Supplementary Information: Coherently amplifying photon production from vacuum with a dense cloud of accelerating photodetectors

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In the following, we derive the RWA Hamiltonian (3) starting from the relativistic cavity field-coupled-N detector action (1). We then derive from the Lindblad master equation (4), approximate equations for the second moments of the cavity and TLSs degrees of freedom via a cumulant expansion.

DERIVATION OF THE RWA HAMILTONIAN FROM THE FIELD ACTION

The starting model action for the cavity field-N detector system is given by

$$\begin{aligned} S &= -\int d^{3+1}x \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \sum_{i=1}^{N} \int d\tau_i \left\{ \frac{m_0}{2}\left[(\partial_{\tau_i} Q_i)^2 - \omega_{d0}^2 Q_i^2\right] - \frac{g}{4!} Q_i^4 \right\} \\ &+ \lambda_0 \sum_{i=1}^{N} \int d^{3+1}x Q_i(\tau_i)\Phi(t,\vec{r})\delta^{3+1}\left[x^\mu - z_i^\mu(\tau_i)\right], \end{aligned} \tag{S1}$$

where the ith detector's worldline is $z_i^\mu(t) = (t, L_x/2 + A\cos(\Omega_m t + \phi_i), L_y/2, L_z/2)$, and τ_i denotes the ith detector's proper time. With the relation $S = \int dt L$, the system Lagrangian in the laboratory frame is

$$\begin{aligned} L &= -\int d^3 r \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \sum_{i=1}^{N} \left\{ \frac{m_0}{2}\left[\frac{dt}{d\tau_i}(\partial_t Q_i)^2 - \frac{d\tau_i}{dt}\omega_{d0}^2 Q_i^2\right] - \frac{d\tau_i}{dt}\frac{g}{4!}Q_i^4 \right\} \\ &+ \frac{\lambda_0}{c}\sum_{i=1}^{N}\frac{d\tau_i}{dt}\int d^3 r Q_i(t)\Phi(t,\vec{r})\delta^3(\vec{r}-\vec{r}_i(t)), \end{aligned} \tag{S2}$$

with the Lorentz factor $d\tau_i/dt = \sqrt{1-\xi^2\sin^2(\Omega_m t + \phi_i)}$, where $\xi = \Omega_m A/c$.

Performing the Legendre transformation on the Lagrangian (S2), we obtain the following cavity field-detector system Hamiltonian:

$$\begin{aligned} H &= \int d^3 r \frac{1}{2}\left[\Pi^2 + (\vec{\nabla}\Phi)^2\right] + \sum_{i=1}^{N}\frac{d\tau_i}{dt}\left[\frac{P_i^2}{2m_0} + \frac{m_0}{2}\omega_{d0}^2 Q_i^2 + \frac{g}{4!}Q_i^4\right] \\ &- \frac{\lambda_0}{c}\sum_{i=1}^{N}\frac{d\tau_i}{dt}\int d^3 r Q_i(t)\Phi(t,\vec{r})\delta^3(\vec{r}-\vec{r}_i(t)), \end{aligned} \tag{S3}$$

where $\Pi(t,\vec{r}) = \dot{\Phi}(t,\vec{r})/c^2$ and $P_i(t) = m_0\dot{Q}_i(t)d\tau_i/dt$ are the momenta conjugate to $\Phi(t,\vec{r})$ and $Q_i(t)$, respectively. The first term in Eq. (S3) is the free cavity scalar field Hamiltonian, the second term is the free detector Hamiltonian, and the third term is the interaction Hamiltonian.

Quantizing by replacing the position and momentum coordinates with their corresponding operators, we assume weak anharmonic potential energy terms for the detector degrees of freedom, so that the free detector energy eigenvalues E_n, $n = 0, 1, 2, \ldots,$ can be approximated using first order perturbation theory. The approximate energy eigenstates for the free detectors are then harmonic oscillator Fock states: $|n_0, n_1, \ldots, n_N\rangle$, $n_i = 0, 1, 2, \ldots$. With the parametric resonance condition applied between the detectors' center of mass oscillation frequency, cavity mode frequency, and transition frequency between the ground and excited detectors' energy levels (see below), we can truncate the Hilbert space of each detector to its two lowest energy eigenstates $|n_i\rangle$, $n_i = 0, 1$, associated with ground and first excited energy eigenvalues E_0, E_1, respectively. Therefore, defining $\tilde{\omega}_d = (E_1 - E_0)/\hbar$, the free detectors' position operators and free Hamiltonian are expressed in terms of Pauli operators:

$$Q_i = \sqrt{\frac{\hbar}{2m_0\omega_{d0}}}\sigma_i^x, \quad \text{(S4)}$$

$$H_{\text{det}} = \frac{\hbar\tilde{\omega}_d}{2}\sum_{i=1}^{N}\frac{d\tau_i}{dt}\sigma_i^z. \quad \text{(S5)}$$

