Existence of solutions for a class of quasilinear elliptic equations involving the p-Laplacian

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Existence of solutions for a class of quasilinear elliptic equations
involving the \( p \)-Laplacian

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Abstract

This paper is concerned with the existence of solutions for the quasilinear elliptic equations

\[-\Delta_p u - \Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u + V(x)|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N,\]

where \( \alpha \geq 1, 1 < p < N, p^* = Np/(N-p) \), \( \Delta_p \) is the \( p \)-Laplace operator and the potential \( V(x) > 0 \) is a continuous function. In this work we mainly focus on nontrivial solutions. When \( 2\alpha p < q < p^* \), we establish the existence of nontrivial solutions by using Mountain-Pass lemma; when \( q \geq 2\alpha p^* \), by using a Pohozaev type variational identity, we prove that the equation has no nontrivial solutions.

1 Introduction and main results

This paper is concerned with the following type of quasilinear elliptic equations:

\[-\Delta_p u - \Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u + V(x)|u|^{p-2}u = |u|^{q-2}u \quad \text{in} \ \mathbb{R}^N, \tag{1.1}\]

where \( \alpha \geq 1, 1 < p < N, p^* = Np/(N-p) \), \( \Delta_p \) is the \( p \)-Laplace operator and the potential \( V : \mathbb{R}^N \to \mathbb{R} \) is a continuous function.

The equation (1.1) is a class of important elliptic equations and it arises in various branches of mathematical physics. For instance, when \( \alpha = 1 \) and \( p = 2 \), solutions of the equation (1.1) are standing waves of the following quasilinear Schrödinger equation

\[i\partial_t \Phi + \Delta \Phi + k\Delta h(|\Phi|^2)h'(|\Phi|^2)\Phi + h_1(|\Phi|^2)\Phi - W(x)\Phi = 0, \tag{1.2}\]

where \( h_1, h : \mathbb{R}^+ \to \mathbb{R} \) are real functions, \( W : \mathbb{R}^N \to \mathbb{R} \) is a given potential, \( \Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \) and \( k \) is a real constant. Corresponding to different forms of \( h \), the quasilinear equation (1.2) can describe very important models in mathematical physics and it have attracted interest from many scholars in the past decades. The semilinear case corresponding to \( k = 0 \) has been studied widely by many scholars (see [5, 17, 33] and references given there). Kurihara [21] utilized the special case \( h(s) = s \) in plasma physics for the superfluid film equation. When \( h(s) = (1+s)^{\frac{1}{2}} \), (1.2) can model the self-channeling of a high power ultra short laser in matter (see [8, 9, 11, 13, 31]). The equation (1.2) has also important applications in fluid mechanics and plasma physics (see references [21, 27]). It is also worth noting that, the equation (1.2) appears in Heisenberg ferromagnets and magnons theory (see references [4, 30, 34, 20]). Additionally, the equation (1.2) is also closely connected to the theory of dissipative quantum mechanics [18] and condensed matter [26].

Due to its importance, the equation (1.2) has drawn much attention from many mathematicians since 1970s. If we consider solutions of the form \( \Phi(x,t) = u(x)\exp(-i\omega t) \), the so-called standing

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wave solutions, then this type of solutions satisfy (1.2) with the condition \( h(s) = s \) if and only if \( u \) solves the following elliptic equation

\[
-\Delta u - \kappa (|u|^2) u + V(x) u = g(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}
\]

where \( V(x) = W(x) - \vartheta \) is a new potential function, \( \vartheta \) is a constant and \( g(u) = uh_1(u^2) \). In recent years, this type of problems has been extensively studied with suitable conditions of \( g \) and \( V \), including the compact case, radial symmetry, periodic case and potential well case \([6, 7, 12, 22]\). Liu and Wang studied the solution of (1.3) for \( k = 1/2 \) in \([22]\), where under certain conditions of \( V \) they reformulated the quasi-linear case to a semilinear problem through change of variables and studied the existence of positive solution in an Orlicz space. The same method is used by Colin and Jeanjean in \([12]\) to deal with the functions space \( H^1(\mathbb{R}^N) \). More precisely, they introduced \( u = h^{-1}(u) \) where \( h(t) \) satisfies:

\[
h'(t) = (1 + 2|h(t)|^2)^{-\frac{1}{2}} \text{ for } 0 \leq t < +\infty \text{ and } -h(t) = h(-t) \text{ for } -\infty < t < 0.
\]

They established the existence of non-trivial solutions by using classical variational methods and critical point theory. This method has widely used in the study of such kinds of quasilinear elliptic problems in the form (1.3) (see for instance \([2, 10, 14, 35]\) and the references given there).

For the nonexistence of nontrivial solutions, that if \( p \geq 2 \) \( 2^* = 4N/(N - 2) \), \( u \) satisfies the conditions \( u^2|\nabla u|^2 \in L^1(\mathbb{R}^N) \), \( g(u) = |u|^{p-2}u \) and \( \nabla V(x) \cdot x \geq 0 \) for all \( x \in \mathbb{R}^N \), by a variational identity given by Pucci and Serrin \([29]\) the equation (1.3) has no positive nontrivial solutions in \( H^1(\mathbb{R}^N) \) (see \([23]\)). When \( p = 2 \), the equation (1.1) has been studied in various cases. When \( \alpha = 1 \), existence of positive ground states of soliton solutions has been studied in \([23, 28]\), the existence of positive nontrivial solutions was established in \([15]\) by change of variables; for \( \alpha > 1/2 \), existence of positive ground states of soliton solutions has been obtained in \([24]\) through a constrained minimization argument, \([22]\) studied the existence of solutions for the following equation

\[
-\Delta u - \Delta(|u|^{2\alpha}) |u|^{2\alpha-2} u + V(x) u = \lambda |u|^{m-1} u \quad \text{in} \quad \mathbb{R}^N,
\]

under certain conditions on the potential \( V \), with \( \lambda > 0 \), \( 2(2\alpha) < m + 1 < 2\alpha 2^* \); when \( \alpha \in (1/2, 1] \), Wu \([35]\) studied the existence of solutions under certain conditions.

For the p-Laplace case with \( 1 < p \leq N \), Severo in \([32]\) established the existence of nontrivial weak solution to

\[
-\Delta_p u - \Delta_p(|u|^2) u + V(x)|u|^{p-2} u = h(u) \quad \text{in} \quad \mathbb{R}^N,
\]

under certain conditions of \( h(u) \) and potential well case on \( V \).

This paper was motivated by the above references, and our main purpose is to prove the existence and nonexistence of nontrivial solutions to problem (1.1) under certain conditions. More precisely, when \( 2 \alpha p < q < p^* \), we establish the existence of nontrivial solutions by variational approach; when \( q \geq 2\alpha p^* \), by using a Pohozaev type identity, we prove that the equation (1.1) has no nontrivial solutions.

Firstly, we give the definition of weak solution to (1.1).

**Definition 1.1.** \( u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \) is called a weak solution of (1.1) if for any \( \phi \in C_0^\infty(\mathbb{R}^N) \) there holds:

\[
\int_{\mathbb{R}^N} [1 + (2\alpha)^{p-1} |u|^{2(\alpha-1)p}] |\nabla u|^{p-2} \nabla u \nabla \phi \, dx
+ (2\alpha - 1)(2\alpha)^{p-1} \int_{\mathbb{R}^N} |\nabla u|^p |u|^{(2\alpha-1)p-2} u \phi \, dx
- \int_{\mathbb{R}^N} (|u|^{q-2} u - V(x)|u|^{p-2} u) \phi \, dx = 0. \tag{1.4}
\]

**Remark 1.1.** It is easy to see that the energy functional associated to (1.1) given by

\[
\varphi(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left( 1 + (2\alpha)^{p-1} |u|^{2(\alpha-1)p} \right) |\nabla u|^p \, dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q dx
\]

is not well defined generally in \( W^{1,p}(\mathbb{R}^N) \). In fact, suppose that \( 1 < p < N \) and \( u \in C_0^1(\mathbb{R}^N \setminus \{0\}) \) is defined as follows:

\[
u(x) = \frac{|x|^{\frac{4-N}{2p}}}{|x|^{\frac{N}{2}}}, \quad x \in B_1 \setminus \{0\}.
\]

