Existence of Intuitionistic Fuzzy Additive Definite Integral by an Optimized Subtraction Operation

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Abstract

The induced aggregation operator from the intuitionistic additive definite integral is an important tool to aggregate information. But, before utilizing any measure, it is essential to examine their existence over the given dataset. In this paper, we intend to examine the existence of intuitionistic fuzzy additive definite integral and hence their induced aggregation operator with respect to the new subtraction operation $\ominus'$. To do so, we first discussed the new subtraction operation $\ominus'$ against the old subtraction operation $\ominus$ over some counter-examples. It is observed that there are some cases in which the old operation $\ominus$ is not defined. Therefore, to ensure the existence of the additive definite integral in a more general setting, we proved some lemma and theorems by using the new subtraction operation and function of bounded variation.

Keywords: Intuitionistic fuzzy sets; Integral and measure; Aggregation operator; Subtraction operation; Bounded variation; Polynomial interpolation.

1. Introduction

In classical mathematics, a dataset is a well-defined collection of objects. If the information about the objects is incomplete or imprecise (or both), the classical set cannot be defined. To handle this uncertainty, we can generalize, the classical set theory by defining the fuzzy set (FS) by replacing the word “well-defined” with human perception as a decision-maker of well-defines [19]. In literature, several variants of FSs e.g. Intuitionistic fuzzy sets [3], General Type-2 fuzzy sets [19], Interval-valued fuzzy sets [4], Vague sets [5] have been established. Attanasov [3,2] came up with the idea of Intuitionistic Fuzzy Sets (IFSs), known as one of the most popular extensions of FSs [16,29,18,28,24,31,11,17].

Intuitionistic fuzzy sets can be discreet or continuous. By continuity, one can address those cases of intuitionistic fuzzy sets, where the variables are in continuous nature. Some researchers significantly contributed to the developments of integral theory and their associated aggregation operators. Choquet and Sugeno integrals are two well-known integrals to address continuous random variables [20,25]. Chunqiao & Chen [29], Studied the property of Choquet integral to approximate the human subjective decision making process. Moreover, there are several researchers who studied these integrals in details. Martinez et. al.
Melin & Martinez [22], Klement et. al. [15], Grabisch & Christophe [10], Chitescu [6], and Khan et al. [13, 14] studied these integrals in different aspects. Developments of integration theory help us to induce new kinds of aggregation operators. For example, Deschrijver & Kerre [8] introduced various lattice aggregation operators and looked at some specific kinds of binary aggregation operators based on unit interval norms. Wei & Zhao [27], induced correlated aggregating operators for intuitionistic fuzzy information to solve multiple attribute group decision making. The above discussed integrals and their induced aggregation operators are based on the Maximum and Minimum operations. However, the selection of the appropriate basic operations is crucial. Therefore, in this paper, we intend to examine the existence of additive definite integral based aggregation operator using the new subtraction operation proposed by Sheng [9].

In Fig. (1) and (2), we have geometrical representation of intuitionistic fuzzy plane. The ordered pair \((\xi_i, \eta_i)\) is the intuitionistic fuzzy values (IFVs) with the membership degree \(\xi_i\) and non-membership degree \(\eta_i\). To aggregate these values, the weighted averaging (WA) operator and the ordered weighted averaging
(OWA) operator are primary tools that are purely based on basic addition and multiplication operations between IFVs. Motivated by these primary operators, and to overcome their limitations, Lei et al. proposed Intuitionistic Fuzzy Additive Definite Integral (IADI) (see [16] and section 3 for details). Utilizing additive definite integral, Xu proposed a new additive definite aggregation operator to aggregate the intuitionistic fuzzy information in [30].

Aggregation of data points is itself a challenging area of research. In this area, we try to aggregate the information using aggregation operators. But, before utilizing any aggregation operators, their existence must be ensured. In [12], Khan et al. proved that the integral proposed by Lei et al. ([16]) fails to exists for Dirichlet type functions (see section (3) and Def. (2.4)). They also provide necessary and sufficient conditions for the existence of IADI using an intuitionistic fuzzy bounded variation. In [9], Sheng discussed some counterintuitive cases in the intuitionistic fuzzy difference operation $\ominus$. To overcome the limitations of $\ominus$, Sheng proposed a new difference operation ($\ominus'$) by optimizing Hamming distance using linear programming. In this paper, we intend to examine the existence of IADI with respect to Sheng’s difference operation ($\ominus'$) and bounded variation.

The content of the paper can be summarized as follows; In section 2, we have some preliminary definitions. In section 3, we discuss the development of IADI and its associated aggregation operator. In the same section, we also discuss the reasoning behind the failure of $\ominus$ in comparison to the newly proposed Sheng’s difference operation $\ominus'$. In section 4, we have proved some lemma and theorems regarding the existence of IADI using Sheng’s difference operation and bounded variation. In section 5, we discussed the applicability of aggregation operator induced by IADI. Finally, we have a conclusion in section 6.

2. Preliminaries

In this section, let us discuss some definitions which are very important for the development of this paper. We will discuss here the definitions of Intuitionistic fuzzy set, Distance measure, Operations on Intuitionistic fuzzy values, Dirichlet type functions, Bounded variation, and Bounded variation based functions.

