Supplementary Informations:
Absence of Thermodynamic Uncertainty Relations with Asymmetric Dynamic Protocols

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I. PROOF OF LEMMA 1

Let us write down a pair of N-point probability distributions, which satisfy the Detailed Fluctuation Theorem (Eq. (2.4) of the main text):

$$p_N(\sigma, q) \;=\; \sum_{i=1}^{N} p_i\, \delta(\sigma - \sigma_i)\delta(q - q_i), \tag{1.1a}$$

$$\bar{p}_N(\sigma, q) \;=\; \sum_{i=1}^{N} p_i\, e^{-\sigma_i}\, \delta(\sigma + \sigma_i)\delta(q + q_i). \tag{1.1b}$$

Normalization of two probability distributions require:

$$\sum_{i=1}^{N} p_i \;=\; 1, \tag{1.2a}$$

$$\sum_{i=1}^{N} p_i\, e^{-\sigma_i} \;=\; 1. \tag{1.2b}$$

Let σ_1 and σ_2 be the smallest and biggest of $\{\sigma_i, i = 1, \cdots, N\}$, and let us choose another point σ_3 such that $\sigma_1 < \sigma_3 < \sigma_2$. (If two or all of $\sigma_1, \sigma_2, \sigma_3$ are equal, we can tune $\sigma_1, \sigma_2, \sigma_3$ as well as p_1, p_2, p_3 such that $\sigma_1 < \sigma_3 < \sigma_2$ and at the same time the normalization conditions (1.2) remain valid, and $\langle \Sigma \rangle$ and $\langle Q \rangle$ remain fixed. This is always possible since we have six parameters to satisfy four independent constraints. Furthermore by choosing the direction of variation, we can always make sure that the variances of Σ and Q do not increase.) Now let us write down $\langle \Sigma \rangle, \langle \Sigma^2 \rangle, \langle Q \rangle, \langle Q^2 \rangle$:

$$\langle Q \rangle \;=\; p_1 q_1 + p_2 q_2 + p_3 q_3 + \sum_{i>3} p_i q_i, \tag{1.3}$$

$$\langle Q^2 \rangle \;=\; p_1 q_1^2 + p_2 q_2^2 + p_3 q_3^2 + \sum_{i>3} p_i q_i^2, \tag{1.4}$$

$$\langle \Sigma \rangle \;=\; p_1 \sigma_1 + p_2 \sigma_2 + p_3 \sigma_3 + \sum_{i>3} p_i \sigma_i, \tag{1.5}$$

$$\langle \Sigma^2 \rangle \;=\; p_1 \sigma_1^2 + p_2 \sigma_2^2 + p_3 \sigma_3^2 + \sum_{i>3} p_i \sigma_i^2. \tag{1.6}$$

We shall redistribute infinitesimally the probability p_3 to p_1, p_2, and at the same time tune σ_3 such that (a) Eqs. (1.2) remain valid, and (b) $\langle \Sigma \rangle$ remains invariant. Let us consider the infinitesimal transformation:

$$p_i \to p_i + dp_i, \quad i = 1, 2, 3, \tag{1.7}$$

$$\sigma_3 \to \sigma_3 + d\sigma_3, \tag{1.8}$$

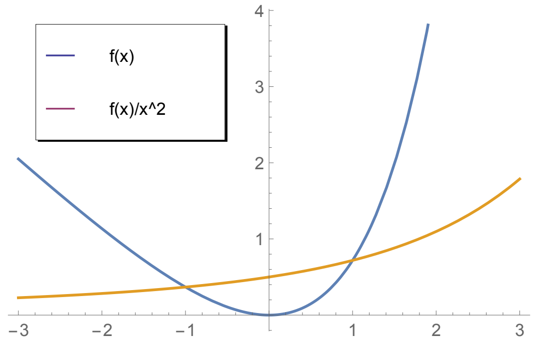

Figure 1: Plots of functions $f(x) = e^x - 1 - x$ and $f(x)/x^2$.

whereas all other p_k, σ_k remain invariant. Let us take the differential of Eqs. (1.2) and (1.5):

$$dp_1 + dp_2 + dp_3 = 0, \tag{1.9}$$

$$e^{-\sigma_1} dp_1 + e^{-\sigma_2} dp_2 + e^{-\sigma_3} dp_3 - e^{-\sigma_3} p_3 d\sigma_3 = 0, \tag{1.10}$$

$$d\langle \Sigma \rangle = \sigma_1 dp_1 + \sigma_2 dp_2 + \sigma_3 dp_3 + p_3 d\sigma_3 = 0. \tag{1.11}$$

Using these equations we can express $dp_2, dp_3, d\sigma_3$ in terms of dp_1. In particular we have

$$dp_2 = -\frac{f(\sigma_3 - \sigma_1)}{f(\sigma_3 - \sigma_2)} dp_1, \tag{1.12}$$

where $f(x)$ is a concave function with positive second derivative:

$$f(x) = e^x - 1 - x. \tag{1.13}$$

As illustrated in Fig. 1, $f(x)$ is non-negative and achieves minimum $f = 0$ at $x = 0$. Hence dp_2 and dp_1 always have different signs, since $\sigma_1 < \sigma_3 < \sigma_2$.

