On Weighted Pál Type (0,2)-interpolation on the Unit Circle

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ON WEIGHTED PÁL TYPE (0,2)-INTERPOLATION ON THE UNIT CIRCLE

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Abstract:
In this paper, we study the explicit representation of weighted Pál-type (0,2)-interpolation on two pairwise disjoint sets of nodes on the unit circle, which are obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n''(x)$ respectively, where $P_n(x)$ stands for n^{th} Legendre polynomial.
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Introduction:

In 1979,P. Turán[10] studied the (0,2) interpolation for getting an approximate solution of differential equation

$$y'' + fy = 0.$$

J. Balázs [6] introduced the weighted (0, 2)-interpolation on the zeros ofultraspherical polynomial $P_n^{(\alpha)}(x), \alpha > -1$. In 1961,O. Kiš [8] initated the Lacunary interpolation on the unit circle.He considered (0,2) interpolation on the unit circle and established the convergence theorem.After that several mathematician have considered (0,2) interpolation viz. on the unit circle, infinite interval and on the real line.In 1996 , Siqing Xie [11]considered (0,1,3) interpolation on the vertically projected nodesonto the unit circle. He claimed the regularity, explicit representation and convergence of (0,1,3) –interpolation.In 2003H.P. Dikshit [7]considered the Pál type interpolation on non –uniformly distributed nodes on the unit circle . After that author[1] (with K.K.Mathur) [1] considered the weighted $(0,2)^*$-interpolation on the set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ on the unit circle and established a convergence theorem for that interpolatory polynomial. In 2012, she [2,3] considered weighted (0;0,2) and (0,2;0)interpolation on projected nodes onto the unit circle,obtained the regularity, fundamental polynomial and established a convergence theorem.Later on author[1] (with M.Shukla) [4] considered (0,2)-interpolation on the nodes, which are obtained by projecting vertically the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial, obtained the explicit forms and established a convergence theorem for the same. Recently, authors [5]considered weighted (0,2)-interpolation on the nodes, which are obtained by projecting vertically the zeros of the$(1-x^2)P_n'(x)$onto the unit circle, established a convergence theorem for the same.
These have motivated us to consider (0;0,2) interpolation on two pairwise disjoint set of nodes on the unit circle.
Let,

$$(1.1)\ Z_n = \begin{Bmatrix} z_0 = 1, \quad z_{2n+1} = -1, \\ z_k = cos\theta_k + i\, sin\theta_k, \\ z_{n+k} = \bar{z_k}\ ,\ k = 1(1)n \end{Bmatrix}$$

$$(1.2) T_n = \begin{Bmatrix} t_k = cos\varphi_k + i\, sin\varphi_k, \\ t_{n+k} = \bar{t_k}\ ,\ k = 1(1)n-2 \end{Bmatrix}$$

be two set of nodes.In which the Lagrange data is prescribed on the first set of nodes whereas Lacunary data on the other one.We obtained regularity, explicit forms and established a convergence theorem of the interpolatory polynomials.
In section 2,we give some Preliminaries, in section3, we describe the problem and regularity, in section 4 and section 5, we presentthe explicit forms and convergence of weighted Pál-type (0,2)-interpolation on the unit circle respectively.

2. Preliminaries:

In this section, we shall usesome well-known results, which are :
The differential equation satisfied by $P_n(x)$is

(2.1)$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$

(2.2)$W(z) = \prod_{k=1}^{2n}(z - z_k) = K_n P_n\left(\frac{1+z^2}{2z}\right)z^n$

(2.3) $R(z) = (z^2-1)\, W(z)$

(2.4) $H(z) = \prod_{k=1}^{2n-4}(z - t_k) = K_n^* P_n''\left(\frac{1+z^2}{2z}\right)z^{n-2}$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of$W(z)$, R$(z)$and H(z) are respectively given as:

(2.5) $L_{1k}(z) = \frac{W(z)}{(z-z_k)\,W'(z_k)}$, $k = 1(1)2n$

(2.6) $L_k(z) = \frac{R(z)}{(z-z_k)\,R'(z_k)}$, $k = 0(1)2n+1$

(2.7) $l_k(z) = \frac{H(z)}{(z-t_k)H'(t_k)}$, $k = 1(1)2n-4$

(2.8) $J_{1k}(z) = \int_0^z t\ l_k(t)\ dt$

(2.9) $J_k(z) = \int_0^z H(t)\ dt$

which satisfies,

(2.10) $J_k(-z) = -\ J_k(z)$

(2.11) $W'(z_k) = \frac{1}{2}K_n(z_k^2-1)z_k^{n-2}P_n'(x_k)$

(2.12) $W''(z_k) = K_n[(n-1)(z_k^2-1)-1]z_k^{n-3}P_n'(x_k)$

(2.13) $W'(t_k) = \frac{1}{2}K_n\frac{n\{(n+3)(t_k^2-1)+4\}}{(t_k^2+1)}t_k^{n-1}P_n(x_k^*)$,

(2.14) $W''(t_k) = \frac{1}{2}K_n\frac{n(n-1)\{(n-1)(t_k^2-1)-1\}}{(t_k^2+1)}t_k^{n-2}P_n(x_k^*)$,

(2.15) $H'(t_k) = \frac{K_n^*}{2}(t_k^2-1)t_k^{n-4}P_n'''(x_k^*)$,

(2.16) $H''(t_k) = K_n^*\{(n-5)(t_k^2-1)-5\}t_k^{n-5}P_n'''(x_k^*)$,

We will also use the following well known inequalities:

For, $-1 < x < 1$

(2.17) $(1-x^2)^{1/4}|P_n(x)| \le \sqrt{\frac{2}{\pi}}n^{-1/2}$,

(2.18) $(1-x^2)^{3/4}|P_n'(x)| \le \sqrt{2}\,n^{1/2}$,

(2.19) $|P_n(x)| \le 1$,

Let $x_k = cos\theta_k (k = 1,2,\dots\dots,n)$ are the zeros of n^{th} Legendre polynomial $P_n(x)$,with $1 > x_1 > x_2 > \cdots\dots\dots\dots x_n > -1$, then

(2.20) $(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$

(2.21) $\left|P_n^{(r)}(x_k)\right| \sim k^{-r-\frac{1}{2}}\ n^{2r},\ r = 0,1,2,3.$

For more details, one can see [9].

3. The Problem and Regularity:

Let $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=1}^{2n-4}$ be two disjoint set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n''(x)$ onto the unit circle respectively, where $P_n(x)$ stands for n^{th} Legendre polynomial. Here we are interested to determine the following polynamial $Q_n(z)$ of degree $\le 6n-7$ satisfying the conditions:

$$\begin{cases} Q_n(z_k) = \alpha_k\ , & k = 0(1)2n+1 \\ Q_n(t_k) = \beta_k\ , & k = 1(1)2n-4 \\ [p(z)Q_n(z)]''_{z=t_k} = \gamma_k\ , & k = 1(1)2n-4 \end{cases}$$

where, α_k, β_k, γ_k arearbitrary complex constants andweight function $p(z) = z^{n(n-3)/2}(z^2-1)^{7/2}(z^2+1)^{-n(n+1)/2}$.

Theorem 3.1: $Q_n(z)$ is regular on Z_n and T_n.

Proof: It is sufficient, if we show that the unique solution of (3.1) is

$$Q_n(z) \equiv 0,$$

when all data $\alpha_k = \beta_k = \gamma_k = 0$.

In this case, we have

$Q_n(z) = W(z)H(z)q(z)$,

where, q (z) is a polynomial of degree $\le 2n-3$.

Obviously, $Q_n(z_k) = 0$, for $k = 1(1)2n$,

and also $Q_n(t_k) = 0$, for $k = 1(1)2n-4$.

Now, from

$$[p(z)Q_n(z)]''_{z=t_k} = 0$$

we get,

$$q'(t_k) = 0$$

Therefore,we have

$$q'(z) = aH(z)$$
$$q(z) = a\,J_k(z) + b$$

where, $J_k(z)$ is given in (2.9).

