Spatial Models for Infants HIV/AIDS Incidence Using an Integrated Nested Laplace Approximation Approach - Appendix

For the i-th district, the count of HIV cases y_i is modelled as $y_i \sim$ Poisson $(E_i \theta_i)$, where the mean θ_i is the risk of infection and E_i is the expected number of infants infected with HIV in district i, i = 1,2,...,47.
The linear predictor is defined on the logarithmic scale:

$$\eta_i = \beta_0 + \sum_{m=1}^{M} \beta_m t_{mi} + u_i,$$

where β_0 denotes HIV outcome rate for all the 47 counties and u_i are the spatially structured random effects. The latent (unobserved) variables, denoted by $\mathbf{x} = (\beta_0, \boldsymbol{\beta}, \boldsymbol{f}, \boldsymbol{\varepsilon})$ are assigned a Gaussian prior whereas $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ denotes the vector of K hyperparameters. Assuming conditional independence, the likelihood of the data is given by

$$p(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta}) = \prod_{i}^{n} p(y_i/x_i, \boldsymbol{\theta}).$$

Assume a multivariate Normal prior on $\mathbf{x}$ with mean $\mathbf{0}$ and precision matrix $\boldsymbol{Q}(\boldsymbol{\theta})$ (inverse covariance). That is,

$$p(\boldsymbol{x}|\boldsymbol{\theta}) \sim Normal(\mathbf{0}, \boldsymbol{Q}^{-1}),$$

with density function given by $p(\boldsymbol{x}|\boldsymbol{\theta}) = (2\pi)^{-n/2} \mid \boldsymbol{Q}(\boldsymbol{\theta}) \mid^{1/2} exp(-\frac{1}{2}\boldsymbol{x}'\boldsymbol{Q}(\boldsymbol{\theta})\boldsymbol{x})$. Assuming that $y_i, i = 1, 2, \ldots, n$ are conditionally independent, the joint posterior distribution of $\boldsymbol{x}$ and $\boldsymbol{\theta}$ is given by

$$p(\boldsymbol{x}, \boldsymbol{\theta}|\mathbf{y}) = p(\boldsymbol{\theta}) \times p(\boldsymbol{x}|\boldsymbol{\theta}) * p(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta}).$$

Turning to the continuous spatial field, the linear predictor is defined on the logarithmic scale:

$$\eta_i = \log(\pi) = \beta_0 + \sum_{m=1}^{M} \beta_m t_{mi} + S(x_i),$$

where β_0 is the intercept, β are the covariates effects, S = S(x_1),..., S(x_n) is a gaussian process,

$$\mu(x) = E[S(x)] \tag{1}$$

$$Cov(X(s), X(t)) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)}(\kappa \mid\mid t - s \mid\mid)^{\nu} K_{\nu}(\kappa \mid\mid t - s \mid\mid), \tag{2}$$

where || . || denotes the Euclidean distance in R^d, K_ν is the Bessel function of the second kind and order ν >0. $\kappa > 0$ is a scale parameter with the dimensions of distance and σ^2 is the marginal variance. The smoothness parameter ν ¿ 0 is a shape parameter that is usually fixed due to poor identifiability and it determines the differentiability of the underlying process.

Inference with INLA

INLA is a deterministic Bayesian inference approach which works by approximating the marginal posterior distributions for the parameter vectors. Marginal distributions are obtained by factoring out a term from the joint distribution. We need to solve the following integrals:

$$p(x_i|\boldsymbol{y}) = \int p(x_i, \boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta}. \tag{3}$$

$$p(\boldsymbol{\theta}_k|\boldsymbol{y}) = \int p(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta}_{-k}. \tag{4}$$

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Using the conditional probability rules (5) becomes

$$p(x_i|\boldsymbol{y}) = \int p(x_i, \boldsymbol{\theta}|\boldsymbol{y}) * p(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta}. \tag{5}$$

We need to compute

(i) $p(\boldsymbol{\theta}|\boldsymbol{y})$ from which all the $p(\boldsymbol{\theta}_k|\boldsymbol{y})$ can be obtained;

(ii) $p(x_i, \boldsymbol{\theta}|\boldsymbol{y})$. This is required to compute the posterior marginals of the latent field.

INLA performs numerical approximation to the posteriors of interest based on the Laplace approximations [1] and it proceeds in three steps:

1 Compute the posterior marginals of the hyperparameters p($\boldsymbol{\theta}|\boldsymbol{y}$)

2 Build a Laplace approximation (or or its simplified version) to p($x_i|\boldsymbol{\theta}$,$\mathbf{y}$).

3 Numerical integration (finite sum) to obtain approximations of the marginal of interest x_i

INLA can be combined with Stochastic partial differential equation (SPDE) for analysis of point reference data. This approach consists in representing a Gaussian Field with its covariance matrix as a Gaussian Markov Random Field [2] which in turn produces substantial computational advantages. Gaussian random fields are characterized by covariance matrices which tend to be dense when n is large. This makes computations involving these matrices slow or even infeasible. This problem is known as "the big n problem". To circumvent this problem, [3] provided an explicit link between Gaussian fields with the Matérn covariance function for certain choices of the smoothing parameter (ν) and GMRFs. This approach extends the work of [4] who approximated a Gaussian field when $\nu \rightarrow 0$ in the Matérn correlation function. Gaussian field with Matérn covariance is a stationary solution to the linear fractional Stochastic partial differential equation (SPDE) [5].

$$(\kappa^2 - \triangle)^{\frac{\alpha}{2}}(\tau x(s)) = \omega(s) \qquad u \in R^d, \qquad \alpha = \nu + \frac{d}{2}, \qquad \kappa > 0, \nu > 0, \tag{6}$$

where $\triangle$ is the Laplacian given by:

$$\triangle = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}. \tag{7}$$

κ is the scale parameter, α controls the smoothness, τ controls the variance and ω is spatial Gaussian white noise process.

By solving a certain SPDE, [3] showed that a GRF with a Mat$\acute{e}$rn correlation function and $\nu = 1$ or $\nu = 2$ has a GMRF representation.

Author details

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