Then, \( u \in W^{1,p}(\mathbb{R}^N) \), but \( \| |u|^{2(\alpha-1)} \nabla u \|_{L^p} = +\infty \).
To deal with this problem, we apply a method developed by Liu et al. [22], Colin-Jeanjean [12] for the case $p = 2, \alpha = 1$ and Severo [32] for the case $\alpha = 1, 1 < p \leq N$. More precisely, we set $w = h^{-1}(u)$, with $h$ satisfying
\begin{align*}
    \begin{cases}
        h'(t) = [1 + (2\alpha)^{p-1}]|h(t)|^{p(2\alpha - 1)\frac{1}{p} - 1}, & t \in [0, +\infty) \\
        h(0) = 0; & h(t) = -h(-t), \quad t \in (-\infty, 0].
    \end{cases}
\end{align*}
(1.5)
For any $w \in W^{1,p}(\mathbb{R}^N)$, let $u = h(w)$ and it follows that
\begin{align*}
    \varphi(u) &= \frac{1}{p} \int_{\mathbb{R}^N} \left(1 + (2\alpha)^{p-1}|h(w)|^{(2\alpha - 1)p}\right) |\nabla(h(w))|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p dx \\
    &\quad - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx \\
    &= \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla w|^p + V(x)|h(w)|^p) dx - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx.
\end{align*}
(1.6)
We set
\begin{align*}
    \psi(w) := \frac{1}{p} \left(\int_{\mathbb{R}^N} |\nabla w|^p dx + \int_{\mathbb{R}^N} V(x)|h(w)|^p dx\right) - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx.
\end{align*}
(1.6)
It is obvious that the functional $\psi$ is well-defined on $W^{1,p}(\mathbb{R}^N)$ under suitable assumptions on the potential $V$, which would be stated later. Furthermore, it is easy to check that $\psi \in C^1(W^{1,p}(\mathbb{R}^N), \mathbb{R})$ and for any $\Psi \in C_0^\infty(\mathbb{R}^N)$, we see that
\begin{align*}
    \langle \psi'(w), \Psi \rangle &= \int_{\mathbb{R}^N} |\nabla w|^{p-2}\nabla w \nabla \Psi dx + \int_{\mathbb{R}^N} V(x)|h(w)|^{p-2}h(w) \cdot h'(w) \cdot \Psi dx \\
    &\quad - \int_{\mathbb{R}^N} |h(w)|^{q-2}h(w) \cdot h'(w) \cdot \Psi dx.
\end{align*}
Hence, the existence of nontrivial solutions to (1.1) is equivalent to the following equation:
\begin{align*}
    -\Delta_p w &= [-V(x)|h(w)|^{p-2}h(w) + |h(w)|^{q-2}h(w)h'(w)](x) \quad x \in \mathbb{R}^N.
\end{align*}
(1.7)
In the following, we shall mainly focus on the study of (1.7). For the potential function $V$, we consider the two cases:
\begin{enumerate}
    \item[(V_1)] There exists $V_0$ such that $V(x) \geq V_0 > 0$ for any $x \in \mathbb{R}^N$.
    \item[(V_2)] $V(x) \to V_\infty$ as $|x| \to \infty$ and $V(x) \leq V_\infty < \infty$ for any $x \in \mathbb{R}^N$.
\end{enumerate}
For simplicity, in the following we denote $W^{1,p}(\mathbb{R}^N)$ by $X_p$.

Our main results are stated as follows:

**Theorem 1.1.** Let $\alpha \geq 1$, $1 < p < N$, $p^* = Np/(N - p)$ and $2\alpha p < q < p^*$. Assume that the assumptions (V_1) - (V_2) hold. Then the equation (1.1) has a nontrivial weak solution $u \in X_p$.

**Theorem 1.2.** Let $\alpha \geq 1$, $1 < p < N$, $p^* = Np/(N - p)$ and $q \geq 2\alpha p^*$. Assume that either (V_1) or (V_2) hold and $\nabla V(x) \cdot x \geq 0$ for any $x \in \mathbb{R}^N$. Then the equation (1.1) has no nontrivial solution $u \in X_p$.

**Remark 1.2.** The main difficulties to study (1.1) are lack of compactness due to the unboundedness of the domain and the nonlinearity due to the presence of the non-homogeneous expression $\Delta_p(|u|^{2\alpha})|u|^{2\alpha - 2}u$, which prevent us to work directly with the functional $\varphi$ defined previously in above. To deal with these problems, we change the variable $h(w) = u$ we reformulate our original problem into a new one, whose corresponding functional $\psi$ is well defined on $X_p$ and of $C^1(X_p, \mathbb{R})$, and also satisfies a geometric hypotheses of the Mountain–Pass theorem.

The remaining part of this paper is organized as follows. In Section 2, we first give some properties for the change of variables $h : \mathbb{R} \to \mathbb{R}$ defined in (1.5) and then establish some preliminary results. In Section 3, we present an auxiliary problem and some connected results. Section 4 is devoted to the proof of Theorem 1.1. In Section 5, we establish the non-existence of non-trivial solutions to (1.1) under certain conditions.
2 Preliminary results

This section will provide preliminary tools, which are important for establishing our main results. We firstly give some important properties of $h$.

Lemma 2.1. The function $h(t)$ enjoys the following properties:

(h1) $h$ is uniquely defined, $C^2$ and invertible,
(h2) $|h(t)| \leq |t|$, for all $t \in \mathbb{R}$;
(h3) $\frac{h(t)}{t} \to 1$ as $t \to 0$;
(h4) $|h(t)| \leq (2\alpha)^{\frac{p-1}{p}} |t|^\frac{1}{p}$, for all $t \in \mathbb{R}$;
(h5) $h(t) \leq 2\alpha \cdot t h'(t) \leq 2\alpha \cdot h(t)$, for all $t \in [0, \infty)$
(h6) $h(t)^{-\frac{1}{p}} \to a$ as $t \to +\infty$, $a > 0$.
(h7) There exists a positive constant $C > 0$ such that:

$$\begin{cases} |h(t)| \geq C|t|^\frac{1}{p}, & |t| \geq 1, \\ |h(t)| \geq C|t|, & |t| \leq 1. \end{cases}$$

Proof. The proof of this lemma can be found in [32] for $\alpha = 1$. The items (h1) – (h2) are immediate by the definition of $h$. To prove (h3), utilizing Mean Value Theorem for integrals, we have

$$h(t) = \int_0^t \left(1 + (2\alpha)^{p-1}|h(s)|^{(2\alpha-1)p}\right)^{-\frac{1}{p}} ds \quad = t \left(1 + (2\alpha)^{p-1}|h(\zeta)|^{(2\alpha-1)p}\right)^{-\frac{1}{p}},$$

where $0 < \zeta < t$. Since $h(0) = 0$, it follows that

$$\lim_{t \to 0} \frac{h(t)}{t} = \lim_{\zeta \to 0} \left[1 + (2\alpha)^{p-1}|h(\zeta)|^{(2\alpha-1)p}\right]^{-\frac{1}{p}} = 1.$$

Next, from $[1 + (2\alpha)^{p-1}|h(t)|^{(2\alpha-1)p}]^\frac{1}{p} h'(t) = 1$, for $t \in (0, +\infty)$, it may be concluded that

$$\int_0^t h'(s)\left[1 + (2\alpha)^{p-1}|h(s)|^{(2\alpha-1)p}\right]^\frac{1}{p} ds = t.$$

Utilizing the change of variables $z = h(s)$, we deduce that

$$t = \int_0^{h(t)} \left[1 + (2\alpha)^{p-1}|z|^{(2\alpha-1)p}\right]^\frac{1}{p} dz \geq (2\alpha)^{\frac{p-1}{p}} \frac{(h(t))^{2\alpha}}{2\alpha} = (2\alpha)^{-\frac{1}{p}} (h(t))^{2\alpha},$$

hence, for $t \in [0, +\infty)$. (h4) is proved. For $t \in (-\infty, 0)$, it is sufficient to use the fact that $h(t) = -h(-t)$. To establish the first inequality of (h5), we need to show that for all $t \geq 0$

$$h(t) \left(1 + (2\alpha)^{p-1}|h(t)|^{(2\alpha-1)p}\right)^\frac{1}{p} \leq 2\alpha t.$$

For this purpose, we study the function

$$g_1(t) = 2\alpha t - h(t)[1 + (2\alpha)^{p-1}|h(t)|^{(2\alpha-1)p}]^\frac{1}{p},$$

where $g_1 : \mathbb{R}^+ \to \mathbb{R}$, $g_1(0) = 0$, and

$$g'_1(t) = (2\alpha - 1)(h'(t))^p > 0,$$

so, $h(t)/2\alpha \leq th'(t)$ is proved. For the second inequality of (h5), we define

$$g_2(t) = t - h(t)[1 + (2\alpha)^{p-1}|h(t)|^{(2\alpha-1)p}]^\frac{1}{p},$$

where $g_2 : \mathbb{R}^+ \to \mathbb{R}$, $g_2(0) = 0$, and

$$g'_2(t) = -\frac{(2\alpha - 1)(2\alpha)^{p-1}|h(t)|^{(2\alpha-1)p}}{1 + (2\alpha)^{p-1}|h(t)|^{(2\alpha-1)p}} < 0.$$
Obviously, \( g_2(t) \leq g_2(0) = 0 \). This follows (h5) for all \( t \geq 0 \). Similar arguments apply to the case \( h(t) \leq th'(t) \leq h(t)/2\alpha \) for all \( t \in (-\infty, 0] \). Now, by (h3) it follows that \( \lim_{t \to 0^+} h(t)/t^{1/2\alpha} = 0 \). From inequality (h5), for any \( t > 0 \) we obtain

\[
\left( \frac{h(t)}{t^{1/2\alpha}} \right)' = \frac{2\alpha h'(t)t - h(t)}{2\alpha t^{1/2\alpha + 1}} \geq 0.
\]

Consequently, \( h(t)t^{-\alpha/2} \) is a nondecreasing function for any \( t \in (0, +\infty) \). Employing item (h4), we see that \( |h(t)|t^{-\alpha/2} \leq (2\alpha)^{\frac{\alpha}{2p}-\frac{1}{2}} \), this follows (h6). The property (h7) is a direct consequence of limits (h3) and (h6).

The following lemma relates the solutions of (1.7) to those of (1.1):

**Lemma 2.2.** Suppose that \( w \in X_p \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \) is a critical point of \( \psi \). Then \( h(w) = w \in X_p \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \) is a weak solution of the equation (1.1).