We consider a 3D cavity 'box' with side lengths L_x, L_y, and L_z, and impose the Dirichlet boundary conditions $\Phi(t, 0, y, z) = \Phi(t, L_x, y, z) = 0$, $\Phi(t, x, 0, z) = \Phi(t, x, L_y, z) = 0$, and $\Phi(t, x, y, 0) = \Phi(t, x, y, L_z) = 0$. The cavity quantum field operator can be decomposed in terms of classical normal mode solutions and associated creation and annihilation operators $a_{\vec{n}}$, $a_{\vec{n}}^\dagger$ as follows:

$$\Phi(t, \vec{r}) = \sum_{\vec{n}}\sqrt{\frac{2\hbar c}{2L_xL_yL_z|k_{\vec{n}}|}}\sin(k_{nx}x)\sin\left(k_{ny}y\right)\sin(k_{nz}z)\left(a_{\vec{n}}(t) + a_{\vec{n}}^\dagger(t)\right), \quad \text{(S6)}$$

with $k_{ni} = n_i\pi/L_i$, $i = x, y, z$. Under the resonance condition (see below), the cavity field is truncated to the single mode with vector label $\vec{n} = (2, 1, 1)$. The corresponding mode frequency is $\omega_c = ck_c$, $k_c = \sqrt{(2\pi/L_x)^2 + (\pi/L_y)^2 + (\pi/L_z)^2}$.

Substituting (S4-S6) into Hamiltonian (S3) and dropping the subscript $2, 1, 1$ on $a_{2,1,1}^{(\dagger)}$ for notational convenience, we then obtain the single-mode, relativistic parametrically driven Dicke Hamiltonian (2):

$$H = \hbar\omega_c a^\dagger a + \frac{\hbar\tilde{\omega}_d}{2}\sum_{i=1}^{N}\frac{d\tau_i}{dt}\sigma_i^z + \hbar\tilde{\lambda}\sum_{i=1}^{N}\frac{d\tau_i}{dt}\sin\left[k_cA\cos(\Omega_m t + \phi_i)\right](a + a^\dagger)\sigma_i^x, \quad \text{(S7)}$$

where $\tilde{\lambda} = \sqrt{2}\hbar\lambda_0/\sqrt{m_0\omega_{d0}cL_xL_yL_z|\vec{k}_{2,1,1}|}$.

Hamiltonian (S7) expressed in terms of J^z, $J^\pm$, $\xi = \Omega_m A/c$, and with the detectors' (TLSs') phases all set zero ($\phi_i = 0$), is

$$H = \hbar\omega_c a^\dagger a + \hbar\tilde{\omega}_d\frac{d\tau}{dt}J^z + \hbar\tilde{\lambda}\frac{d\tau}{dt}\sin\left[\frac{\omega_c\xi}{\Omega_m}\cos(\Omega_m t)\right](\hat{a} + \hat{a}^\dagger)J^x, \quad \text{(S8)}$$

here the reciprocal Lorentz factor is $d\tau/dt = \sqrt{1 - \xi^2\sin^2(\Omega_m t)}$. Applying the Fourier series expansion to $d\tau/dt$ and the Jacobi-Anger expansion to the $\sin\left[\omega_c\xi\cos(\Omega_m t)/\Omega_m\right]$ term, we obtain

$$\frac{d\tau}{dt} = \sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}\binom{2n}{n}\left(\frac{\xi}{2}\right)^{2n} + 2\sum_{n=1}^{\infty}\sum_{n'=1}^{n}(-1)^{n-n'}\binom{\frac{1}{2}}{n}\binom{2n}{n-n'}\left(\frac{\xi}{2}\right)^{2n}\cos\left(2n'\Omega_m t\right), \quad \text{(S9)}$$

$$\sin\left[\frac{\omega_c\xi}{\Omega_m}\cos(\Omega_m t)\right] = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}\left(\frac{\omega_c\xi}{\Omega_m}\right)\cos\left[(2n+1)\Omega_m t\right], \quad \text{(S10)}$$