We can then calculate the differential of $\langle \Sigma^2 \rangle$ and express in terms of dp_1:

$$\begin{aligned} d\langle \Sigma^2 \rangle &= \frac{(\sigma_3 - \sigma_1)^2 (\sigma_3 - \sigma_2)^2}{f(\sigma_3 - \sigma_2)} \\ &\times \left[\frac{f(\sigma_3 - \sigma_2)}{(\sigma_3 - \sigma_2)^2} - \frac{f(\sigma_3 - \sigma_1)}{(\sigma_3 - \sigma_1)^2} \right] dp_1. \end{aligned} \tag{1.14}$$

As illustrated in Fig. 1, the function $f(x)/x^2$ is strictly monotonically increasing. Since $\sigma_3 - \sigma_2 < 0$ and $\sigma_3 - \sigma_1 > 0$, the difference in the bracket in the RHS of Eq. (1.14) is strictly negative. We shall chose $dp_1 > 0$, then we have $dp_2 < 0$, and $d\langle \Sigma^2 \rangle < 0$, and hence the variance $\langle \delta \Sigma^2 \rangle$ decreases.

Let us now take care of the other variable Q. Let us take the differential of Eq. (1.3) and set it to zero:

$$d\langle Q\rangle = \sum_{i=1}^{3} p_i dq_i + \sum_{i=1}^{3} q_i dp_i = 0. \tag{1.15}$$

With dp_2, dp_3 determined previously in terms of dp_1, this equation defines a plane in the three dimensional space spanned by dq_1, dq_2, dq_3. Now let us take the differential of Eq. (1.4) and set it to zero:

$$d\langle Q^2\rangle = \sum_i 2p_i q_i dq_i + \sum_i q_i^2 dp_i = 0, \tag{1.16}$$

which defines another plane in the same space. Assuming q_1, q_2, q_3 are not all identical, (If this condition is not satisfied, we can adjust q_1, q_2, q_3 such that they become different, whilst $\langle Q\rangle$ remains fixed.) these two planes are not parallel, and hence intersect each other on a straight line, which divided the plane Eq. (1.16) into two halves. As illustrated in Fig. 2, in one half, $d\langle Q^2\rangle$ is positive, whereas in the other half, $d\langle Q^2\rangle$ is negative. All we need is to pick an arbitrary point (dq_1, dq_2, dq_3) in the negative half plane, such that $d\langle Q^2\rangle$ is negative.

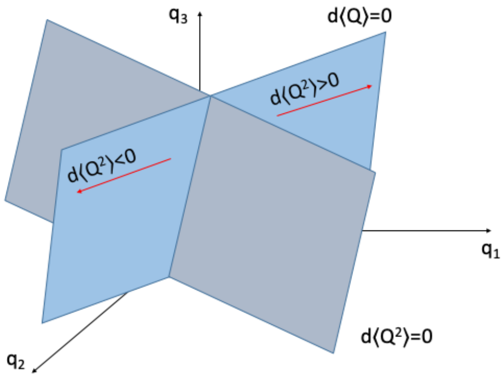

Figure 2: Illustration of the redistribution process to reduce the variance of Q.

Constructed as above, the redistribution process tunes the parameters $p_1, p_2, p_3, \sigma_3, q_1, q_2, q_3$ such that (i) Eq. (**??**) and the normalization conditions (1.2) remain valid, (ii) p_1 increases whereas p_2 decreases, (iii) the averages $\langle\Sigma\rangle$ and $\langle Q\rangle$ remain fixed, (iv) the variances $\langle\delta\Sigma^2\rangle$ and $\langle\delta Q^2\rangle$ decrease. Doing this iteratively either p_2 or p_3 will eventually become zero, and we can remove the point from the support of $p_N(\sigma, q)$ (and also remove the dual point from the support of $\bar{p}_N(\sigma, q)$). As a consequence we obtain a pair of $(N-1)$-point distributions $p_{N-1}(\sigma, q), \bar{p}_{N-1}(\sigma, q)$, such that $p_{N-1}(\sigma, q)$ has the same averages $\langle \Sigma \rangle$ and $\langle Q \rangle$, but smaller variances $\langle \delta\Sigma^2 \rangle$ and $\langle \delta Q^2 \rangle$ comparing to $p_N(\sigma, q)$. Keep doing this, we eventually obtain a pair of 2-point distributions $p_2(\sigma, q), \bar{p}_2(\sigma, q)$, such that $p_2(\sigma, q)$ has the same averages $\langle \Sigma \rangle$ and $\langle Q \rangle$, but smaller variances $\langle \delta\Sigma^2 \rangle$ and $\langle \delta Q^2 \rangle$ comparing to original distribution $p_N(\sigma, q)$.