**Proof.** By preceding lemma we see that \( |w|^p \geq |h(w)|^p = |u|^p \) and \( |\nabla w|^p \geq |h'(w)\nabla w|^p = |\nabla u|^p \). Thus \( u \in X_p \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \). As \( w \) is a critical point of \( \psi \), we conclude that for any \( \Psi \in C^\infty_0(\mathbb{R}^N) \)

\[
\int_{\mathbb{R}^N} |\nabla w|^{p-2} \nabla w \nabla \Psi \, dx = \int_{\mathbb{R}^N} |h(w)|^{q-2} h(w) - V(x) |h(w)|^{p-2} h(w) h'(w) \cdot \Psi \, dx. \tag{2.1}
\]

Since \( (h^{-1})'(t) = \frac{1}{h'(h^{-1}(t))} \), it easily seen that

\[
(h^{-1})'(t) = [1 + (2\alpha)^{-1} |h(h^{-1}(t))|^{(2\alpha-1)p}]^{\frac{1}{2}} = [1 + (2\alpha)^{-1} |t|^{(2\alpha-1)p}]^{\frac{1}{2}}, \tag{2.2}
\]

and

\[
\nabla w = (h^{-1})'(u) \nabla u = [1 + (2\alpha)^{-1} |u|^{(2\alpha-1)p}]^{\frac{1}{2}} \nabla u. \tag{2.3}
\]

Moreover, for any \( \phi \in C^\infty_0(\mathbb{R}^N) \), we see that

\[
h'(w)^{-1} \phi = [1 + (2\alpha)^{-1} |u|^{(2\alpha-1)p}]^{\frac{1}{2}} \phi \in X_p. \tag{2.4}
\]

It follows that

\[
\nabla (h'(w)^{-1} \phi) = (2\alpha - 1)(2\alpha)^{-1} |u|^{(2\alpha-1)p} \nabla w + [1 + (2\alpha)^{-1} |u|^{(2\alpha-1)p}]^{\frac{1}{2}} \nabla u \phi.
\]

Setting \( \Psi = h'(w)^{-1} \phi \) in (2.1) and employing (2.3)–(2.4) we get (1.4), which is the desired conclusion. \( \square \)

The following two lemmas are proved in the corresponding references, which are extremely useful in the subsequent sections.

**Lemma 2.3.** [36, Lemma A.1] Let \( U \) be an open subset of \( \mathbb{R}^N \) and \( p \in [1, \infty) \). If \( v_n \to u \) in \( L^p(U) \), there exist a subsequence \( \{v_{n_k}\} \) of \( \{v_n\} \) and \( g \in L^p(U) \) such that, \( u_{n_k}(x) \to u(x) \) almost everywhere on \( U \) and

\[
|u(x)|, |u_n(x)| \leq g(x) \quad \text{a.e. on } U.
\]

**Lemma 2.4.** [25] Let \( p \in (1, \infty) \) and \( q \in [1, \infty) \) with \( q \neq p^* = Np/(N - p) \) if \( p < N \). Suppose that \( u_n \) is bounded in \( L^q(\mathbb{R}^N) \), \( \nabla u_n \) is bounded in \( L^p(\mathbb{R}^N) \) and for some \( R > 0 \) there holds:

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_n|^q \, dx = 0.
\]

Then \( u_n \to 0 \) as \( n \to \infty \) in \( L^\vartheta(\mathbb{R}^N) \) for \( \vartheta \in (q, p^*) \).

Before state the last lemmas in this section, we introduce the following definition:
Definition 2.1. Let $X$ be a real Banach space and $I : X \to \mathbb{R}$ a functional of class $C^1(X, \mathbb{R})$. A sequence $\{w_n\} \subset X$ is said to be Cerami sequence for the functional $I$ at the level $c > 0$ if the sequence $\{w_n\}$ satisfies:

$$I(w_n) \to c, \quad (1 + \|w_n\|_X)\|I'(w_n)\|_X \to 0, \quad \text{as } n \to \infty.$$ 

When each Cerami sequence contains a convergent subsequence in $X$, then the functional $I$ is said to satisfy the Cerami condition.

Lemma 2.5. Assume that the assumptions $(V_1) - (V_2)$ hold. Let $1 < p < N$ and $2\alpha p < q < p^*$. Then any Cerami sequence $\{w_n\}$ for the functional $\psi$ at the level $c > 0$ is bounded in $X_p$.

Proof. Let $\{w_n\} \subset X_p$ is a Cerami sequence for $\psi$ at the level $c > 0$, that is $\psi(w_n) \to c$ and $(1 + \|w_n\|_{X_p})\|\psi'(w_n)\|_{X_p} \to 0$ as $n \to \infty$.

In the following we show that $\{w_n\}$ is bounded in $X_p$. For this purpose, we first prove that

$$\int_{\mathbb{R}^N} |\nabla w_n|^p dx + \int_{\mathbb{R}^N} V(x)|h(w_n)|^p dx \leq C. \quad (2.5)$$

Since $\{w_n\}$ is a Cerami sequence in $X_p$, there exists a positive constant $c$ so that

$$\psi(w_n) = \frac{1}{p} \left( \int_{\mathbb{R}^N} |\nabla w_n|^p dx + \int_{\mathbb{R}^N} V(x)|h(w_n)|^p dx \right) - \frac{1}{q} \int_{\mathbb{R}^N} |h(w_n)|^q dx = c + o(1) \quad \text{as } n \to \infty,$$

and also $$(1 + \|w_n\|_{X_p})\|\psi'(w_n)\| = o(1) \quad \text{as } n \to \infty.$$ 

For any $\phi \in X_p$, we obtain

$$\psi'(w_n)\phi = \int_{\mathbb{R}^N} |\nabla w_n|^{p-2} \nabla w_n \nabla \phi dx + \int_{\mathbb{R}^N} V(x)h(w_n)|^{p-2}h(w_n)[1 + (2\alpha)^p - 1]|h(w_n)|^{p(2\alpha - 1)} - 1/p \phi dx \quad (2.6)$$

Choose $\phi_n := h(w_n)[h'(w_n)]^{1 - \frac{1}{p}} = \frac{1}{[1 + (2\alpha)^p - 1]|h(w_n)|^{(2\alpha - 1)p} - 1/p h(w_n)]$. Applying (h2) and (h5), we infer that $|\phi_n| \leq 2\alpha|w_n|$ and $|\nabla \phi_n| \leq 2\alpha|\nabla w_n|$, which show $\|\phi_n\|_{X_p} \leq 2\alpha\|w_n\|_{X_p}$. Then, letting $n \to \infty$ we see that

$$\langle \psi'(w_n), \phi_n \rangle \leq \|\phi_n\|_{X_p} \|\phi_n\|_{X_p} \leq C(1 + \|w_n\|_{X_p})\|\psi'(w_n)\| \to 0.$$ 

Set $\phi = \phi_n$ in (2.6), then we deduce that

$$a(1) = \langle \psi(w_n), \phi_n \rangle$$

$$= \int_{\mathbb{R}^N} \left( 1 + (2\alpha - 1)(2\alpha)^{p-1}|h(w_n)|^{p(2\alpha - 1)}[1 + (2\alpha)^p - 1]|h(w_n)|^{p(2\alpha - 1)} - 1/p \right) \nabla w_n|^p dx$$

$$+ \int_{\mathbb{R}^N} V(x)h(w_n)|^{p} dx - \int_{\mathbb{R}^N} h(w_n)|^q dx,$$

as $n \to \infty$. Since $q > 2\alpha p > p$, it follows that

$$c + a(1) = \psi(w_n) - \frac{1}{q} \langle \psi'(w_n), \phi_n \rangle$$

$$= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla w_n|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|h(w_n)|^p dx$$

$$- \frac{1}{q} \int_{\mathbb{R}^N} \left( 1 + (2\alpha - 1)(2\alpha)^{p-1}|h(w_n)|^{p(2\alpha - 1)}[1 + (2\alpha)^p - 1]|h(w_n)|^{p(2\alpha - 1)} - 1/p \right) \nabla w_n|^p dx \quad (2.7)$$

$$- \frac{1}{q} \int_{\mathbb{R}^N} V(x)|h(w_n)|^p dx$$

$$\geq \left( \frac{1}{p} - \frac{2\alpha}{q} \right) \int_{\mathbb{R}^N} |\nabla w_n|^p dx + \left( \frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^N} V(x)|h(w_n)|^p dx.$$
as $n \to \infty$. Thus, from (2.7) we deduce that
\[
\int_{\mathbb{R}^N} V(x)|h(w_n)|^p \, dx \quad \text{and} \quad \|\nabla w_n\|_p^p
\]
are bounded and so is $\int_{\mathbb{R}^N} |\nabla w_n|^p + \int_{\mathbb{R}^N} V(x)|h(w_n)|^p \, dx$, that is, (2.5) is satisfied. It remains to prove that $\int_{\mathbb{R}^N} |w_n|^p \, dx$ is bounded. Clearly, we have
\[
\int_{\mathbb{R}^N} |w_n|^p \, dx = \int_{U_n} |w_n|^p \, dx + \int_{(U_n)^c} |w_n|^p \, dx,
\]
where $U_n = \{x \in \mathbb{R}^N : |w_n(x)| > 1\}$ and $(U_n)^c = \mathbb{R}^N \setminus U_n$. Since $|h(t)| \geq C|t|$ for all $|t| \leq 1$, we see that
\[
\int_{(U_n)^c} |w_n|^p \, dx \leq \frac{1}{V_0 C^p} \int_{(U_n)^c} V(x)|h(w_n)|^p \, dx \leq \frac{1}{V_0 C^p} \int_{\mathbb{R}^N} V(x)|h(w_n)|^p \, dx \leq \frac{M}{V_0 C^p}.
\]
By the assumption (V2) and Sobolev inequality we infer that
\[
\int_{U_n} |w_n|^p \, dx \leq \int_{U_n} V(x)|w_n|^p \, dx \leq V_\infty \int_{U_n} |w_n|^p \, dx \leq \int_{\mathbb{R}^N} |w_n|^p \, dx \leq CV_\infty M^{p^*/p}.
\]
Combining the last two inequalities yields
\[
\int_{\mathbb{R}^N} |w_n|^p \, dx \leq CV_\infty M^{p^*/p} + \frac{M}{V_0 C^p}.
\]
Then the proof is complete.