where the $J_{(2n+1)}(z)$ are Bessel functions of the first kind. Keeping only terms up to second harmonics in Ω_m, Eqs. (S9) and (S10) become approximately

$$\frac{d\tau}{dt} \approx D_0 + D_2 \cos(2\Omega_m t) \tag{S11}$$

$$\frac{d\tau}{dt} \sin\left[\frac{\omega_c \xi}{\Omega_m} \cos(\Omega_m t)\right] \approx C_1 \cos(\Omega_m t), \tag{S12}$$

where the ξ dependent D_0 and D_2 coefficients can be read off from Eq. (S9) and the ξ, Ω_m dependent coefficient $C_1 = 2J_1(\omega_c \xi/\Omega_m)$ can be read off from Eq. (S10). Substituting Eqs. (S11), (S12) into Hamiltonian Eq. (S8), we obtain

$$H = \hbar\omega_c a^\dagger a + \hbar\left[\omega_d + \tilde{\omega}_d D_2 \cos(2\Omega_m t)\right] J^z + \hbar\tilde{\lambda} C_1 \cos(\Omega_m t)(a^\dagger + a)(J^+ + J^-), \tag{S13}$$

where $\omega_d = \tilde{\omega}_d D_0$ is the renormalized detector oscillator frequency. Transforming to the rotating frame via the unitary operator $U_{\mathrm{RF}}(t) = \exp\left(i\omega_c a^\dagger a t + iJ^z\left[\omega_d t + \tilde{\omega}_d D_2 \sin(2\Omega_m t)/2\Omega_m\right]\right)$, the cavity mode and detector annihilation operators pick up time-dependent phase terms as follows:

$$\begin{aligned} a(t) &\rightarrow e^{-i\omega_c t} a(t), \\ \sigma_i^-(t) &\rightarrow e^{-i\left[\omega_d t + \frac{\tilde{\omega}_d D_2}{2\Omega_m}\sin(2\Omega_m t)\right]} \sigma_i^-(t). \end{aligned} \tag{S14}$$

The system Hamiltonian (S13) then becomes in the interaction picture:

$$H_I = \hbar\tilde{\lambda} C_1 \cos(\Omega_m t)(e^{i\omega_c t} a^\dagger + e^{-i\omega_c t} a)\left[e^{i\omega_d t} e^{iB\sin(2\Omega_m t)} J^+ + e^{-i\omega_d t} e^{-iB\sin(2\Omega_m t)} J^-\right] \tag{S15}$$

where $B = \tilde{\omega}_d D_2/2\Omega_m < 1$.
Making use of the Jacobi-Anger expansion again such that

$$e^{\pm iB\sin(2\Omega_m t)} \approx J_0(B) \pm 2iJ_1(B)\sin(2\Omega_m t) \tag{S16}$$

and substituting Eq. (S16) into Eq. (S15), we arrive at the following expression for the system Hamiltonian:

$$\begin{aligned} H_I \approx\ & \hbar\tilde{\lambda} C_1 \cos(\Omega_m t)\Big\{ e^{i(\omega_c+\omega_d)t}\left[J_0(B) + 2iJ_1(B)\sin(2\Omega_m t)\right] a^\dagger J^+ \\ & + e^{-i(\omega_c+\omega_d)t}\left[J_0(B) - 2iJ_1(B)\sin(2\Omega_m t)\right] a J^- \\ & + e^{i(\omega_c-\omega_d)t}\left[J_0(B) - 2iJ_1(B)\sin(2\Omega_m t)\right] a^\dagger J^- \\ & + e^{-i(\omega_c-\omega_d)t}\left[J_0(B) + 2iJ_1(B)\sin(2\Omega_m t)\right] a J^+ \Big\}. \end{aligned} \tag{S17}$$

Imposing the parametric resonance condition $\Omega_m = \omega_c + \omega_d$ and combining the $\cos(\Omega_m t)$ term with the first two terms within the braces, we retain the time-independent terms and drop the oscillating terms at integer multiples of Ω_m (RWA). As a result, we recover the approximate time independent Hamiltonian (3):

$$H \approx \hbar\lambda(a^\dagger J^+ + aJ^-), \tag{S18}$$

where we have dropped the subscript I and the renormalized coupling constant $\lambda = \frac{1}{2}\tilde{\lambda} C_1\left[J_0(\tilde{\omega}_d D_2/2\Omega_m) - J_1(\tilde{\omega}_d D_2/2\Omega_m)\right]$. Figure 1 compares the average photon number dynamics for the relativistic, parametrically driven Dicke Hamiltonian Hamiltonian (2) [Eq. (S7)] with the dynamics for the RWA Hamiltonian (3) [Eq. (S18)]. The plots are obtained by numerically solving for some example parameter values the Lindblad master equation (4) using QuTiP [1]. From the figure, we can see that the

