499.23</td>
<td>485.66</td>
</tr>
<tr>
<td></td>
<td>561.79</td>
<td>548.71</td>
<td>537.00</td>
<td>526.73</td>
<td>517.90</td>
<td>511.62</td>
<td>507.58</td>
<td>515.81</td>
<td>508.51</td>
</tr>
</tbody>
</table>
**Fig. 5** The total travel cost with each seat capacity in the case of different travel demand

Fig. 5 is the total travel cost of all passengers with different seat capacity. We can see that the total travel cost will increase significantly with greater travel demand. When passenger demand is different, the cost curve is obvious. When seat capacity is small \((c_{\text{seat}} \leq 32)\), increasing seat capacity can reduce the total travel cost. But, if seat capacity is increased to a certain extent, then continue to increase seat capacity, the total travel cost shows an upward trend. Although it has not been possible to clearly compare the changes of the equilibrium cost of per passenger, the change of the total travel cost can explain the feasibility of reducing the passenger equilibrium cost by changing the bus model and adjusting seat capacity.

It can also be seen from Table 7 that when the maximum capacity is constant and the travel demand is large, the fewer seats should be designed so as to reserve more standing space. The optimal seat capacity under different travel demand is marked. For example, when travel demand is 200 and 210, seat capacity can be as much as 40, so that more passengers can have seats, but only standing space for 4 people is reserved. When travel demand is 240, 245, 250 and 255, seat capacity should be designed as 32 in order to reserve more standing space (standing space for 16 people).

**6 Conclusion**

From the perspective of passenger comfort, this paper establishes a many-to-one
equilibrium model. We analyze the travel characteristics of commuters during the morning peak hours and the evolution of passenger flow with different seat capacity. Furthermore, we separately consider the crowding costs of seated and standing passengers in a vehicle, so that the result obtained by the model is more realistic, but it will not change the user equilibrium theorem. Because of the capacity constrain, commuters cannot avoid crowding, and further, they need to bear the additional costs of standing without a seat or failure to board. Finally, this article proposes a mathematical equilibrium problem to set an optimal number of seats while minimizing the total travel costs. A bi-level programing model is developed to solve the problem.

Then, it can be seen that the early/late penalties make passengers tend to travel together, which leads to the overcrowding of desirable vehicles in peak-period. Besides, when bus size is constant, a reasonable seat capacity can make passengers travel more comfortably, and at the same time make passengers tend to travel scattered (The optimal seat capacity is given in Sect. 4). In the end, passengers traveling closer to the destination will be mixed with passengers farther away, but passengers who are farther away will choose to arrive early or late to avoid the crowding effects. Due to the difference in the crowding cost between seated and standing passengers, the travel cost can be reduced to a certain extent by changing the bus model on the basis of seat availability. Therefore, changing the number of seats is also a solution that can improve travel comfort. Based on seat reliability, passengers have certain rules to follow. The operator can research the travel behavior in depth, and develop a desirable operation solution. It has certain reference value for improving the level of service in public transportation.

Author Contributions Conceptualization: Huayan Shang, Xin Xing; Methodology: Huayan Shang, Guangbin Tian, Tieqiao Tang; Formal analysis and investigation: Guangbin Tian, Xin Xing; Writing - original draft preparation: Guangbin Tian, Xin Xing; Writing - review and editing: Huayan Shang, Hongrui Chu, Tieqiao Tang; Supervision: Huayan Shang, Hongrui Chu.

Funding This research was jointly supported by grants from the National Natural Science Foundation of China (71971144, 72101165) and the Beijing Municipal Education Commission Foundation (SZ201910038021).
Data Availability All the datasets generated and analysed during the current study are available in this paper.

Ethical Declarations

Conflict of interest The authors declared that they have no conflicts of interest to this work.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Not Applicable.

References

Consultancy MVA (2008) Valuation of overcrowding on rail services. Prepared for Department for Transport
for scheduling and pricing. Transp Res Part B 71:1–18


Appendix A: The derivation

A.1 The early/late arrival penalty

Assuming that vehicles 1, 2, ..., m, m+1, ..., k are dispatched from the depot, only vehicle m+1 can arrive at workplace W on time. The early/late arrival penalty for passengers taking vehicle j to the workplace W can be given by:

\[
\theta(j) = \begin{cases} 
\eta_1 h(m+1-j), & j \in \{1, \ldots, m\} \\
0, & j = m+1 \\
\eta_2 h(j-(m+1)), & j \in \{m+2, \ldots, k\} 
\end{cases}
\]  

(A1)

\( h \) is departure interval, representing the difference between departure times of successive vehicles from the depot. \( \eta_1 \) and \( \eta_2 \) are respectively the schedule delay penalties of the early and late arrival time, \( h(m+1-j) \) and \( h(j-(m+1)) \) represent the early and late arrival time. Because the loss suffered by arriving late passengers is obviously greater than those arriving early, so there is \( \eta_1 < \eta_2 \).