Since by preceding lemma $\{w_n\} \subset X_p$ is a bounded Cerami sequence for $\psi$ at the level $c > 0$, there is a positive constant $N$ and $w \in X_p$, and a subsequence $\{w_{n_i}\} \subset \{w_n\}$ still denoted by $\{w_n\}$, such that $N \geq \|w\|_{X_p}$ and $N \geq \|w_n\|_{X_p}$. Moreover,
\[
\begin{aligned}
&w_n \rightharpoonup w \text{ weakly in } X_p, \\
&w_n \to w \text{ in } L^q_{\text{loc}}(\mathbb{R}^N), 1 \leq q < p^*, \\
&w_n(x) \to w(x) \quad \text{a.e. in } \mathbb{R}^N, \\
&h(w_n) \to h(w) \text{ in } L^q_{\text{loc}}(\mathbb{R}^N), 1 \leq q < p^*.
\end{aligned}
\tag{2.8}
\]

**Lemma 2.6.** Let $2\alpha p < q < 2\alpha p$ and $1 < p < N$. Suppose that the assumptions (V1) – (V2) hold. If the sequence $\{w_n\}$ is a Cerami sequence satisfying (2.8), then there holds:
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |h(w_n)|^{q-2} h(w_n) h'(w_n) \, w_n \, dx = \int_{\mathbb{R}^N} |h(w)|^{q-2} h(w) h'(w) \, w \, dx.
\tag{2.9}
\]
Furthermore, $\forall \epsilon > 0$, there exists $\rho_0 = \rho_0(\epsilon) \geq 1$ such that if $\rho \geq \rho_0$, there hold:
\[
\lim_{n \to \infty} \int_{B_2 \rho} \left[|\nabla w_n|^p + V|h(w_n)|^p \right] \, dx \leq \epsilon
\tag{2.10}
\]
\[
\lim_{n \to \infty} \int_{B_2 \rho} \left[|\nabla w_n|^p + V|h(w_n)|^{p-2} h(w_n) h'(w_n) \right] \, dx \leq \epsilon
\tag{2.11}
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V|h(w_n)|^p \, dx = \int_{\mathbb{R}^N} V|h(w)|^p \, dx,
\tag{2.12}
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V|h(w_n)|^{p-2} h(w_n) h'(w_n) w_n \, dx = \int_{\mathbb{R}^N} V|h(w)|^{p-2} h(w) h'(w) w \, dx.
\tag{2.13}
\]
1. **Proof.** Using (h5) of Lemma 2.1, for all \( n \in \mathbb{N} \) we deduce that
\[
|h(w_n)|^{q-2}h(w_n)h'(w_n)w_n | \leq |h(w_n)|^q, \quad x \in \mathbb{R}^N. \tag{2.14}
\]

2. From (h4) of Lemma 2.1 and Sobolev inequality, we infer that
\[
\iint_{\mathbb{R}^N} |h(w_n)|^q dx \leq \int_{\mathbb{R}^N} |h(w_n)|^{2\alpha_p^*} dx \leq (2\alpha_p^*)^\frac{q}{p} \int_{\mathbb{R}^N} |w_n|^{p'} dx \leq M, \tag{2.15}
\]
for some constant \( M > 0 \). Furthermore, since \( h(w_n) \rightarrow h(w) \) in \( L^q_{\text{loc}}(\mathbb{R}^N) \), going if necessary to a subsequence, we may assume that \( h(w_n) \rightarrow h(w) \) a.e. on \( \mathbb{R}^N \). Now, we claim that for any \( \epsilon > 0 \) there exists \( \rho_0 = \rho_0(\epsilon) > 0 \) so that \( \rho \geq \rho_0 \) and
\[
\iint_{(B_\rho)^c} |h(w_n)|^q dx \leq \frac{\epsilon}{2}.
\]
Indeed, let \( g(s) = |s|^{q-2} s \), we have \( g(s) = o(|s|^{p-1}) \) as \( s \rightarrow 0 \). Moreover, since \( q < 2\alpha_p^* \), we infer that \( \lim_{s \to \infty} |g(s)|/(|s|^{2\alpha_p^*} - 1) = 0 \). Thus, for each \( \epsilon > 0 \), there exist \( 0 < s_0 < s_1 \) such that
\[
|g(s)| \leq \epsilon |s|^{p-1} \quad \text{if} \quad |s| \leq s_0 \quad \text{and} \quad |g(s)| \leq \epsilon |s|^{2\alpha_p^* - 1} \quad \text{if} \quad |s| \geq s_1,
\]
which shows
\[
|g(s)| \leq \epsilon \left( |s|^p + |s|^{2\alpha_p^*} \right) + \chi_{[s_0, s_1]}(|s|)g(s) |s|, \quad \forall s \in \mathbb{R},
\]
where \( \chi \) denotes the characteristic function. We then obtain from the last inequality that, for all \( n \in \mathbb{N} \),
\[
|h(w_n)|^q \leq \epsilon \left( V(x)|h(w_n)|^p + |h(w_n)|^{2\alpha_p^*} \right) + \chi_{[s_0, s_1]}(|h(w_n)|)g(h(w_n))h(w_n),
\]
and
\[
\iint_{(B_\rho)^c} |h(w_n)|^q dx \leq \epsilon K(w_n) + \beta \int_{D_n \cap (B_\rho)^c} \chi_{[s_0, s_1]}(|h(w_n)|)dx,
\tag{2.16}
\]
where
\[
K(w_n) = \int_{\mathbb{R}^N} \left( V(x)|h(w_n)|^p + |h(w_n)|^{2\alpha_p^*} \right) dx,
\]
and
\[
D_n = \{ x \in \mathbb{R}^N \mid s_0 \leq |h(w_n(x))| \leq s_1 \}, \quad \beta = \max_{s_0 \leq |s| \leq s_1} |g(s)|.
\]
By Lemma 2.5 and (2.15) we have
\[
K(w_n) = \int_{\mathbb{R}^N} V(x)|h(w_n)|^p dx + \int_{\mathbb{R}^N} |h(w_n)|^{2\alpha_p^*} dx \leq 2M.
\]
In addition, the inequality (2.15) yields
\[
|s_0|^{2\alpha_p^*} |D_n| \leq \int_{D_n} |h(w_n)|^{2\alpha_p^*} dx \leq \int_{\mathbb{R}^N} |h(w_n)|^{2\alpha_p^*} dx \leq M, \quad \text{for all} \quad n \in \mathbb{N},
\]
which implies that
\[
\sup_{n \in \mathbb{N}} |D_n| \leq \frac{M}{|s_0|^{2\alpha_p^*}} < +\infty,
\]
where \( |D_n| = \text{meas}(D_n) \). As in the proof of Proposition 2.1 in [1], we can obtain
\[
\lim_{\rho \to \infty} |D_n \cap (B_\rho)^c| = 0, \quad \text{uniformly in} \quad n \in \mathbb{N}.
\]
Furthermore, by the absolute continuity of the Lebesgue integral, for any \( \epsilon > 0 \), there exists \( \rho_0 \geq 1 \) such that when \( \rho \geq \rho_0 \), there holds:
\[
|D_n \cap (B_\rho)^c| < \frac{\epsilon}{\beta^2}.
\]
which implies
\[
\int_{D_{n}(\rho_{0}) \cap (B_{\rho})^{c}} \chi_{[_{\rho_{0}}, \rho_{1}]}(|h(w_{n})|)dx < \frac{\epsilon}{\beta} \quad \text{for all } n \in \mathbb{N}.
\]

Then, by the last inequality above and (2.16), we see that if \( \rho \geq \rho_{0} \) there holds:
\[
\int_{(B_{\rho})^{c}} |h(w_{n})|^{q} \leq \epsilon(2M + 1), \quad \text{for all } n \in \mathbb{N},
\]

which is our claim. Next, it follows from (2.14) that
\[
\left| \int_{(B_{\rho})^{c}} |h(w_{n})|^{q-2} h(w_{n}) h'(w_{n}) w_{n} dx \right| \leq \frac{\epsilon}{2}, \quad (2.17)
\]

By Fatou’s lemma we see that
\[
\left| \int_{(B_{\rho})^{c}} |h(w)|^{q-2} h(w) h'(w) w dx \right| \leq \frac{\epsilon}{2}, \quad (2.18)
\]

On the other hand, since \( w_{n} \to w \) in \( L^{q}(B_{\rho}) \) and \( h(w_{n}) \to h(w) \) in \( L^{q}(B_{\rho}) \), by Lemma 2.3, up to a subsequence there exist two functions \( g_{1}, g_{2} \in L^{q}(B_{\rho}) \), in such away
\[
|w_{n}| \leq g_{1} \quad \text{a.e. on } B_{\rho} \quad \text{and} \quad |h(w_{n})| \leq g_{2} \quad \text{a.e. on } B_{\rho}.
\]

Therefore, we obtain
\[
\lim_{n \to \infty} \int_{B_{\rho}} |h(w_{n})|^{q-2} h(w_{n}) h'(w_{n}) w_{n} dx = \int_{B_{\rho}} |h(w)|^{q-2} h(w) h'(w) w dx, \quad (2.19)
\]

by Lebesgue Dominated Convergence Theorem. Thus, from (2.17) – (2.19) we get (2.9).