**Definition 2.1.** [3]: Let $X$ be the incomplete and imprecise universe of discourse. An Intuitionistic fuzzy set (IFS) $A$ proposed by Atanassov, is constructed as:

$$A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}.$$  \hspace{1cm} (1)

Here $\mu_A(x)$ is a membership value of $x$ assigned by some decision maker $A$. Similarly, $\nu_A(x)$ is the non-membership value of $x$ with respect to $A$. The linear sum of $\mu_A(x)$ and $\nu_A(x)$ are closed in $[0,1]$, i.e., $0 \leq w_A(x) + u_A(x) \leq 1$. Moreover, the hesitancy degree is defined as $\pi_A(x) = 1-(w_A(x) + u_A(x))$, which
represents the hesitation of the decision maker. And the doublet \((\mu_A(x), \nu_A(x))\) (or simply \((\mu, \nu)\)) is called intuitionistic fuzzy value (IFV).

**Definition 2.2.** \([3]\): A function \(d_{IFS} : IFS \times IFS \rightarrow [0, 1]\) is called intuitionistic fuzzy distance measure between two IFS \(A\) and \(B\) if it holds the following conditions:

(i) \(0 \leq d_{IFS}(A, B) \leq 1\) (Non-negativity).

(ii) \(d_{IFS}(A, B) = 0 \iff A = B\) (Reflexivity).

(iii) \(d_{IFS}(A, B) = d_{IFS}(B, A)\) (Symmetricity).

(iv) For \(A \subseteq B \subseteq C \in IFS(X) \Rightarrow d_{IFS}(A, C) \geq d_{IFS}(A, B)\) and \(d_{IFS}(A, C) \geq d_{IFS}(B, C)\) (Triangle inequality).

**Definition 2.3.** \([9]\) For \(a, a_1, a_2 \in X\), let \(x = (\mu, \nu), x_1 = (\mu_1, \nu_1)\) and \(x_2 = (\mu_2, \nu_2)\) be the corresponding IFVs. Then, basic intuitionistic fuzzy operations are defined as follows:

(i) \(a_1 \oplus a_2 = (\mu_1 + \mu_2 - \mu_1 \mu_2, \nu_1 \nu_2)\)

(ii) \(a_1 \ominus a_2 = \begin{cases} \frac{(1 - \mu_2)}{(1 - \mu_1)} \frac{(1 - \nu_2)}{(1 - \nu_1)}, & 0 \leq \frac{\mu_1}{\mu_2} \leq \frac{1 - \mu_1}{1 - \nu_1} < 1 \\ (0, 1), & \text{otherwise} \end{cases}\)

(iii) \(\lambda x = (1 - (1 - \mu)^\lambda, \nu^\lambda)\)

(iv) \(x^\lambda = (\mu^\lambda, 1 - (1 - \nu)^\lambda)\).

**Definition 2.4.** \([12, 7]\): A non-negative real valued function \(g : IFS \times IFS \rightarrow \mathbb{R}^+\) is said to be of Dirichlet type function, if

\[ g(\mu, \nu) = \begin{cases} f_1, & \mu, \nu \in \mathbb{Q}([0, 1]) \\ f_2, & \mu, \nu \in \mathbb{Q}^c([0, 1]) \end{cases} \]

Where, \(\mathbb{Q}([0, 1]), \mathbb{Q}^c([0, 1])\) are the rational and irrational numbers from \([0, 1]\), and \(f_1 \neq f_2 > 0\) are arbitrarily chosen real valued positive functions of \(\mu\) and \(\nu\).

**Definition 2.5.** \([1]\) The \(p\)-summable total variation \(V_p(f)\) of a function \((f)\) over a partition is defined as:

\[ V_p(f) = \left( |f(x_1)|^p + \sum_{i=2}^{n} |\Delta f(x_i)|^p \right)^{\frac{1}{p}} \]

where, \(\Delta\)-transform of \(f\) is defined as \(\Delta f(x_i) = f(x_i) - f(x_{i-1})\).

**Definition 2.6.** \([1]\) A function \(f\) is said to be of bounded variation (BV) function on a chosen interval \([0, 1]\) iff its total variation is finite, i.e.

\[ f \in BV([0, 1]) \iff \sup_{n} V_p(f) < +\infty \]

3. Intuitionistic fuzzy additive definite integral proposed by Lei et al.

Let us first discuss some fundamentals of intuitionistic fuzzy additive definite integral (IADI). Let \(D\) be some region in the intuitionistic fuzzy plane (see Fig. (1) and (2)). And, let \(\alpha^-\) and \(\alpha^+\) be the infimum and supremum IFVs in \(D\) such that;
\[ \alpha^- = \{ \inf_{\alpha \in D} (\mu_\alpha), \sup_{\alpha \in D} (\nu_\alpha) \} \]
\[ \text{and } \alpha^+ = \{ \sup_{\alpha \in D} (\mu_\alpha), \inf_{\alpha \in D} (\nu_\alpha) \} \]

From Fig. (2b), we can have two partitions of the intervals
\[ I_1 = [\mu_0 = \inf_{\alpha \in D} (\mu_\alpha), \mu_n = \sup_{\alpha \in D} (\mu_\alpha)] \text{and } I_2 = [\nu_0 = \inf_{\alpha \in D} (\nu_\alpha), \nu_n = \sup_{\alpha \in D} (\nu_\alpha)] \]
such that:
\[ P_{I_1} : \mu_0 = \inf_{\alpha \in D} (\mu_\alpha) < \mu_1 < \mu_2, \ldots, \mu_n = \sup_{\alpha \in D} (\mu_\alpha) \]
\[ P_{I_2} : \nu_0 = \inf_{\alpha \in D} (\nu_\alpha) < \nu_1 < \nu_2, \ldots, \nu_n = \sup_{\alpha \in D} (\nu_\alpha) \]