Now for $z = 1\&-1$, we get

$$a = b = 0$$

Therefore, $Q_n(z) \equiv 0$

Hence the theorem follows.

4. Explicit Representation Of Interpolatory Polynomials:

We shall write $Q_n(z)$ satisfying (3.1) as:

(4.1) $$Q_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n-4} \beta_k B_k(z) + \sum_{k=1}^{2n-4} \gamma_k C_k(z)$$

where, $A_k(z)$, $B_k(z)$ and $C_k(z)$are unique polynomial ,each of degree at most $6n-7$ satisfying the conditions:

For $k = 0(1)2n+1$

(4.2) $$\begin{cases} A_k(z_j) = \delta_{jk}, & j = 0(1)2n+1 \\ A_k(t_j) = 0, & j = 1(1)2n-4 \\ [p(z)A_k(z)]''_{z=t_j} = 0, & j = 1(1)2n-4 \end{cases}$$

For $k = 1(1)2n-4$

(4.3) $$\begin{cases} B_k(z_j) = 0, & j = 0(1)2n+1 \\ B_k(t_j) = \delta_{jk} & j = 1(1)2n-4 \\ [p(z)B_k(z)]''_{z=t_j} = 0, & j = 1(1)2n-4 \end{cases}$$

For $k = 1(1)2n-4$

(4.4) $$\begin{cases} C_k(z_j) = 0, & j = 0(1)2n+1 \\ C_k(t_j) = 0, & j = 1(1)2n-4 \\ [p(z)C_k(z)]''_{z=t_j} = \delta_{jk}, & j = 1(1)2n-4 \end{cases}$$

Theorem 4.1: For $k = 1(1)2n-4$, we have

(4.5) $C_k(z) = W(z)\, H(z)\{C_k J_{1k}(z) + C_{1k} J_k(z) + C_{2k}\}$

where,

(4.6) $C_k = \frac{1}{t_k p(t_k) w(t_k) H'(t_k)}$

(4.7) $C_{1k} = -\, C_{1k} \frac{\{J_{1k}(1) - J_{1k}(-1)\}}{2J_k(1)}$

(4.8) $C_{2k} = -\, C_{1k} \frac{\{J_{1k}(1) + (-1)\}}{2}$

Theorem 4.2: For,$k = 1(1)2n-4$, we have

(4.9)$B_k(z) = \frac{(z^2-1) l_k^2(z) W(z)}{(t_k^2-1) W(t_k)} + \frac{W(z)H(z)}{(t_k^2-1) H'(t_k) W(t_k)} \{S_k(z) + b_{1k} J_k(z) + b_{2k}\} + b_k C_k(z)$

where,

(4.10)$S_k(z) = -\int_0^z \frac{(t^2-1)[l_k'(t) - l_{1k}'(t_k) l_k(t)]}{(t-t_k)}\, dt$

(4.11) $b_{1k} = -\frac{\{S_k(1) - S_k(-1)\}}{2J_k(1)}$

(4.12) $b_{2k} = -\frac{\{S_k(1) + S_k(-1)\}}{2}$

(4.13) $b_k = -2\frac{p(t_k)}{(t_k^2-1)} - 4t_k l_k'(t_k) \frac{p(t_k)}{(t_k^2-1)} - 2p(t_k)\{l_k'(t_k)\}^2$

Theorem 4.3: For$k = 1(1)2n$, we have

(4.14) $A_k(z) = \frac{(z^2-1) L_{1k}(z) H^2(z)}{(z_k^2-1) H^2(z_k)} + \frac{H(z)W(z)}{H^3(z_k)(z_k^2-1) W'(z_k)} \{M_k(z) + a_{1k} J_k(z) + a_{2k}\}$

where,

(4.15)$M_k(z) = -\int_0^z \frac{[(t^2-1)H'(t)H(z_k) - (z_k^2-1)H'(z_k)H(t)]}{(t-z_k)}\, dt$