Next, to prove (2.10), using the idea in reference [2] we choose the function \( \xi = \xi_{\rho} \in C^{1}(\mathbb{R}^{N}) \), with \( \rho > 1 \) such that
\[
\xi_{\rho} = \begin{cases} 
0, & \text{in } B_{\rho}^{c} \\
1, & \text{in } (B_{2\rho})^{c}
\end{cases} \quad \text{and} \quad \xi_{\rho} \in [0, 1], \ |\nabla \xi_{\rho}| \leq 2\rho^{-1} \quad \text{in } \mathbb{R}^{N}.
\]

Choose \( \Phi_{n} = h(w_{n})|h'(w_{n})|^{-1} \), since \( |\Phi_{n}| \leq |w_{n}| \) and \( \{w_{n}\} \) is a bounded Cerami sequence in \( X_{p} \), we deduce that the sequence \( \{\xi_{\rho} \Phi_{n}\} \) is bounded in \( X_{p} \) as well. Thus, from \( \langle \psi'(w_{n}), \xi_{\rho} \Phi_{n} \rangle = o(1) \) as \( n \to \infty \), we have
\[
\int_{\mathbb{R}^{N}} \left( (2\alpha - 1)(2\alpha)^{-1} |h(w_{n})|^{(2\alpha - 1)p}[1 + (2\alpha)^{-1} |h(w_{n})|^{(2\alpha - 1)p}]^{-1} \right) |\nabla w_{n}|^{p} \xi_{\rho} dx
+ \int_{\mathbb{R}^{N}} |\nabla w_{n}|^{p} \xi_{\rho} dx + \int_{\mathbb{R}^{N}} |\nabla w_{n}|^{p-2} \nabla w_{n} \nabla \xi_{\rho} \Phi_{n} dx
= \int_{\mathbb{R}^{N}} |h(w_{n})|^{p} \xi_{\rho} dx + o(1).
\]

From (2.17) we see that
\[
\int_{\mathbb{R}^{N}} |h(w_{n})|^{q} \xi_{\rho} dx < \frac{\epsilon}{2}, \quad (2.21)
\]

for large \( \rho \). Accordingly, from (2.20) – (2.21) with the definition of \( \xi_{\rho} \), it may be concluded that
\[
\int_{(B_{\rho})^{c}} \left[ |\nabla w_{n}|^{p} + V |h(w_{n})|^{p} \right] \xi_{\rho} dx
\]
\[
\leq \int_{U_{\rho}} |\nabla w_{n}|^{p-2} |\nabla w_{n}||\nabla \xi_{\rho}| |w_{n}| dx + \int_{\mathbb{R}^{N}} |h(w_{n})|^{q} \xi_{\rho} dx + o(1)
\]
\[
\leq 2\rho^{-1} \int_{U_{\rho}} |\nabla w_{n}|^{p-1} |w_{n}| dx + \frac{\epsilon}{2} + o(1).
\]

9
as \( n \to \infty \), where \( U_\rho = B_{2\rho} \setminus B_\rho \). First we have by the Hölder inequality

\[
\int_{U_\rho} |\nabla w_n|^{p-1}|w_n|dx \leq \left[ \int_{U_\rho} |w_n|^p dx \right]^\frac{p-1}{p} \left[ \int_{U_\rho} |\nabla w_n|^p dx \right]^\frac{1}{p} \leq \|w_n\|_{X_\rho}^{-1} \left[ \int_{U_\rho} |w_n|^p dx \right]^\frac{1}{p}.
\]

Since the sequence \( \{w_n\} \) is bounded in \( X_\rho \) and \( w_n \to w \) in \( L^p(U_\rho) \), we see that

\[
\lim_{n \to \infty} \int_{U_\rho} |\nabla w_n|^{p-1}|w_n|dx \leq \Lambda^{p-1} \left[ \int_{U_\rho} |w|^p dx \right]^\frac{1}{p}. \tag{2.23}
\]

On the other hand, from (2.22) and (2.23), it follows that

\[
\lim_{n \to \infty} \int_{(B_{2\rho})^c} \left[ |\nabla w_n|^p + V|\xi|^{p-1} \right] \left( |w_n|^p + V|w|^{p-1} \right) dx \leq 2\rho^{-1}\Lambda^{p-1} \left[ \int_{U_\rho} |w|^p dx \right]^\frac{1}{p} + \epsilon^2.
\]

Therefore, since \( |w|^p \in L^1(\mathbb{R}^N) \), there is \( \rho_0 = \rho_0(\epsilon) \geq 1 \) such that if \( \rho \geq \rho_0 \), there holds

\[
2\rho^{-1}\Lambda^{p-1} \left[ \int_{U_\rho} |w|^p dx \right]^\frac{1}{p} < \frac{\epsilon}{2}.
\]

This estimate gives (2.10). Next, from (2.10) we conclude that

\[
\lim_{n \to \infty} \int_{(B_{2\rho})^c} V|h(w_n)|^p dx \leq \epsilon \quad \text{and} \quad \int_{(B_{2\rho})^c} V|h(w)|^p dx \leq \epsilon,
\]

it follows that

\[
\left| \int_{\mathbb{R}^N} (V|h(w_n)|^p - V|h(w)|^p) dx \right| \leq \epsilon + \left| \int_{B_{2\rho}} (V|h(w_n)|^p - V|h(w)|^p) dx \right|. \tag{2.24}
\]

In addition, as \( w_n \to w \) in \( L^p(B_{2\rho}) \) and \( h(w_n) \to h(w) \) in \( L^p(B_{2\rho}) \), by Lebesgue Dominated Convergence Theorem and Lemma 2.3, we get

\[
\lim_{n \to \infty} \int_{B_{2\rho}} V|h(w_n)|^p dx - \int_{B_{2\rho}} V|h(w)|^p dx = 0. \tag{2.25}
\]

Thereby, from (2.24) and (2.25) it follows easily that

\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^N} (V|h(w_n)|^p - V|h(w)|^p) dx \right| \leq 2\epsilon,
\]

and this gives (2.12). We next show that (2.13) holds. Employing (h5) of Lemma 2.1 and (2.21) yields

\[
\int_{\mathbb{R}^N} |h(w_n)|^{q-2} h(w_n) h'(w_n) w_n \xi^p dx \leq \int_{\mathbb{R}^N} |h(w_n)|^q \xi^p dx < \frac{\epsilon}{2}, \tag{2.26}
\]

for large \( \rho \). Moreover, from \( \langle \psi'(w_n), \xi^p \rangle = o(1) \), as \( n \to \infty \) we have

\[
\int_{\mathbb{R}^N} |\nabla w_n|^p \xi^p dx + \int_{\mathbb{R}^N} V|h(w_n)|^{p-2} h(w_n) h'(w_n) w_n \xi^p dx \\
+ \int_{\mathbb{R}^N} |\nabla w_n|^p \nabla w_n \nabla \xi^p w_n dx - \int_{\mathbb{R}^N} |h(w_n)|^{q-2} h(w_n) h'(w_n) w_n \xi^p dx \tag{2.27}
= o(1).
\]

Once that \( |\nabla \xi^p| \leq 2\rho^{-1} \) in \( \mathbb{R}^N \), from (2.26) – (2.27) together with the Hölder inequality and analysis similar to that in the proof of (2.10) show that

\[
\int_{(B_{\rho})^c} \xi^p \left( |\nabla w_n|^p dx + V|h(w_n)|^{p-2} h(w_n) h'(w_n) w_n dx \right) dx \\
\leq 2\rho^{-1} \int_{U_\rho} |\nabla w_n|^{p-1}|w_n| dx + \int_{\mathbb{R}^N} |h(w_n)|^q \xi^p dx + o(1) \\
\leq 2\rho^{-1} \|w_n\|_{L^p(\mathbb{R}^N)} \|w_n\|_{L^p(U_\rho)} + \int_{\mathbb{R}^N} |h(w_n)|^q \xi^p dx + o(1) \\
\leq 2\rho^{-1}\Lambda^{p-1} + \frac{\epsilon}{2} + o(1), \quad \text{as} \quad n \to \infty
\]
for large $\rho$. Thus, for each $\epsilon > 0$ there is a constant $\rho_0 \geq 1$ such that if $\rho \geq \rho_0$,
\[
\lim_{n \to \infty} \int_{(B_{2\rho})^c} \left( |\nabla w_n|^p + V|h(w_n)|^{p-2}h(w_n)h'(w_n)w_n \right) dx \leq \epsilon.
\]
3 From this we conclude that
\[
\lim_{n \to \infty} \int_{(B_{2\rho})^c} V|h(w_n)|^{p-2}h(w_n)h'(w_n)w_n dx \leq \epsilon,
\]
which implies
\[
\int_{(B_{2\rho})^c} V|h(w)|^{p-2}h(w)h'(w)wdx \leq \epsilon.
\]
4 As $w_n \to w$ in $L^p(B_{2\rho})$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$, we conclude that
\[
\lim_{n \to \infty} \int_{B_{2\rho}} V|h(w_n)|^{p-2}h(w_n)h'(w_n)w_n dx = \int_{B_{2\rho}} V|h(w)|^{p-2}h(w)h'(w)wdx,
\]
by Dominated Convergence Theorem and Lemma 2.3. Consequently, we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |V|h(w_n)|^{p-2}h(w_n)h'(w_n)w_n - V|h(w)|^{p-2}h(w)h'(w)w|dx \leq 2\epsilon.
\]
Therefore, (2.13) is satisfied, and the proof is complete. 