Now, we have \( P_{I_1} \) as a partition of \( I_1 \) and \( P_{I_2} \) as partition of \( I_2 \) in the region \( D \). In this case, we have \( k \) number of small region \( \delta_i \) \( \{ i = 0, 1, 2, \ldots, k - 1 \} \) (see Fig. (2c)). Let \( d(x_1, x_2) \) be the Euclidean distance between the points \( x_1, x_2 \) in the small regions \( \delta_i \), then we define a real number \( \Gamma \) as follows:
\[ \Gamma = \max_{0 \leq i \leq k-1} \{ \sup \{d(x_1, x_2) : x_1, x_2 \in \delta_i \} \} \]

Let \( (\xi_i, \eta_i) \) be an IFV in the \( i^{th} \) region \( \delta_i \), and \( f \) be a real valued non-negative intuitionistic fuzzy function passing through the point \( (\xi_i, \eta_i) \). Then, \( f(\xi_i, \eta_i)(\xi_i, \eta_i)\Delta \delta_i \) is also an IFV corresponding to the IFV \( (\xi_i, \eta_i) \) in the \( i^{th} \) region \( \delta_i \). It is to be notice, for any IFV \( \alpha_i \in \delta_i \), we can find supremum and infimum of \( \alpha_i \) in \( \delta_i \). Let \( (\alpha_0^i) \) and \( (\alpha_1^i) \) be the suprimum and infimum values of \( \alpha_i = (\xi_i, \eta_i) \), and \( f_{\sup}(\alpha_0^i) \) and \( f_{\inf}(\alpha_1^i) \) be the supremum and infimum values of \( f \) in the \( i^{th} \) region \( \delta_i \). Then, with respect to the \( i^{th} \) region \( \delta_i \) of \( D \), we have three IFVs;
\[ f_{\inf}(\alpha_0^i)\alpha_0^i\Delta \delta_i, f_{\sup}(\alpha_1^i)\alpha_1^i\Delta \delta_i, \text{ and } f(\alpha_i)\alpha_i\Delta \delta_i \]
such that
\[ f_{\inf}(\alpha_0^i)\alpha_0^i\Delta \delta_i \leq f(\alpha_i)\alpha_i\Delta \delta_i \leq f_{\sup}(\alpha_1^i)\alpha_1^i\Delta \delta_i \]

Now, on adding all corresponding values for each region \( \delta_i \) of \( D \), we have:
\[ \oplus_{i=0}^{k-1} f_{\inf}(\alpha_0^i)\alpha_0^i\Delta \delta_i, \oplus_{i=0}^{k-1} f_{\sup}(\alpha_1^i)\alpha_1^i\Delta \delta_i \text{ and } \oplus_{i=0}^{k-1} f(\alpha_i)\alpha_i\Delta \delta_i \]

, or without loss of generality, it is same as:
\[ \oplus_{i=1}^{k} f_{\inf}(\alpha_0^i)\alpha_0^i\Delta \delta_i, \oplus_{i=1}^{k} f_{\sup}(\alpha_1^i)\alpha_1^i\Delta \delta_i \text{ and } \oplus_{i=1}^{k} f(\alpha_i)\alpha_i\Delta \delta_i \]

. In limiting case, the above expressions can be written as follows;
\[ \lim_{l \to 0} \oplus_{i=1}^{k} f_{\inf}(\alpha_0^i)\alpha_0^i\Delta \delta_i, \lim_{l \to 0} \oplus_{i=1}^{k} f_{\sup}(\alpha_1^i)\alpha_1^i\Delta \delta_i, \text{ and } \lim_{l \to 0} \oplus_{i=1}^{k} f(\alpha_i)\alpha_i\Delta \delta_i \]
Utilising these expressions, Lei et al. proposed intuitionistic fuzzy additive definite integral (IADI) to aggregate real valued non negative intuitionistic fuzzy functions by proposing the following three sums; lower Darboux sum ($LD_{f_k}$), upper Darboux sum ($UD_{f_k}$) and Riemann sum ($RS_{f_k}$) over the region $D$ as follows:

$$LD_{f_k} = \bigoplus_{i=1}^{k} f_{\text{int}}(\alpha^0_i)\alpha^0_i \Delta \delta_i$$

$$UD_{f_k} = \bigoplus_{i=1}^{k} f_{\text{sup}}(\alpha^1_i)\alpha^1_i \Delta \delta_i$$

$$RS_{f_k} = \bigoplus_{i=1}^{k} f(\alpha_i)\alpha_i \Delta \delta_i$$

If $LD_{f_k} = UD_{f_k} = RS_{f_k}$, i.e.,

$$\bigoplus_{i=1}^{k} f_{\text{int}}(\alpha^0_i)\alpha^0_i \Delta \delta_i = \bigoplus_{i=1}^{k} f_{\text{sup}}(\alpha^1_i)\alpha^1_i \Delta \delta_i = \bigoplus_{i=1}^{k} f(\alpha_i)\alpha_i \Delta \delta_i$$

(2)

or

$$UD_{f_k} \ominus LS_{f_k} \rightarrow (0,1)$$

(3)