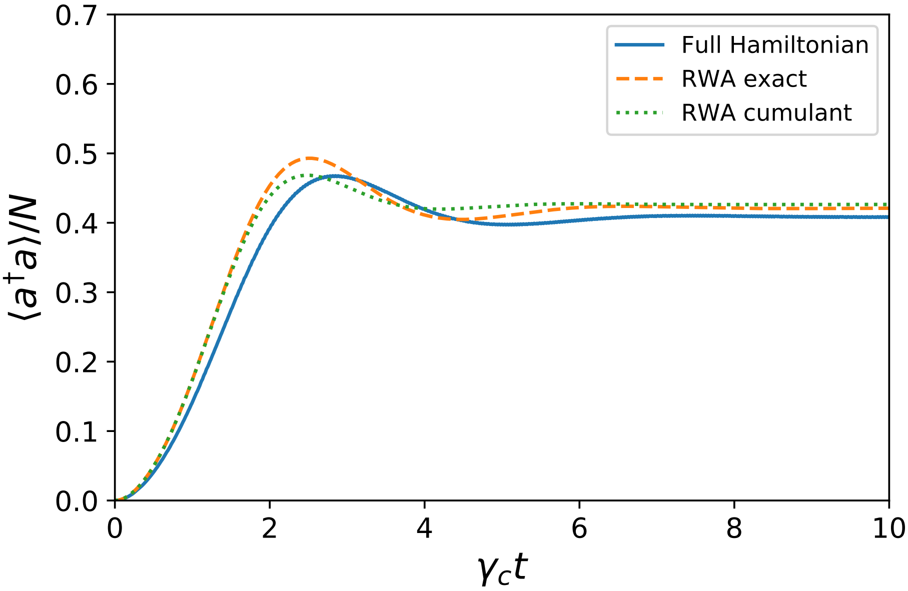

FIG. 1: Dynamical evolution of $\langle a^\dagger a\rangle/N$ for seven TLSs coupled to independent environments for highly relativistic center of mass acceleration, and with phases ϕ_i randomly chosen between 0 and 2π. The solid line is for the relativistic, parametrically driven Dicke Hamiltonian (2) [Eq. (S7)] and dashed line for the RWA Hamiltonian (3) [Eq. (S18)], both calculated numerically using QuTiP [1]; the dotted line is for the cumulant expansion approximation (S22) to the RWA Hamiltonian (3) [Eq. (S18)] . The chosen example parameters are $\omega_c = \omega_{d0} = 1.0$, $\xi = 0.8$, $\gamma_c = \gamma_d = 0.02$, and $\lambda_0 = 0.0148$. The renormalized coupling $\lambda = 0.01$, corresponding to $N_{\rm crit} = 1$.

Lindblad master equation (4) with RWA Hamiltonian (3) [Eq. (S18)] gives a good approximation to the average photon number dynamics for the parametrically driven Dicke Hamiltonian (2) [Eq. (S7)], even for relativistic motion with $\xi \sim 1$. Furthermore, the dynamics is relatively insensitive to the individual TLS's phases ϕ_i.

Under nonrelativistic conditions $\xi \ll 1$ relevant for possible experimental realizations, we have $D_0 \approx 1$, $D_2 \approx 0$ (with $d\tau/dt \approx 1$), and $C_1 \approx \omega_c\xi/\Omega_m$, so that the renormalized coupling is $\lambda \approx \tilde{\lambda}\omega_c\xi/(2\Omega_m)$; the Lindblad master equation (4) with RWA Hamiltonian (3) [Eq. (S18)] gives an even more accurate approximation to the average photon number dynamics for the parametrically driven Dicke Hamiltonian Hamiltonian (2) [Eq. (S7)] in the nonrelativistic regime than for the relativistic regime (Fig. 1).