A.2 The cost relationship between two successive stations

(1) \( q_{ji} \leq SA_{ji} \)

\( C_{ji}^{\text{seat}} \) and \( C_{j(i+1)}^{\text{seat}} \) respectively denote the travel costs of seated passengers from station \( H_j \) and \( H_{i+1} \) taking vehicle \( j \). We have \( C_{ji}^{\text{seat}} = \varphi_0 \sum_{\alpha=1}^{k} T_{\alpha} + \theta(j) + e_{ji}^{\text{seat}} \) from Eq. (3). Then the travel costs of seated passengers at the first stop and the second stop are respectively expressed as follows:

\[
C_{j1}^{\text{seat}} = \varphi_0 \sum_{\alpha=1}^{3} T_{\alpha} + \theta(j) + e_{j1}^{\text{seat}} = \varphi_0 (T_1 + T_2 + T_3) + \theta(j) + e_{j1}^{\text{seat}}
\]  

(A2)

\[
C_{j2}^{\text{seat}} = \varphi_0 \sum_{\alpha=2}^{3} T_{\alpha} + \theta(j) + e_{j2}^{\text{seat}} = \varphi_0 (T_2 + T_3) + \theta(j) + e_{j2}^{\text{seat}}
\]  

(A3)

So there has:

\[
C_{j1}^{\text{seat}} - C_{j2}^{\text{seat}} = T_1 \varphi_0 + e_{j1}^{\text{seat}} - e_{j2}^{\text{seat}}
\]  

(A4)
Generally, there are
\[ C_{ji(i-1)}^{\text{seat}} - C_{ji}^{\text{seat}} = T_{(i-1)}^j \varphi_0 + e_{ji(i-1)}^{\text{seat}} - e_{ji}^{\text{seat}} \] (A5)

(2) \( q_{ji} > SA_{ji} \)

\( C_{ji}^{\text{std}} \) and \( C_{ji(i-1)}^{\text{std}} \) respectively denote the travel costs of standing passengers from station \( H_j \) and \( H_{j-1} \) taking vehicle \( j \). We have

\[ C_{ji}^{\text{std}} = \sum_{a=1}^{3} \varphi_{ji}^{\text{std}} \left( \sum_{o=1}^{a} q_{jo}^{\text{std}} \right) T_a + \theta(j) + e_{ji}^{\text{std}} \] from Eq. (3). Then the travel costs of standing passengers at the first, second and third stop are respectively expressed as follows:

\[ C_{j1}^{\text{std}} = \sum_{a=1}^{3} \varphi_{j1}^{\text{std}} \left( \sum_{o=1}^{a} q_{jo}^{\text{std}} \right) T_a + \theta(j) + e_{j1}^{\text{std}} \] (A6)

\[ C_{j2}^{\text{std}} = \sum_{a=2}^{3} \varphi_{j2}^{\text{std}} \left( \sum_{o=1}^{a} q_{jo}^{\text{std}} \right) T_a + \theta(j) + e_{j2}^{\text{std}} \] (A7)

\[ C_{j3}^{\text{std}} = \sum_{a=3}^{3} \varphi_{j3}^{\text{std}} \left( \sum_{o=1}^{a} q_{jo}^{\text{std}} \right) T_a + \theta(j) + e_{j3}^{\text{std}} \] (A8)

So there has:

\[ C_{j1}^{\text{std}} - C_{j2}^{\text{std}} = \varphi_{j1}^{\text{std}} \left( q_{j1}^{\text{std}} \right) T_1 + e_{j1}^{\text{std}} - e_{j2}^{\text{std}} \] (A9)

\[ C_{j2}^{\text{std}} - C_{j3}^{\text{std}} = \varphi_{j2}^{\text{std}} \left( q_{j1}^{\text{std}} + q_{j2}^{\text{std}} \right) T_2 + e_{j2}^{\text{std}} - e_{j3}^{\text{std}} \] (A10)

Generally, there are

\[ C_{ji(i-1)}^{\text{std}} - C_{ji}^{\text{std}} = \varphi_{ji}^{\text{std}} \left( \sum_{o=1}^{i-1} q_{jo}^{\text{std}} \right) T_{(i-1)} + e_{ji(i-1)}^{\text{std}} - e_{ji}^{\text{std}} \] (A11)

A.3 Theorem

**Theorem 4.** At equilibrium, for any \( i \in \{1, 2, \ldots, s - 1\} \), if \( S_i \neq \emptyset \), then \( P_{i+1} = \emptyset \).