Intuitionistic fuzzy additive definite integral (IADI) exists and is defined as;

$$\lim_{\Gamma \to 0} \bigoplus_{i=1}^{k} f(\alpha_i)\alpha_i \Delta \delta_i = \iint_{D} f(\alpha)\alpha \Delta \delta$$

or

$$\lim_{\Gamma \to 0} \bigoplus_{i=1}^{k} f(\mu_i, \nu_i)(\mu_i, \nu_i) \Delta \delta_i = \iint_{D} f(\mu, \nu)(\mu, \nu)d\mu d\nu$$

(4)

Using Def. (2,3), the induced aggregation operator can be defined as;

$$\iint_{D} f(\mu, \nu)(\mu, \nu)d\mu d\nu = \left[1 - \prod_{i=1}^{n} (1 - \mu_i)f(\mu_i, \nu_i)\Delta \delta_i, \prod_{i=1}^{n} \nu_i f(\mu_i, \nu_i)\Delta \delta_i\right]$$

4. Intuitionistic fuzzy difference operation proposed by DW Sheng

In [9], Sheng discussed some counter intuitive cases in the intuitionistic fuzzy difference operation $\ominus$. Difference operation between IFVs is defined as deconvolution of equations using intuitionistic fuzzy addition ($\oplus$) and intuitionistic fuzzy multiplication operations ($\otimes$). In other words, let $x, y, z$ are IFVs. Then, $x = y \oplus z$ implies $z = x \ominus y$. From Def. (2,3), let $x = (\mu_1, v_1)$ and $y = (\mu_2, v_2)$ be the IFVs, and there exists an IFV $z = (\mu, v)$ such that $x = y \ominus z$ or $(\mu_1, v_1) = (\mu_2 + \mu - \mu_2\mu, v_2 \nu)$. By comparing both sides, we can easily calculate $\mu = \frac{\mu_1 - \mu_2}{1 - \mu_2}$ and $v = \frac{v_1}{v_2}$. For the existence of the IFV $z = (\frac{\mu_1 - \mu_2}{1 - \mu_2}, \frac{v_1}{v_2})$, it is essential that the following inequality must hold;

$$0 \leq \frac{v_1}{v_2} \leq 1 - \frac{\mu_1}{1 - \mu_2} \leq 1$$

(5)
But, there are several other IFVs in the universe of discourse for which this inequality may not hold. For example, let \( x = (0.5, 0.3) \) and \( y = (0.4, 0.5) \) be two IFVs. Then, \( x \ominus y = (0.167, 0.6) \). But, if \( y' = (0.4, 0.2) \), Def. (2.3) will fail to find \( x \ominus y \) (Def. (2.3) will assign it a certain value \((0, 1)) \). To overcome the counterintuitive cases of intuitionistic fuzzy difference operation \( \ominus \), using linear programming and Hamming distance measure, DW Sheng proposed a new intuitionistic fuzzy difference operation \( \ominus' \) \([9]\). Abstractly, Sheng’s difference between two IFVs \( x = (\mu_1, v_1) \) and \( y = (\mu_2, v_2) \) is defined as;

\[
x \ominus' y = \begin{cases} 
(0, \frac{v_1}{v_2}), & \text{if } \mu_1 \leq \mu_2, v_1 \leq v_2, \\
(\frac{\mu_1 - \mu_2}{1 - \mu_2}, \frac{v_1}{v_2}), & \text{if } 0 \leq \frac{v_1}{v_2} \leq \frac{1 - \mu_1}{1 - \mu_2} < 1, \\
(\frac{\mu_1 - \mu_2}{1 - \mu_2}, 1 - \frac{v_1}{v_2}), & \text{if } \mu_1 > \mu_2, \frac{1 - \mu_1}{1 - \mu_2} < \frac{v_1}{v_2}, \\
(0, 1), & \text{if } \mu_1 \leq \mu_2, v_1 > v_2.
\end{cases}
\]  

(6)

Sheng’s difference \( \ominus' \), calculates the difference between \( x = (0.5, 0.3) \) and \( y' = (0.4, 0.2) \) as \( x \ominus' y' = (0.167, 0.63) \) which is not equal to the certain value \((0, 1)\). Let us take the following example to illustrate the working of the new difference operation.

**Example 4.1.** Let \( x = (0.2, 0.5), y_1 = (0.3, 0.4), y_2 = (0.3, 0.6), y_3 = (0.1, 0.7), y_4 = (0.15, 0.45) \) be the IFVs. Then, from Eq. (6), Sheng’s difference between \( x \) and \( y_i \) are calculated as;

\[
\begin{align*}
x \ominus' y_1 & = (0, 1) \\
x \ominus' y_2 & = (0.0, 0.5) = (0, 0.833) \\
x \ominus' y_3 & = (0.2 - 0.1, 0.5) = (0.111, 0.714) \\
x \ominus' y_4 & = (0.2 - 0.15, 0.1 - 0.2) = (0.059, 0.941).
\end{align*}
\]

Let us consider another example to illustrate the motivation of this paper.