3 Auxiliary problem

To prove Theorem 1.1 we shall use some classical results due to L. Jeanjean-K.Tanaka in [19] and Do O'-Medeiros [16] for the following elliptic equation
\[
-\Delta_{\rho}u = f(u) \quad \text{in} \ \mathbb{R}^N. \tag{3.1}
\]
The functional associated to the equation (3.1) is defined by
\[
J(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} F(u) dx,
\]
where $F(s) := \int_0^s f(t)dt$. It is shown in [16] that under the assumptions (F1) – (F3) below, the functional $J$ is well-defined on $X_p$ and $J \in C^1(X_p, \mathbb{R})$. The authors in [16] (see also [19] for $p = 2$) consider the following assumptions on $f(s)$:
(F1) $f \in C(\mathbb{R}, \mathbb{R})$ is continuous and odd;
(F2) For $1 < p < N$, $-\infty < \liminf_{s \to 0^+} \frac{f(s)}{sp^{p-1}} \leq \limsup_{s \to 0^+} \frac{f(s)}{sp^{p-1}} = -\nu < 0$;
(F3) When $1 < p < N$ there holds
\[
\lim_{s \to +\infty} \frac{f(s)}{sp^{p-1}} = 0, \quad p^* = \frac{Np}{N-p};
\]
(F4) There exists a constant $\eta > 0$ such that $F(\eta) > 0$.
Recalling that a solution $w(x) \in X_p$ of the equation (3.1) is called a least energy solution if and only if $J(w) = \mu$, where
\[
\mu := \inf \{J(u) : u \in X_p \setminus \{0\} \} \quad \text{is a solution of the equation \ (3.1)}. \tag{3.2}
\]
By a ground state solution (or least energy) of (3.1) we mean a minimizer of $\mu$. Hence, if $w$ is any nontrivial solution of the equation (3.1) and $v$ is a minimizer of (3.2) then there holds, $J(v) \leq J(w)$.
The following results are proved in the corresponding references.
Theorem 3.1. [16, Theorem 1.4] (see also [5]) Under the assumptions $(F1) - (F4)$ with $1 < p < N$, problem (3.1) has a least energy solution $v$ which is positive, that is, for all $x \in \mathbb{R}^N$, $v(x) > 0$.

Theorem 3.2. [16, Theorem 1.8] (see also [19]) Let $(F1) - (F4)$ and assume $1 < p < N$. Setting
\[ \Lambda = \{ \gamma \in C([0, 1], X_p) : \gamma(0) = 0, J(\gamma(1)) < 0 \} \]
and
\[ \inf_{\gamma \in \Lambda} \max_{t \in [0, 1]} J(\gamma(t)) = b, \]
then $\Lambda \neq \emptyset$ and $\mu = b$, where $\mu > 0$ is defined in (3.2). That is, the Mountain-Pass value gives the least energy level. Moreover, for any least energy solution $v$ of (3.1), there exists a path $\gamma \in \Lambda$ such that $v \in \gamma([0, 1])$ and
\[ J(v) = \max_{t \in [0, 1]} J(\gamma(t)). \]

Remark 3.1. In [16] for $1 < p < N$ and in [19] for $p = 2$, it was also proved that under the assumptions $(F1) - (F3)$ there exist $\beta > 0, \delta > 0$ such that
\[ J(w) \geq \beta \|w\|_{X_p}^p \quad \text{if} \quad \|w\|_{X_p} \leq \delta. \]

4 Proof of Theorem 1.1

To prove Theorem 1.1 we first show that the energy functional $\psi$ has the Mountain-Pass Geometry, in the sense that $\Gamma \neq \emptyset$, and the Mountain-Pass value is defined by
\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \psi(\gamma(t)) > 0 \]
where
\[ \Gamma = \{ \gamma \in C([0, 1], X_p) : \gamma(0) = 0, \psi(\gamma(1)) < 0 \}. \]

For this purpose, we utilize some results relevant to the auxiliary problem.

Lemma 4.1. Let $\alpha > 1$ and $1 < p < N$. Under the assumptions $(V_1) - (V_2)$ and $2\alpha p < q < p^*$, the energy functional $\psi$ has the Mountain-Pass Geometry.

Proof. Let the functional corresponding to the equation $-\Delta_p w = f_0(w)$
\[ \psi_0(w) = \frac{1}{p} \left( \int_{\mathbb{R}^N} |\nabla w|^p dx + \int_{\mathbb{R}^N} V_0|h(w)|^p dx \right) - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx \]
and the functional corresponding to the equation $-\Delta_p w = f_\infty(w)$ be
\[ \psi_\infty(w) = \frac{1}{p} \left( \int_{\mathbb{R}^N} |\nabla w|^p dx + \int_{\mathbb{R}^N} V_\infty|h(w)|^p dx \right) - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx, \]
where the nonlinearities $f_\infty, f_0$ are defined as follows:
\[ f_\infty(w) := |h(w)|^{p-2}h(w)h'(w) - V_\infty|h(w)|^{p-2}h(w)h'(w), \]
and
\[ f_0(w) := |h(w)|^{p-2}h(w)h'(w) - V_0|h(w)|^{p-2}h(w)h'(w). \]

By the assumptions $(V_1) - (V_2)$, it is obvious that $\psi_0(w) \leq \psi(w) \leq \psi_\infty(w)$ for all $w \in X_p$. Moreover, it is easy to check that the nonlinearities $f_0$ and $f_\infty$ satisfy the hypotheses $(F1) - (F3)$. Hence, by Remark 3.1 we deduce that there exist $\beta > 0$ and $\delta > 0$ such that
\[ \psi(w) \geq \psi_0(w) \geq \beta \quad \text{if} \quad \|w\|_{X_p} \leq \delta. \]
To be more precise,
\[
\psi_0(w) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla w|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V_0|h(w)|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx \\
\geq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla w|^p dx + \frac{C}{p} \int_{\mathbb{R}^N} V_0|w|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx \\
\geq \frac{C_1}{p} \|w\|_{X_p}^p - \frac{C_2}{q} \|w\|_{X_p}^q,
\]
by Lemma 2.1 and Sobolev embedding theorem. Since \( q > p \), for \( \|w\|_{X_p} = \delta \) small enough we conclude that
\[
\psi_0(w) \geq \frac{C_1}{p} \delta^p - \frac{C_2}{q} \delta^q = \beta > 0.
\]
Furthermore, as \( f_\infty \) satisfies (F1) – (F3), employing Theorem 3.2 to the functional \( \psi_\infty \) we deduce that there exists \( c \in X_p \) with \( \|c\| > \delta \) so that \( \psi_\infty(c) < 0 \), which shows that \( \psi(c) < 0 \). That is, for \( e := tw, t > 0 \), we infer that
\[
\psi_\infty(tw) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla w|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V_\infty|h(tw)|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} |h(tw)|^q dx \\
\leq \frac{1}{p} \|w\|^p - \frac{C}{q} t^\alpha \int_{\mathbb{R}^N} |w| \frac{1}{p} dx,
\]
by the Lemma 2.1 and Sobolev embedding theorem. Since \( q/2\alpha > p \), we have
\[
\psi_\infty(tw) \to -\infty, \; t \to +\infty.
\]
Therefore, by Theorem 3.2 we see that \( \Gamma \neq \emptyset \), where
\[
\Gamma = \{ \gamma \in C([0, 1], X_p) : \gamma(0) = 0, \psi(\gamma(1)) < 0 \}.
\]
This completes the proof. \( \square \)

**Remark 4.1.** We recall that \( \lim_{s \to +\infty} s^{q^*/q} = \infty \) since \( p < q \). Thus, there is a constant \( \eta > 0 \) so that \( F_\infty(\eta) > 0 \) and \( F_0(\eta) > 0 \), where
\[
\int_0^s f_\infty(t)dt = F_\infty(s) = -\frac{1}{p} V_\infty|h(s)|^p + \frac{1}{q} |h(s)|^q
\]
and
\[
\int_0^s f_0(s)dt = F_0(s) = -\frac{1}{p} V_0|h(s)|^p + \frac{1}{q} |h(s)|^q.
\]
Hence, \( f_0 \) and \( f_\infty \) satisfy (F4) as well. Since the nonlinearities \( f_0 \) and \( f_\infty \) satisfy the hypotheses (F1) – (F4), thus, as a consequence of Theorem 3.1 the equations
\[
-\Delta_p w = f_0(w) \; \text{in} \; \mathbb{R}^N \; \text{and} \; -\Delta_p w = f_\infty(w) \; \text{in} \; \mathbb{R}^N
\]
have least energy solutions in \( X_p \), which are positive.

**Lemma 4.2.** Suppose that \( (V_1) - (V_2) \) and \( 2\alpha p < q < p^* \) hold. Let \( \{w_n\} \subset X_p \) is a Cerami sequence for \( \psi \) at the level \( c > 0 \) satisfying (2.8). Then, passing to a subsequence, \( w_n \to w \) in \( X_p \) and \( w \neq 0 \) is a critical point of \( \psi \), that is \( \psi'(w) = 0 \).