**Proof.** We use contradiction to proof the theorem. Suppose that when \( i \in \{1, 2, \ldots, s - 1\} \), there exists \( i \) if \( S_i \neq \emptyset \), then \( P_{i+1} \neq \emptyset \). That is, there exists \( l \) such
that $q_{i(j+1)}^{\text{seat}} > 0$. Suppose that for any $j \in S_i$, if $S_i \neq \emptyset$, then $q_{ji}^{\text{std}} > 0$. According to the equilibrium conditions, we have $C_{ji}^{\text{std}} = C_i^{\text{std}}$ and $e_{ji}^{\text{std}} \geq 0$. For station $H_i$, we then obtain $C_{ji}^{\text{std}} \geq C_i^{\text{std}}$ and $e_{ji}^{\text{std}} = 0$ such that $C_{ji}^{\text{std}} \geq C_j^{\text{std}}$. For station $H_{i+1}$, because $q_{i(j+1)}^{\text{seat}} > 0$, there has $C_{i(j+1)}^{\text{seat}} = C_{i(j+1)}^{\text{std}}$ and $e_{i(j+1)}^{\text{seat}} \geq 0$ and $C_{j(i+1)}^{\text{std}} \geq C_{i(j+1)}^{\text{std}}$ and $e_{j(i+1)}^{\text{std}} = 0$.

Then, we obtain $C_{j(i+1)}^{\text{std}} \geq C_{i(j+1)}^{\text{std}}$. And since $q_{ji}^{\text{std}} > 0$, there has $C_{ji}^{\text{std}} \geq C_{ji}^{\text{std}} = C_{ji}^{\text{std}}$.

According to Eq. (4), we then get

$$C_{j(i+1)}^{\text{std}} + \varphi^{\text{std}} \left( \sum_{o=1}^{j} q_{jo}^{\text{std}} \right) T_i + e_{ji}^{\text{std}} - e_{j(i+1)}^{\text{seat}} \leq C_{i(j+1)}^{\text{seat}} + \varphi_j T_i + e_{ji}^{\text{std}} - e_{i(j+1)}^{\text{seat}}.$$ Since $e_{ji}^{\text{std}} - e_{j(i+1)}^{\text{seat}} \geq 0$.

Then, we obtain $C_{j(i+1)}^{\text{std}} < C_{i(j+1)}^{\text{seat}}$. This contradicts with $C_{j(i+1)}^{\text{std}} \geq C_{i(j+1)}^{\text{std}}$ got above. So, for any $i \in \{1, 2, ..., s-1\}$, if $S_i \neq \emptyset$, then $P_{i+1} = \emptyset$.

**Theorem 5.** At equilibrium, for station $H_i$ $(i > \nu)$ and any $j \in S_i$, if $q_{ji}^{\text{std}} > 0$,

then $\sum_{o=1}^{i-1} q_{jo}^{\text{std}} > 0$.

**Proof.** We use contradiction to proof the theorem. Suppose that there exists $i$ and $j$ such that $q_{ji}^{\text{std}} > 0$ and $\sum_{o=1}^{i-1} q_{jo}^{\text{std}} = 0$, then $q_{j(i+1)}^{\text{std}} = 0$. According to the equilibrium expression (5), if $q_{ji}^{\text{std}} > 0$, then $C_{ji}^{\text{std}} = C_i^{\text{std}}$. From Eq. (4), it can be obtained

$$C_{j(i+1)}^{\text{std}} - C_{ji}^{\text{std}} = \varphi^{\text{std}} \left( \sum_{o=1}^{j} q_{jo}^{\text{std}} \right) T_i + e_{ji}^{\text{std}} - e_{j(i+1)}^{\text{std}}.$$ By assumption, if $\sum_{o=1}^{i-1} q_{jo}^{\text{std}} = 0$, then $e_{j(i+1)}^{\text{std}} = 0$ and $e_{ji}^{\text{std}} = 0$. So, we get $C_{ji}^{\text{std}} \geq C_{ji}^{\text{std}}$.

At equilibrium, passengers at station $H_{i-1}$ will choose to take at least one bus in order to go to work. Suppose that $l \in \{1, 2, ..., k\}$ and $l \neq j$, so $q_{j(l+1)}^{\text{std}} > 0$ and
\[
\sum_{i=1}^{i-1} q_{lo}^{\text{std}} > 0. \text{ According to the equilibrium conditions, we get } C_{i(i-1)}^{\text{std}} = C_{i-1}^{\text{std}} \text{ and } e_{i(i-1)}^{\text{std}} \geq 0, \\
C_{li}^{\text{std}} \geq C_{i}^{\text{std}} \text{ and } e_{li}^{\text{std}} = 0 \text{ and } e_{i(i-1)}^{\text{std}} \geq e_{li}^{\text{std}}. \text{ Combining Eq. (4), we then obtain} \\
C_{i-1}^{\text{std}} = C_{i(i-1)}^{\text{std}} = \varphi^{\text{std}} \left( \sum_{o=1}^{i} q_{lo}^{\text{std}} \right) T_{i-1} + C_{li}^{\text{std}} + e_{i(i-1)}^{\text{std}} - e_{li}^{\text{std}} > C_{li}^{\text{std}}, \text{ so } C_{i-1}^{\text{std}} > C_{i}^{\text{std}}. \text{ This contradicts with } C_{i}^{\text{std}} \geq C_{i-1}^{\text{std}} \text{ got above. So, the hypothesis is not true and the original proposition is true.}
\]