**Example 4.2.** Let \( x_i = (\frac{1}{n}, 1 - \frac{1}{n}) \) be the IFVs for each \( n \in [1, \infty) \), and \( y = (\mu, \nu) \) be an IFV. The intuitionistic fuzzy difference between \( x_i \) and \( y \) can be calculated as;

\[
x_i \ominus y = \begin{cases} 
\left( \frac{1 - \nu}{n(1 - n)}, \frac{n - 1}{n \nu} \right), & n \leq \frac{1}{\mu}, n \leq \frac{1}{1 - \nu} \\
(0, 1), & \text{otherwise}
\end{cases}
\]

The above expression maps every points between \( n \geq \frac{1}{\mu}, n \leq \frac{1}{1 - \nu} \) and \( n < \frac{1}{\mu}, \nu < (1 - \mu) \) to a certain IFV \((0, 1)\). That is, there are infinite number of IFVs which has a fixed difference with \( y \), which is not a general
case. But, by using Sheng’s difference operation, difference between $x_i$ and $y$ are calculated as:

\[ x_i \ominus' y = \begin{cases} 
(0, \frac{n-1}{n\nu}) & n \geq \frac{1}{\mu}, n \leq \frac{1}{1-\nu} \\
(1-\frac{n\mu}{n(1-\mu)}, \frac{n-1}{n\nu}) & n < \frac{1}{\mu}, n < \frac{1}{1-\nu} \\
(\frac{1-n\mu}{n(1-\mu)}, \frac{n-1}{n\nu}) & n < \frac{1}{\mu}, \nu < (1-\mu) \\
(0,1) & n \geq \frac{1}{\mu}, n > \frac{1}{1-\nu} 
\end{cases} \]

Which implies, Sheng’s difference operation $\ominus'$ can handle more cases in compare to the intuitionistic fuzzy difference $\ominus$.

We analyses the above example in different region. We can divide the existence analysis of IADI of Example 4.2 into four different cases:

I: \( \left( \text{When } n \geq \frac{1}{\mu}, \text{ and } n \leq \frac{1}{1-\nu} \right) \): In this case, $x_i \ominus y = (0, 1)$ i.e. for infinitely many values of the range $n \geq \frac{1}{\mu}, n \leq \frac{1}{1-\nu}$, $x_i \ominus y$ assigned to a certain value $(0, 1)$. Therefore, we cannot ensure the existence of IADI in this case. On the other hand, $x_i \ominus' y = (0, \frac{n-1}{n\nu})$ i.e. $\ominus'$ assigns different values for different IFVs. To handle this case, we ensured the existence of IADI and hence its associated aggregation operator in a general way in Section 5.

II \( \left( \text{When } n < \frac{1}{\mu}, n < \frac{1}{1-\nu} \right) \): Here, $x_i \ominus' y = x_i \ominus y = \left(1-\frac{n\mu}{n(1-\mu)}, \frac{n-1}{n\nu}\right)$. Existence of this case is discussed in [12].

III \( \left( \text{When } n < \frac{1}{\mu}, \text{ and } \nu < (1-\mu) \right) \): In this case also $x_i \ominus y = (0, 1)$. But, $x_i \ominus' y = \left(1-\frac{n\mu}{n(1-\mu)}, \frac{n-1}{n\nu}\right)$. Thus, we have to verify the existence of IADI.

IV \( \left( \text{When } n \geq \frac{1}{\mu}, \text{ and } n > \frac{1}{1-\nu} \right) \): In this case, $x_i \ominus' y = x_i \ominus y = (0, 1)$. Therefore, in section 5, we intend to examine the existence of IADI $\lim_{\Gamma \to 0} \oplus f(\mu_i, \nu_i)(\mu_i, \nu_i)\Delta \delta_i$ in more general setting using the new difference operation proposed by Sheng ($\ominus'$).

5. Existence of intuitionistic fuzzy additive integral using the new difference operation $\ominus'$.

Here, we present a sufficient condition for the existence of intuitionistic fuzzy additive definite integral. Because the integral’s existence is heavily reliant on the specified function. As a result, we’ll employ the concept of the function of bounded variation to ensure that the IADI exists. Bounded variation (BV) is a method used in classical mathematics to calculate the approximate arc length of measurable functions (see
Def. 2.3). The notation $BV([0,1])$ is used in this study to refer the set of all bounded variation functions between $[0,1]$. Now, by replacing the condition of the function $f$ with a positive real-valued bounded variation function and keeping all of the assumptions of IADI discussed in section 3, Lei et al.’s IADI can be represented as:

$$\lim_{\Gamma \to 0} \oplus_{i=1}^{k} f(\alpha_i) \Delta \delta_i = \int \int_{D} f(\alpha) \alpha \Delta \delta$$  \hspace{1cm} (7)

or

$$\lim_{\Gamma \to 0} \oplus_{i=1}^{k} f(\mu_i, \nu_i) (\mu_i, \nu_i) \Delta \delta_i = \int \int_{D} f(\mu, \nu) (\mu, \nu) d\mu d\nu$$  \hspace{1cm} (8)

To ensure the existence of this integral by using the Sheng’s difference operation, let us state the following lemma;

**Lemma 5.1.** Let $f \in BV([0,1])$ then, there exist two real valued increasing functions $f_1, f_2$ such that $f = f_1 - f_2$.

The proof of this lemma is available in [24]. Now, in the view of Lemma (5.1), we proved Lemma (5.2) as follows;

**Lemma 5.2.** Let $f$ is a non negative real valued bounded variation increasing function. Then, the integral $\int \int_{D} f(\mu, \nu) (\mu, \nu) d\mu d\nu$ exists.