CUMULANT EXPANSION

We start from the dynamics of the cavity-TLS system as described by the Lindblad master equation:

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \gamma_c\mathcal{L}_a[\rho] + \sum_{i=1}^{N}\gamma_d\mathcal{L}_{\sigma_i^-}[\rho], \tag{S19}$$

where the Lindblad superoperators are defined by $\mathcal{L}_A[\rho] = A\rho A^\dagger - \frac{1}{2}A^\dagger A\rho - \frac{1}{2}\rho A^\dagger A$. Consider the unitary transformation $G = \exp[i\theta(a^\dagger a - J^z + N/2)]$, which leaves Hamiltonian (3) [Eq. (S18)] invariant. The master equation for the transformed system density matrix $\tilde{\rho} = G\rho G^\dagger$ is given by

$$\begin{aligned}\dot{\tilde{\rho}} &= -\frac{i}{\hbar}G[H,\rho]G^\dagger + \gamma_c G\mathcal{L}_a[\rho]G^\dagger + \gamma_d \sum_{i=1}^{N} G\mathcal{L}_{\sigma_i^-}[\rho]G^\dagger \\ &= -\frac{i}{\hbar}[H,\tilde{\rho}] + \gamma_c\mathcal{L}_a[\tilde{\rho}] + \gamma_d\sum_{i=1}^{N}\mathcal{L}_{\sigma_i^-}[\tilde{\rho}], \end{aligned} \tag{S20}$$

where we have applied the transformation relations $G^\dagger a G = e^{i\theta}a$ and $G^\dagger\sigma_i^- G = e^{-i\theta}\sigma_i^-$. Assume that the system state is initialized as a product state of the individual ground states of the isolated cavity and N isolated TLSs: $|\psi(0)\rangle = |0\rangle_c \otimes |-,-,\ldots,-\rangle_s$. The fact that $\tilde{\rho}(0) = \rho(0)$ and the transformed equation (S20) coincides with Eq. (S19), indicates that $\tilde{\rho}(t) = \rho(t)$ for the whole time range: G is a symmetry of the cavity-TLS system and initial state. As a consequence, only operators that are invariant under the G transformation give non-vanishing expectation values, and thus for the first and second moments, we need consider only the following non-zero moments: $\langle\sigma_1^z\rangle$, $\langle a^\dagger a\rangle$, $\langle a\sigma_1^-\rangle$, $\langle a^\dagger\sigma_1^+\rangle$, and $\langle\sigma_1^+\sigma_2^-\rangle$; all TLS's give the same moment values (a consequence of having the same assumed coupling and damping rates), which allows us to replace $\sigma_i^{z(+,-)}$ and $\sigma_i^+\sigma_j^-$ $(i \neq j)$ with $\sigma_1^{z(+,-)}$ and $\sigma_1^+\sigma_2^-$, respectively. We have verified numerically for a few N TLSs using QuTiP [1] that the following noninvariant moments that would otherwise appear in the moment equations vanish: i.e., $\langle a^2\rangle = \langle aJ^+\rangle = \langle J^+J^+\rangle = 0$.

With Eq. (S19), the dynamical differential equations for the second moments can be obtained through a cumulant approximation [2]. These equations also include the nonvanishing third moment terms $\langle a^{(\dagger)}\sigma_2^{\pm}\sigma_1^z\rangle$ and $\langle a^\dagger a\sigma_1^z\rangle$. The usual way to approximate these third moments is to set to zero the corresponding third order cumulants and obtain an approximate expression involving a sum of products of first and second order moments. However, we instead utilize an alternative cumulant approximation that is found to be more accurate. To proceed, we first rewrite σ_1^z through the identity $\sigma_1^z = 2\sigma_1^+\sigma_1^- - 1$, and substitute into the third moments to obtain the sum of a fourth moment term and one second moment term (note that alternative substitutions such as $\sigma_1^z = 1 - 2\sigma_1^-\sigma_1^+$ give poorer approximations than the former, normal ordered choice). By approximating the fourth moments through setting the fourth cumulants to zero, we obtain the following improved third-moment approximations:

$$\begin{aligned}\langle a\sigma_2^-\sigma_1^z\rangle &= \langle a\sigma_2^-\rangle\langle\sigma_1^z\rangle + 2\langle a\sigma_1^-\rangle\langle\sigma_2^-\sigma_1^+\rangle \\ \langle a^\dagger\sigma_2^+\sigma_1^z\rangle &= \langle a^\dagger\sigma_2^+\rangle\langle\sigma_1^z\rangle + 2\langle a^\dagger\sigma_1^+\rangle\langle\sigma_1^-\sigma_2^+\rangle \\ \langle a^\dagger a\sigma_1^z\rangle &= \langle a^\dagger a\rangle\langle\sigma_1^z\rangle + 2\langle a^\dagger\sigma_1^+\rangle\langle a\sigma_1^-\rangle. \end{aligned} \tag{S21}$$