(4.16) $a_{1k} = -\frac{\{M_k(1) - M_k(-1)\}}{2J_k(1)}$

(4.17) $a_{2k} = -\frac{\{M_k(1) + M_k(-1)\}}{2}$

For$k = 0, 2n+1$, we have

(4.18) $A_k(z) = H(z)W(z)\{a_{1k}^* J_k(z) + a_{2k}^*\}$

(4.19) $a_{1k}^* = \frac{1}{W(1)H(1)\{J_k(1) - J_k(-1)\}}$

(4.20) $a_{2k}^* = -\frac{J_k(-1)}{W(1)H(1)\{J_k(1) - J_k(-1)\}}$

5. Estimation Of Fundamental Polynomials:

Lemma: Let$A_k(z), B_k(z)$ and $C_k(z)$ be defined in section4.Then for $|z| \le 1$

(5.1)$\sum_{k=0}^{2n+1} |p(z)A_k(z)| \le cn^2 \log n$,

(5.2)$\sum_{k=1}^{2n-4} |p(z)B_k(z)| \le cn^2 \log n$,

(5.3)$\sum_{k=1}^{2n-4} |p(z)C_k(z)| \le c \log n$,

where, c is a constant independent of n and z.

Proof: Usingthe conditions from (2.17)-(2.21),we get the result.

6.Convergence:

In this section, we prove the main theorem:

Theorem: Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Let the arbitrary γ_k 's be such that

(6.1) $|\gamma_k| = O\left(n^2 \omega\left(f, \frac{1}{n}\right)\right)$

Then $\{Q_n(z)\}$ defined by

(6.2) $$Q_n(z) = \sum_{k=0}^{2n+1} f(z_k)A_k(z) + \sum_{k=1}^{2n-4} f(t_k)B_k(z) + \sum_{k=1}^{2n-4} \gamma_k C_k(z)$$

satisfies the relation,

(6.3) $|p(z)\{Q_n(z) - f(z)\}| = O(\omega_2(f, n^{-1}) \log n)$,

where, $\omega_2(f, n^{-1})$ be the modulus of continuity of $f(z)$.

To prove the theorem , we shall need the followings :

Remark : Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, and $f' \in Lip\alpha, \alpha > 1$, then the sequence $\{Q_n(z)\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, which follows from (6.3) as

(6.4) $\omega_2(f, n^{-1}) = O(n^{-1-\alpha}), \alpha > 1$.

There exist a polynomial $F_n(z)$ of degree $\leq 6n - 7$ satisfying Jackson's inequality.

(6.5) $|f(z) - F_n(z)| \leq c\omega_2(f, n^{-1}),\ z = e^{i\theta} (0 \leq \theta < 2\pi)$

And also an inequality due to O.Kiš [8].

(6.6) $\left|F_n^{(m)}(z)\right| \leq cn^m \omega_2(f, n^{-1}),\ m \epsilon I^+$.

Proof: Since $Q_n(z)$ be is uniquely determined polynomial of degree $\leq 6n - 7$ and the polynomial $F_n(z)$ satisfying (6.5) and (6.6) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k)A_k(z) + \sum_{k=1}^{2n-4} F_n(t_k)B_k(z) \sum_{k=1}^{2n-4} F_n''(t_k)C_k(z)$$

Then

$$|p(z)\{Q_n(z) - f(z)\}| \leq |p(z)\{Q_n(z) - F_n(z)\}| + |p(z)||\{F_n(z) - f(z)\}|$$

$$\leq \sum_{k=0}^{2n+1} |f(z_k) - F_n(z_k)|\,|p(z)A_k(z)|$$

$$+ \sum_{k=1}^{2n-4} |f(t_k) - F_n(t_k)|\,|p(z)B_k(z)|$$

$$+ \sum_{k=1}^{2n-4} \{|\gamma_k| + |F_n''(t_k)|\}\,|p(z)B_k(z)|$$

$$+|p(z)||F_n(z) - f(z)|$$

Using (6.1),(6.4)-(6.6) and Lemma, we get (6.3).

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