Given is, $f$ a non negative real valued bounded variation increasing function. For two IFVs $\alpha$ and $\beta$ such that $\alpha \leq \beta$ we have $f(\alpha) \leq (\beta)$. Thus, $f_{sup} = f(\alpha^{+})$, $f_{inf} = f(\alpha^{-})$ ($\alpha^{+}$ and $\alpha^{-}$ are the suprimum and infimum IFVs). Then, by the definition of lover darboux sum (lds) and upper darboux sum (uds) we have;

$$UD_{f_k} \ominus LD_{f_k} = \oplus_{i=1}^{k} f_{sup}(\alpha_i^{+}) \Delta \delta_i \ominus \oplus_{i=1}^{k} f_{inf}(\alpha_0^{+}) \Delta \delta_i \leq \oplus_{i=1}^{k} \left( f(\alpha) \alpha_i^{+} \Delta \delta_i \ominus f(\alpha) \alpha_0^{+} \Delta \delta_i \right)$$

for $\alpha_i^{+} = (\xi_{\alpha_i^{+}}, \eta_{\alpha_i^{+}})$, $\alpha_0^{+} = (\xi_{\alpha_0^{+}}, \eta_{\alpha_0^{+}})$, and definition (2.2), we have;

$$= \left\{ 0, \frac{\prod_{i=1}^{k} (1-\xi_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (1-\xi_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i} \right\}$$

Again, by definition (2.2) we have;

$$= \begin{cases} 0, \frac{\prod_{i=1}^{k} (1-\xi_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (1-\xi_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \frac{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}{\prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i}, \prod_{i=1}^{k} (\eta_{\alpha_i^{+}}) f(\alpha^{+}) \Delta \delta_i} \right\}$$

if $\mu_1 \leq \mu_2, v_1 \leq v_2,$

if $0 \leq \frac{v_1}{v_2} \leq \frac{1-\mu_1}{1-\mu_2} < 1,$

if $\mu_1 > \mu_2, \frac{1-\mu_1}{1-\mu_2} < \frac{v_1}{v_2},$

if $\mu_1 \leq \mu_2, v_1 > v_2.
From the definitions of $\delta$, as $\delta \to 0$ we have:

$$\left( \prod_{i=1}^{k} (1 - \xi_{\alpha_{i}^{+}}) f(\alpha^{+}) / \prod_{i=1}^{k} (1 - \xi_{\alpha_{i}^{-}}) f(\alpha^{-}) \right) \to 1$$

Without loss of the generality, let $\Delta_{i} = \Delta_{\delta}$, as $\Delta_{\delta} \to 0$ we have:

$$\left( \prod_{i=1}^{k} (1 - \xi_{\alpha_{i}^{+}}) f(\alpha^{+}) / \prod_{i=1}^{k} (1 - \xi_{\alpha_{i}^{-}}) f(\alpha^{-}) \right) \to 1$$

Therefore,$\n
$$UD_{f_{k}} \ominus' LD_{f_{k}} = \begin{cases} 
(0, 1), & \text{if } \mu_{1} \leq \mu_{2}, v_{1} \leq v_{2}, \\
(0, 1), & \text{if } 0 \leq \frac{v_{1}}{v_{2}} \leq \frac{1 - \mu_{1}}{\mu_{2}} < 1, \\
(0, 1), & \text{if } \mu_{1} > \mu_{2}; \frac{1 - \mu_{1}}{\mu_{2}} < \frac{v_{1}}{v_{2}}, \\
(0, 1), & \text{if } \mu_{1} \leq \mu_{2}, v_{1} > v_{2}
\end{cases}$$

In each situation, $UD_{f_{k}} \ominus' LD_{f_{k}} \to (0, 1)$. As $LD_{f_{k}} \leq RS_{f_{k}} \leq UD_{f_{k}}$, this implies, $LD_{f_{k}} = RS_{f_{k}} = UD_{f_{k}}$. Therefore, with respect to Sheng’s difference operation (\ominus'), the integral $\iint_{D} f(\mu, \nu)(\mu, \nu) d\mu d\nu$ exists. \qed

**Lemma 5.3.** Let $f, g$ are real valued increasing functions. Then, $\iint_{D} (f - g)(\mu, \nu)(\mu, \nu) d\mu d\nu$ exists

Let $h$ be another function defined as $h = f - g$. Then,

$$h_{\text{sup}} = f_{\text{sup}} - g_{\text{inf}}, \quad h_{\text{inf}} = f_{\text{inf}} - g_{\text{sup}}$$

From the definitions of $LD_{f_{k}}$ and $UD_{f_{k}}$, we have:

$$UD_{f_{k}} \ominus' LD_{f_{k}} = \oplus_{i=1}^{k} h_{\text{sup}}^{i}(\alpha_{i}^{+}) \alpha_{i}^{+} \Delta_{i} \ominus' \oplus_{i=1}^{k} h_{\text{inf}}^{i}(\alpha_{i}^{-}) \alpha_{i}^{-} \Delta_{i}$$

As, $f$ and $g$ both are increasing functions. Then, $g_{\text{sup}}^{i} = g(\alpha^{+}), f_{\text{sup}}^{i} = f(\alpha^{+}), g_{\text{inf}}^{i} = g(\alpha^{-})$ and
\[ f_{i\inf} = f(\alpha^-). \] This implies;

\[
UD_{f_k} \ominus' LD_{f_k} = \bigoplus_{i=1}^k (f(\alpha^+)-g(\alpha^-)) (\alpha_i^-) \alpha_i^+ \Delta \delta_i \ominus' \bigoplus_{i=1}^k (f(\alpha^-)-f(\alpha^+)) (\alpha_i^-) \alpha_i^+ \Delta \delta_i
\]