**Proof.** Since \( \{w_n\} \) is bounded in \( X_p \), we can assume, going if necessary to a subsequence \( w_n \to w \) in \( X_p \). To prove the weak limit \( w \) is a critical point of \( \psi \), we only need to show that \( \psi'(w) = 0 \).
Since the set \( C_{00}^\infty(\mathbb{R}^N) \) is dense in \( X_p \), therefore it is sufficient to prove that \( \langle \psi'(w), \Psi \rangle = 0 \) for any \( \Psi \in C_{00}^\infty(\mathbb{R}^N) \). It is easily seen that
\[
\langle \psi'(w_n), \Psi \rangle - \langle \psi'(w), \Psi \rangle = \int_{\mathbb{R}^N} \left[ |\nabla w_n|^{p-2} \nabla w_n - |\nabla w|^{p-2} \nabla w \right] \nabla \Psi dx \\
+ \int_{\mathbb{R}^N} \left[ V(x)|h(w_n)|^{p-2} h(w_n)h'(w_n) - V(x)|h(w)|^{p-2} h(w)h'(w) \right] \Psi dx \\
+ \int_{\mathbb{R}^N} \left[ |h(w)|^{q-2} h(w)h'(w) - |h(w_n)|^{q-2} h(w_n)h'(w_n) \right] \Psi dx.
\]
Furthermore, by similar argument as that in the proof of Lemma 2.6 it shows that

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} V(x) h(w_n) |h(w_n)|^{p-2} h(w_n) h'(w_n) \Psi dx = \int_{\mathbb{R}^N} V(x) |h(w)|^{p-2} h(w) h'(w) \Psi dx, \quad \text{(4.1)} \]

and

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} |h(w_n)|^{q-2} h(w_n) h'(w_n) \Psi dx = \int_{\mathbb{R}^N} |h(w)|^{q-2} h(w) h'(w) \Psi dx. \quad \text{(4.2)} \]

From (2.8) it may be concluded that

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla w_n|^{p-2} \nabla w_n \nabla \Psi dx = \int_{\mathbb{R}^N} |\nabla w|^{p-2} \nabla w \nabla \Psi dx \quad \text{(4.3)} \]

for any \( \Psi \in C_0^\infty(\mathbb{R}^N) \). Therefore, from (4.1) – (4.3) and Lebesgue Dominated convergence theorem it infers that

\[ \lim_{n \to \infty} \langle \psi'(w_n), \Psi \rangle - \langle \psi'(w), \Psi \rangle = 0, \]

for any \( \Psi \in C_0^\infty(\mathbb{R}^N) \). Thus, we have \( \langle \psi'(w), \Psi \rangle = 0 \) since \( \psi'(w_n) \to 0 \). That is, \( \psi'(w) = 0 \) and \( w \) is a critical point of \( \psi \).

We next prove that \( w \) is nontrivial. On the contrary, suppose that the claim is not true, that is \( w = 0 \). For convenience, the proof will be divided into three steps:

**Step 1:** Using the idea in [12], we claim that \( \{ w_n \} \) is also a Cerami sequence for the functional \( \psi_{\infty} \), at the level \( c > 0 \), that is

\[ \psi_{\infty}(w_n) \to c, \quad \| \psi'_{\infty}(w_n) \| \to 0, \quad \text{as} \quad n \to \infty. \]

Indeed, applying \( V(x) \to V_{\infty} \) as \( |x| \to \infty \) and (h2) of Lemma 2.1 we see that

\[ \psi_{\infty}(w_n) - \psi(w_n) = \frac{1}{p} \int_{\mathbb{R}^N} |h(w_n)|^p (V_{\infty} - V(x)) dx \leq \frac{1}{p} \int_{\mathbb{R}^N} |w_n|^p (V_{\infty} - V(x)) dx \to 0, \]

since \( w_n \to 0 \) in \( L^p_{\text{loc}}(\mathbb{R}^N) \) as \( n \to +\infty \). Furthermore, for the similar arguments as above, we infer that

\[ \| \psi'_{\infty}(w_n) - \psi'(w_n) \| \leq \sup \| \psi'_{\infty}(w_n) - \psi'(w_n) \| \leq \sup \int_{\mathbb{R}^N} |\nabla \psi_{\infty}(w_n)| dx \leq \left[ \int_{\mathbb{R}^N} |\nabla \psi_{\infty}(w_n)|^{p'} dx \right]^{\frac{1}{p-1}} \to 0 \quad \text{as} \quad n \to \infty. \]

Here, \( \| \cdot \| \) denotes the norm in \( L^p(\mathbb{R}^N) \). From the above it follows that

\[ \| \psi'_{\infty}(w_n) \| \to 0, \quad \text{as} \quad n \to +\infty, \]

which shows that \( \{ w_n \} \) is a Cerami sequence of the functional \( \psi_{\infty} \).

**Step 2:** We claim that, there exist two positive constants \( \delta, R \) and \( \{ y_n \} \) in \( \mathbb{R}^N \) such that

\[ \lim_{n \to \infty} \int_{B_R(y_n)} |w_n|^p dx \geq \delta > 0. \quad \text{(4.4)} \]

On the contrary, suppose that the claim is not true, that is

\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |w_n|^p dx = 0, \quad \text{for all} \quad R > 0. \]

By Lemma 2.4, we see that \( w_n \to 0 \) in \( L^q(\mathbb{R}^N) \) for \( p < q < p^* \) and \( 1 < p < N \). Thus, it is easily seen from (h2) and (h5) of Lemma 2.1 that

\[ \int_{\mathbb{R}^N} |h(w_n)|^{q-2} h(w_n) h'(w_n) w_n dx \leq \int_{\mathbb{R}^N} |h(w_n)|^q dx \leq \int_{\mathbb{R}^N} |w_n|^q dx \to 0, \quad \text{(4.5)} \]
as \( n \to +\infty \). On the other hand, since \( \psi'(w_n)w_n \to 0 \) as \( n \to +\infty \), we have
\[
\int_{\mathbb{R}^N} \left[ |\nabla w_n|^p + V(x)|h(w_n)|^{p-2}h(w_n)h'(w_n)w_n \right] dx \\
= \int_{\mathbb{R}^N} |h(w_n)|^{q-2}h(w_n)h'(w_n)w_n dx \to 0,
\]
which together with (4.5) shows that
\[
\int_{\mathbb{R}^N} \left[ |\nabla w_n|^p + V(x)|h(w_n)|^{p-2}h(w_n)h'(w_n)w_n \right] dx \to 0, \quad \text{as } n \to +\infty.
\]
Applying (h5) of Lemma 2.1 and (2.12) to the latest statement above, there holds
\[
\int_{\mathbb{R}^N} \left[ |\nabla w_n|^p + V(x)|h(w_n)|^p \right] dx \to 0, \quad \text{as } n \to +\infty,
\]
which implies
\[
\psi(w_n) = \frac{1}{p} \int_{\mathbb{R}^N} \left[ |\nabla w_n|^p + V(x)|h(w_n)|^p \right] dx - \frac{1}{q} \int_{\mathbb{R}^N} |h(w_n)|^q dx \to 0, \quad \text{as } n \to +\infty.
\]
This contradicts the fact that \( \psi(w_n) \to c > 0 \). Therefore, we get the claim.

**Step 3:** Let \( \tilde{w}_n(x) = w_n(x + y_n) \). Since \( \{ w_n \} \) is a Cerami sequence for \( \psi_\infty \), it is clear that \( \{ \tilde{w}_n \} \) is a Cerami sequence for \( \psi_\infty \) as well. By a similar argument as in the case \( \{ w_n \} \), going if necessary to a subsequence, we may assume \( \tilde{w}_n \to \tilde{w} \) in \( X_p \) with \( \psi_\infty(\tilde{w}) = 0 \). As the sequence \( \{ \tilde{w}_n \} \) is non-vanishing we also see that \( \tilde{w} \neq 0 \). That is, as in the proof of Step 2, there exist \( \delta > 0, R > 0 \) and \( \{ y_n \} \subset \mathbb{R}^N \) so that
\[
\int_{B_R} |\tilde{w}|^p dx = \lim_{n \to \infty} \int_{B_R} |\tilde{w}_n|^p dx = \lim_{n \to \infty} \int_{B_R(y_n)} |w_n|^p dx \geq \delta > 0,
\]
since \( \tilde{w}_n \to \tilde{w} \) in \( L^p(B_R) \). From Lemma 2.1(h5), for all \( n \in \mathbb{N} \) we have
\[
|h(\tilde{w}_n)|^p \geq |h(\tilde{w}_n)|^{p-2}h(\tilde{w}_n)h'(\tilde{w}_n)\tilde{w}_n, \quad x \in \mathbb{R}^N.
\]
Additionally, since \( q > 2\alpha p \) from the item (h5) of Lemma 2.1 we deduce that
\[
\frac{1}{p} |h(\tilde{w}_n)|^{q-2}h(\tilde{w}_n)h'(\tilde{w}_n)\tilde{w}_n \geq \frac{1}{2\alpha p} |h(\tilde{w}_n)|^q \geq \frac{1}{q} |h(\tilde{w}_n)|^q.
\]
Since \( \{ \tilde{w}_n \} \) is a Cerami sequence for \( \psi_\infty \), hence from Fatou’s lemma we have
\[
c = \lim_{n \to \infty} \left( \psi_\infty(\tilde{w}_n) - \frac{1}{p} \psi_\infty'(\tilde{w}_n)\tilde{w}_n \right) \\
= \lim_{n \to \infty} \int_{\mathbb{R}^N} \left( \frac{V_\infty}{p} |h(\tilde{w}_n)|^p - \frac{V_\infty}{p} |h(\tilde{w}_n)|^{p-2}h(\tilde{w}_n)h'(\tilde{w}_n)\tilde{w}_n \right) dx \\
+ \lim_{n \to \infty} \int_{\mathbb{R}^N} \left( \frac{1}{p} |h(\tilde{w}_n)|^{q-2}h(\tilde{w}_n)h'(\tilde{w}_n)\tilde{w}_n - \frac{1}{q} |h(\tilde{w}_n)|^q \right) dx \\
\geq \int_{\mathbb{R}^N} \left( \frac{V_\infty}{p} |h(\tilde{w})|^p - \frac{V_\infty}{p} |h(\tilde{w})|^{p-2}h(\tilde{w})h'(\tilde{w})\tilde{w} \right) dx \\
+ \int_{\mathbb{R}^N} \left( \frac{1}{p} |h(\tilde{w})|^{q-2}h(\tilde{w})h'(\tilde{w})\tilde{w} - \frac{1}{q} |h(\tilde{w})|^q \right) dx \\
= \psi_\infty(\tilde{w}) - \frac{1}{p} \psi_\infty'(\tilde{w})\tilde{w} = \psi_\infty(\tilde{w}).
\]
Consequently, \( \tilde{w} \) is a non-zero critical point of \( \psi_\infty \), which satisfies \( c \geq \psi_\infty(\tilde{w}) \). Thus, we see that \( \mu_\infty \leq c \), where
\[
\mu_\infty := \inf \{ \psi_\infty(w) : w \in X_p \setminus \{0\} \} \text{ is a solution of the equation } -\Delta_p w = f_\infty(w).
\]
Let us denote by $\tilde{v}$ a least energy solution of the equation $-\Delta_p w = f_\infty(w)$ (see Remark 4.1 for more details). Utilizing Theorem 3.2 to the functional $\psi_\infty$, we can find a path $\gamma(t) \in C([0,1], X_p)$ so that