The resulting approximate differential equations for the nonvanishing second moments and $\langle\sigma_1^z\rangle$ are

$$\begin{aligned}\frac{d}{dt}\langle a^\dagger a\rangle &= -\gamma_c\langle a^\dagger a\rangle + 2iN\lambda\langle a\sigma_1^-\rangle \\ \frac{d}{dt}\langle a\sigma_1^-\rangle &= -\frac{\gamma_c+\gamma_d}{2}\langle a\sigma_1^-\rangle - i\lambda\left[(N-1)\langle\sigma_1^+\sigma_2^-\rangle - \langle a^\dagger a\rangle\langle\sigma_1^z\rangle - 2\langle a\sigma_1^-\rangle\langle a^\dagger\sigma_1^+\rangle + \frac{1-\langle\sigma_1^z\rangle}{2}\right] \\ \frac{d}{dt}\langle a^\dagger\sigma_1^+\rangle &= -\frac{\gamma_c+\gamma_d}{2}\langle a^\dagger\sigma_1^+\rangle + i\lambda\left[(N-1)\langle\sigma_1^+\sigma_2^-\rangle - \langle a^\dagger a\rangle\langle\sigma_1^z\rangle - 2\langle a\sigma_1^-\rangle\langle a^\dagger\sigma_1^+\rangle + \frac{1-\langle\sigma_1^z\rangle}{2}\right] \\ \frac{d}{dt}\langle\sigma_1^+\sigma_2^-\rangle &= -\gamma_d\langle\sigma_1^+\sigma_2^-\rangle - 2i\lambda\left[\langle a\sigma_1^-\rangle\langle\sigma_1^z\rangle + 2\langle a\sigma_1^-\rangle\langle\sigma_1^+\sigma_2^-\rangle\right] \\ \frac{d}{dt}\langle\sigma_1^z\rangle &= -\gamma_d(\langle\sigma_1^z\rangle + 1) + 4i\lambda\langle a\sigma_1^-\rangle. \end{aligned} \tag{S22}$$

The above nonlinear dynamical equations must be solved numerically (see Fig. 1), although an approximate analytical solution for the average cavity photon number can be obtained in the long time limit steady state, defined by setting to zero the time derivatives of the moments. In particular, we obtain $\langle a^\dagger a\rangle \approx N\gamma_d/(2\gamma_c)$ when $N \gg N_{\rm crit}$, where $N_{\rm crit} = \gamma_c\gamma_d/(4\lambda^2)$.

If we instead assume the usual, less accurate second moment approximation where the third order cumulants are set to zero, then the second terms on the right hand side of Eqs. (S21) are absent as well as the corresponding terms quadratic in the second moments in Eq. (S22). This then enables the following general analytic solution to be obtained for the average cavity photon number in the long time limit:

$$\langle a^\dagger a\rangle = \frac{N\gamma_d\{(N-N_{\rm crit})(\gamma_c+\gamma_d) - 2\gamma_c + \sqrt{(N-N_{\rm crit})^2(\gamma_c+\gamma_d)^2 + 4\gamma_c\left[(N+N_{\rm crit})(\gamma_c+\gamma_d)-\gamma_c\right]}\}}{4\gamma_c\left[N(\gamma_c+\gamma_d)-\gamma_c\right]} \quad (S23)$$

In the limit N, $N_{\rm crit} \gg 1$, Eq. (S23) simplifies approximately to

$$\langle a^\dagger a\rangle \approx \begin{cases} \frac{N\gamma_d}{N_{\rm crit}(\gamma_c+\gamma_d)} & N \ll N_{\rm crit} \\ \frac{N\gamma_d}{2\gamma_c} & N \gg N_{\rm crit}. \end{cases} \quad (S24)$$

While the more accurate, vanishing fourth cumulant approximation does not give a simple general analytic expression like Eq. (S23), it nevertheless yields the same approximate expressions as (S24) for N, $N_{\rm crit} \gg 1$. Note that the vanishing fourth cumulant approximation gives a more accurate approximation than the vanishing third cumulant approximation for the full dynamical evolution of the average cavity photon number, including the first burst peak dynamics.

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[2] R. Kubo, J. Phys. Soc. Jpn.**17**, 1100 (1962).