Let \( \alpha_i^+ = (\xi_{\alpha_i^+}, \eta_{\alpha_i^+}) \) and \( \alpha_i^- = (\xi_{\alpha_i^-}, \eta_{\alpha_i^-}) \). Then, by using Def. \([2.3]\), we have;

\[
UD_{f_k} \ominus' LD_{f_k} = \left(1 - \prod_{i=1}^k (1 - \xi_{\alpha_i^+}) (f(\alpha^+)-g(\alpha^-)) \Delta \delta_i, \prod_{i=1}^k (\eta_{\alpha_i^-}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i \right)
\]

\[
\ominus' \left(1 - \prod_{i=1}^k (1 - \xi_{\alpha_i^-}) (f(\alpha^-)-g(\alpha^+)) \Delta \delta_i, \prod_{i=1}^k (\eta_{\alpha_i^+}) (f(\alpha^-)-g(\alpha^+)) \Delta \delta_i \right)
\]

By Eq. \([\boldsymbol{0}]\), we have;

\[
UD_{f_k} \ominus' LD_{f_k} =
\]

\[
\left\{ \begin{array}{ll}
0, & \text{if } \mu_1 \leq \mu_2, v_1 \leq v_2, \\
\frac{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}, & \text{if } 0 \leq v_2 v_1 \leq \frac{1}{1-\mu_2}, \\
\left(1 - \frac{\prod_{i=1}^k (1 - \xi_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}{\prod_{i=1}^k (1 - \xi_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}, \right) & \frac{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}, & \text{if } \mu_1 > \mu_2, 1 - \frac{\mu_2}{\mu_2} < v_1 \leq v_2,
\end{array} \right.
\]

\[
(0, 1),
\]

\[
= \left\{ \begin{array}{ll}
0, & \text{if } \mu_1 \leq \mu_2, v_1 \leq v_2, \\
\frac{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}, & \text{if } 0 \leq v_2 v_1 \leq \frac{1}{1-\mu_2}, \\
\left(1 - \frac{\prod_{i=1}^k (1 - \xi_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}{\prod_{i=1}^k (1 - \xi_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}, \right) & \frac{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}{\prod_{i=1}^k (\eta_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta_i}, & \text{if } \mu_1 > \mu_2, 1 - \frac{\mu_2}{\mu_2} < v_1 \leq v_2,
\end{array} \right.
\]

\[
(0, 1),
\]

Without loss of the generality, let \( \Delta \delta_i = \Delta \delta \). For \( \Delta \delta \to 0 \), we have;

\[
\left( \frac{\prod_{i=1}^k (1 - \xi_{\alpha_i}) (f(\alpha^-)-g(\alpha^-)) \Delta \delta}{\prod_{i=1}^k (1 - \xi_{\alpha_i}) (f(\alpha^-)-g(\alpha^-))} \right) \to 1
\]

\[
11
\]
In each case, our equation produces:

$$ UD_{f_k} \oplus' LD_{f_k} = \begin{cases} 
  \langle 0, 1 \rangle, & \text{if } \mu_1 \leq \mu_2, v_1 \leq v_2, \\
  \langle 0, 1 \rangle, & \text{if } 0 \leq \frac{\mu_1}{v_2} \leq \frac{1-\mu_1}{1-v_2} < 1, \\
  \langle 0, 1 \rangle, & \text{if } \mu_1 > \mu_2, \frac{1-\mu_1}{1-v_2} < \frac{\mu_1}{v_2}, \\
  \langle 0, 1 \rangle, & \text{if } \mu_1 \leq \mu_2, v_1 > v_2.
\end{cases} $$

Thus, $UD_{f_k} \oplus' LD_{f_k} \rightarrow (0, 1)$. This implies the existence of $\int\int_{D}(f-g)(\mu, \nu)(\mu, \nu)d\mu d\nu$. □

To prove the existence of IADI, let us prove the following sufficient condition in the form of the theorem using bounded variation and new difference operation;

**Theorem 5.1.** If $f \in BV([0,1])$. Then, $\int\int_{D} f(\mu, \nu)(\mu, \nu)d\mu d\nu$ exists.

**Proof 5.1.** To prove this theorem, we need Lemma [5.1], [5.2], and [5.3]. As stated in theorem, $f \in BV([0,1])$. Then, by Lemma [5.1], there exist two increasing functions $g_1, g_2$ such that:

$$ f = g_1 - g_2 $$

In Lemma [5.2], we proved the IADI $\int\int_{D} f(\mu, \nu)(\mu, \nu)d\mu d\nu$ exists for non-negative real-valued increasing functions. And, in Lemma [5.3], we proved $\int\int_{D} (f_1 - f_2)(\mu, \nu)(\mu, \nu)d\mu d\nu$ exists for the difference of two increasing functions. Therefore, in the view of these Lemmas, for $f \in BV([0,1])$, $\int\int_{D} f(\mu, \nu)(\mu, \nu)d\mu d\nu$ exists. □

In the following section, we explain how IADI is beneficial to aggregate large amount of IFVs. To do so, we artificially constructed the $n$ number of IFVs using the Yager generating function [32]. Letter, these IFVs are aggregated using IADI with different $n$.