$$
\psi_\infty(\gamma(1)) < 0, \gamma(0) = 0, \tilde{v} \in \gamma([0,1]) \quad \text{and}
$$

$$
\max_{0 \leq t \leq 1} \psi_\infty(\gamma(t)) = \psi_\infty(\tilde{v}).
$$

In $(V_2)$, without loss of generality we may assume that $V(x) \not= V_\infty$. Thus, for all $w \in X_p$ and $w \not= 0$, it is easily seen that

$$
\psi(w) < \psi_\infty(w),
$$

which shows

$$
\psi(\gamma(t)) < \psi_\infty(\gamma(t)), \quad \text{for all} \quad t \in (0,1].
$$

It follows that

$$
c \leq \max_{t \in [0,1]} \psi(\gamma(t)) < \max_{t \in [0,1]} \psi_\infty(\gamma(t)) = \psi_\infty(\tilde{v}) = \mu_\infty \leq c.
$$

This is a contradiction. We conclude that $w$ is a nontrivial critical point of the functional $\psi$ in $X_p$. This is the desired conclusion. $\square$

5 Nonexistence of nontrivial solution

This section is devoted to the non-existence of nontrivial solution of the equation (1.1). In order to prove the non-existence of nontrivial solution to (1.1), we only need to show that the equation (1.7) has no nontrivial solution. Firstly, we give the Pohozaev identity corresponding to (1.7).

**Lemma 5.1.** Assume that $w \in X_p$ is a weak solution of the equation (1.7), then $w$ satisfies the following identity:

$$
\mathcal{P}(w) := \frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla w|^p dx + \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p dx
$$

$$
+ \frac{1}{p} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x)|h(w)|^p dx - \frac{N}{q} \int_{\mathbb{R}^N} |h(w)|^q dx = 0.
$$

**Proof.** To prove the lemma, for a nontrivial critical point $w$ of the functional $\psi$ defined by (1.6), we set

$$
w_t(x) = \begin{cases} 
  w \left( \frac{x}{t} \right), & t > 0, \\
  0, & t = 0. 
\end{cases}
$$

It easy to check that, for any $t \in (0, \infty)$, $w_t$ has the following properties

$$
\|\nabla w_t\|_p^p = t^{N-p} \int_{\mathbb{R}^N} |\nabla w|^p dx,
$$

$$
\|h(w_t)\|_p^p = t^N \int_{\mathbb{R}^N} |h(w)|^p dx,
$$

$$
\|h(w_t)\|_q^q = t^N \int_{\mathbb{R}^N} |h(w)|^q dx.
$$

On the other hand, the functional corresponding to (1.7) is given by

$$
\psi(w) = \frac{1}{p} \int_{\mathbb{R}^N} [\nabla w]^p + V(x)|h(w)|^p] dx - \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q dx.
$$

From the above it follows that

$$
\psi(w_t) = \frac{1}{p} \int_{\mathbb{R}^N} [\nabla w_t]^p dx + \int_{\mathbb{R}^N} V(x)|h(w_t)|^p] dx - \frac{1}{q} \int_{\mathbb{R}^N} |h(w_t)|^q dx
$$

$$
= \frac{1}{p} \left( t^{N-p} \int_{\mathbb{R}^N} |\nabla w|^p dx + t^N \int_{\mathbb{R}^N} V(xt)|h(w_t)|^p dx \right) - \frac{1}{q} t^N \int_{\mathbb{R}^N} |h(w)|^q dx.
$$
Since \( w \) is a solution of the equation (1.7), we conclude that
\[
\frac{d}{dt} \psi(w_i)_{|t=1} = \left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx \\
+ \frac{1}{p} \int_{\mathbb{R}^N} (\nabla V \cdot x)|h(w)|^p \, dx = \frac{N}{q} \int_{\mathbb{R}^N} |h(w)|^q \, dx = 0,
\]
and this is precisely the assertion of the lemma.

**Proof of Theorem 1.2.** On the contrary, suppose that \( w \) is a nontrivial solution of problem (1.7) and \( q \geq 2\alpha p^* \). Applying Lemma 5.1 we can rewrite (5.1) as
\[
\left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx + \frac{1}{p} \int_{\mathbb{R}^N} (\nabla V \cdot x)|h(w)|^p \, dx \\
= \frac{1}{q} N \int_{\mathbb{R}^N} |h(w)|^q \, dx.
\]

From \( \langle \psi'(w), w \rangle = 0 \) we see that
\[
\left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx + \frac{1}{p} \int_{\mathbb{R}^N} (\nabla V \cdot x)|h(w)|^p \, dx \\
= \left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} |h(w)|^q \, dx.
\]

Combining (5.2) and (5.3) we obtain
\[
\frac{1}{p} \int_{\mathbb{R}^N} |h(w)|^p (\nabla V(x) \cdot x) \, dx = \\
\left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx + \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx \\
+ \left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} h'(w)|h(w)|^q \, dx + \frac{N}{q} \int_{\mathbb{R}^N} |h(w)|^q \, dx.
\]

For simplicity, we rewrite this expression as
\[
A = B + C,
\]
where
\[
A := \frac{1}{p} \int_{\mathbb{R}^N} |h(w)|^p (\nabla V(x) \cdot x) \, dx,
\]
\[
B := \left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx - \frac{1}{p} \int_{\mathbb{R}^N} \bigg( \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx \bigg) |h(w)|^q \, dx,
\]
and
\[
C := \left( 1 - \frac{N}{p} \right) \int_{\mathbb{R}^N} |h(w)|^q \, dx + \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q \, dx.
\]

In view of the Lemma 2.1(h5), we then obtain the following inequality
\[
B = \left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx - \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx \\
\leq \left( \frac{N}{p} - 1 \right) \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx - \frac{N}{p} \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx \\
= - \int_{\mathbb{R}^N} V(x)|h(w)|^p \, dx \\
< 0.
\]

On the other hand, by the assumptions \( q \geq 2\alpha p^* = 2\alpha N p/(N - p) , 1 < p < N \) and Lemma 2.1, we have
\[
C = \left( 1 - \frac{N}{p} \right) \int_{\mathbb{R}^N} |h(w)|^q \, dx + \frac{1}{q} \int_{\mathbb{R}^N} |h(w)|^q \, dx \\
\leq \left( \frac{p - N}{2\alpha p} + \frac{N}{q} \right) \int_{\mathbb{R}^N} |h(w)|^q \, dx
\]
From (5.6) and (5.7) it follows that
\[
B + C < 0. \tag{5.8}
\]

Since \((\nabla V(x) \cdot x) \geq 0\), it follows straightforward that the left hand side of (5.4) is positive, that is
\[
A = \frac{1}{p} \int_{\mathbb{R}^N} |h(w)|^p (\nabla V(x) \cdot x) dx \geq 0. \tag{5.9}
\]

Thus, from (5.8) and (5.9) it may be concluded that
\[
A > B + C.
\]

This is a contradiction.

\[\square\]

Remark 5.1. It is worth pointing out that, by a similar argument, we can prove a version of Theorems 1.1 and 1.2, in the compact-coercive case. That is, our method in this paper can be adopted to establish the existence and nonexistence of nontrivial solutions to the equation (1.1), under the following condition:
\[
\lim_{|x| \to +\infty} V(x) = +\infty.
\]

In this case, the proof of Theorem 1.1 follows easily, because the map \(w \to h(w)\) is compact from \(X\) into \(L^q(\mathbb{R}^N)\) for \(2\alpha p < q < p^*\) under the above condition, where \(X\) is defined by
\[
X = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty \right\}.
\]

Theorems 1.1 and 1.2 still hold in the radially symmetric case. More precisely, we can establish the existence and nonexistence of nontrivial solutions to the equation (1.1), under the following condition:
\[
V(x) = V(|x|), \quad \text{for any } x \in \mathbb{R}^N.
\]

As for the compact-coercive case, we let
\[
X = W^{1,p}_r(\mathbb{R}^N) = \{ u \in W^{1,p}(\mathbb{R}^N) : u(x) = u(|x|) \}.
\]

We emphasize that, in case \(V(x) = V(|x|), \forall x \in \mathbb{R}^N\), only radially symmetric functions are in this space. In this case, the proof can be handled as above by utilizing the fact that the map \(w \to h(w)\) is compact from \(X\) into \(L^q(\mathbb{R}^N)\) for \(2\alpha p < q < p^*\) (see for instance [15, 22], for more details of continuity and compactness of the map \(w \to h(w)\)). We note that in both cases above, \(X\) is a subspace of \(W^{1,p}(\mathbb{R}^N)\).

Declaration

We have no conflicts of interest to disclose.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References