### 6. Applicability of the Existence of Intuitionistic Fuzzy Additive Definite Integrals

To aggregate the intuitionistic fuzzy information using IADI, we need to follow the following steps;

(i) Interpolate the data points by some non-negative real-valued function ($f$).

(ii) Check the bounded variation property of the interpolated function $f$.

(iii) If $f$ is a bounded variation function. Then, integral $\int\int_{D} f(\mu, \nu)(\mu, \nu)d\mu d\nu$ exists.
(iv) Using Def. (2.3), the IADI can be evaluated as:

\[
\int_{D} f(\mu, \nu)(\mu, \nu) \, d\mu d\nu = \left[ 1 - \prod_{i=1}^{n} (1 - \mu_i) f(\mu_i, \nu_i) \Delta \delta_i \right] \prod_{i=1}^{n} \mu_i f(\mu_i, \nu_i) \Delta \delta_i \]  \hspace{1cm} (9)

here, \( f_i \) is the value of the bounded variation function at the \( i^{th} \) IFV, and \( \Delta \delta_i \) is the diameter of that particular region \( \delta_i \) (see Fig. (2c) and Def. (2.3)).

Assume we have a large group of decision-makers who are asked to weigh the objects in the form of IFVs. Illustratively, we constructed the huge amount of IFVs using Yager generating function (32). Let \( A = \{(\mu_i, \nu_i), \ i = 1,2,\ldots,n\} \) be an IFS with IFVs \((\mu_i, \nu_i), \ i = 1,2,\ldots,n\), where the membership and nonmembership values are defined with the help of Yager generating function;
\[ \mu_i = \left( \frac{1}{t} \right)^i, \quad \text{and} \quad \nu_i = \left( 1 - \left( \frac{1}{t} \right)^i \right) + , \quad t > 0. \] (10)

In Fig. (3), we represent the collection of IFVs of Eq. (10). To aggregate this amount of information by IADI, we first need to interpolate it by some bounded variation base non-negative real-valued function. On applying the polynomial interpolation on \( A \), the interpolated polynomial of the data set \( A \) is calculated as
\[ P(\mu, \nu) : \nu = \mu^2 - 2\mu + 1 \] (see Fig. 4).

As every polynomial satisfies the bounded variation property. Now, from Eq. (9) and (10), the aggregated intuitionistic fuzzy value is evaluated as;
\[
\int \int_D f(\mu, \nu)(\mu, \nu)d\mu d\nu = \left[ 1 - \prod_{i=1}^{n} \left( \left( \frac{1}{t} \right)^i \right) P(\mu, \nu)\Delta \delta_i ; \prod_{i=1}^{n} \left( 1 - \left( \frac{1}{t} \right)^i \right) P(\mu, \nu)\Delta \delta_i \right]. \] (11)

Now using Eq. (11) with different \( n \), aggregated intuitionistic fuzzy values of Eq. (10) are evaluated as;

(i) With \( n = 100 \) & \( t = \frac{1}{2} \), \( \int \int_D f(\mu, \nu)(\mu, \nu)d\mu d\nu = (0.6942, 0.0935) \).

(ii) With \( n = 200 \) & \( t = \frac{1}{2} \), \( \int \int_D f(\mu, \nu)(\mu, \nu)d\mu d\nu = (0.8520, 0.0219) \).

(iii) With \( n = 300 \) & \( t = \frac{1}{2} \), \( \int \int_D f(\mu, \nu)(\mu, \nu)d\mu d\nu = (0.9167, 0.0069) \).

(iv) With \( n = 400 \) & \( t = \frac{1}{2} \), \( \int \int_D f(\mu, \nu)(\mu, \nu)d\mu d\nu = (0.9491, 0.0026) \).

(v) With \( n = 500 \) & \( t = \frac{1}{2} \), \( \int \int_D f(\mu, \nu)(\mu, \nu)d\mu d\nu = (0.9672, 0.0011) \).
From the above results, we can observe that as \( n \) increases the membership value of the aggregated IFV increases and the nonmembership value decreases. This implies the aggregation operator induced from Eq. (11) is well defined. Therefore, IADI illegible to aggregate intuitionistic fuzzy values by using bounded variation-based non-negative real-valued functions efficiently.

7. Conclusion

In the manuscript, our main motto is to ensure the existence of intuitionistic fuzzy additive definite integral (IADI) and hence their induced aggregation operator. Aggregation operator is a very useful tool to aggregate intuitionistic fuzzy information. We showed that there are cases where IADI may not exist. To overcome this limitation, we examined the existence of IADI using bounded variation and the new optimized subtraction operation proposed by Sheng. And proved necessary lemma and theorems wherever it needed. Finally, for the illustration purpose, we constructed examples of large intuitionistic fuzzy values (IFVs) by using Yagen generating function. As every polynomial satisfies the bounded variation property, we interpolated our large IFVs to a polynomial by using the basic polynomial interpolation method. Then by using the aggregation operator induced from IADI, we calculated the aggregated values on different levels. It is to be noticed, for any collection of IFVs, different interpolated functions are possible. Therefore, the aggregated outcome may variate with respect to different interpolations. That is, the output result is highly dependent on how the sample data is interpolated.

8. Compliance with ethical standards

Conflict of interest: We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome. We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us. We understand that the Corresponding Author is the sole contact for the Editorial process (including Editorial Manager and direct communications with the office). He is responsible for communicating with the other authors about progress, submissions of revisions and final approval of proofs. We confirm that we have provided a current, correct email address which is accessible by the Corresponding Author.

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